Master Thesis in Mathematics

Representations of Iwahori-Hecke Algebras Induced from Parabolic Subalgebras

Von parabolischen Unteralgebren induzierte Darstellungen von Iwahori-Hecke-Algebren

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Introduction

The classification of finite simple groups is amongst the most significant results in the field of algebra achieved in the last century. It states that every finite simple group is either cyclic of prime order, an alternating group on at least 5 points, a simple group of Lie type or one of the 26 sporadic simple groups. All of these groups have been studied extensively and their representation theory in particular is of great interest to algebraists.

Iwahori-Hecke algebras occur naturally as endomorphism rings in the representation theory of finite groups of Lie type. It has been shown by Iwahori that these algebras have a presentation depending on a related Weyl group, see [Iwa64]. This led to an abstract definition of Iwahori-Hecke algebras for arbitrary Coxeter groups as all Weyl groups are Coxeter groups.

We will study the representation theory of Iwahori-Hecke algebras, which is closely related to the representation theory of the associated Coxeter groups. In particular, we will be interested in the structure of so-called *imprimitive* representations, which are representations induced from certain proper subalgebras. This construction is analogous to the representation theory of finite groups, where it is well known that we can induce representations from a subgroup to representations of the whole group. For Coxeter groups one is particularly interested in the representations induced from *parabolic* subgroups, that is subgroups generated by a proper subset of the generating set. In [HHM15, Lemma 8.2] Gerhard Hiß, William J. Husen and Kay Magaard showed that the representation obtained by inducing an ordinary representation of a proper parabolic subgroup of a Weyl group is always reducible. We will prove an analogous result for Iwahori-Hecke algebras.

This thesis is organized as follows:

In Chapter 1 we will give definitions of all basic structures we need, including Coxeter groups and, of course, Iwahori-Hecke algebras, as well as some terminology regarding the representation theory of algebras. This is accompanied by first results on the structure of imprimitive representations, in particular on their dimensions and characters. The last section is dedicated to the most important tool for the study of Iwahori-Hecke algebras, the concept of specialisation, and, building on that, the decomposition map.

Most definitions and results in Chapter 1 are taken from [GP00].

In Chapter 2 we study the representation theory of so-called generic Iwahori-Hecke algebras which are the logical starting point when using specialisation. As it turns out, almost everything can be deduced from the behaviour of these generic algebras in characteristic 0. Of particular interest are *equal-parameter* algebras, which we will therefore study in greater detail.

Chapter 3 contains an application of our results so far on one-parameter Iwahori-Hecke algebras of exceptional type in characteristic 0. We will show that under certain weak conditions on the field over which the algebra is defined, all imprimitive representations are reducible. One key ingredient in the proof are the decomposition matrices found in [GJ11] and [GP00], which enable us to use our earlier results. The second important tool is the GAP part of CHEVIE ([S$^+$97], [GHL$^+$96]) and its development version by Jean Michel([Mic15]).

Finally, Chapter 4 translates the results of Chapter 3 to characteristic $\ell > 0$. As it turns out, most of our work has already been done: Several ideas carry over from the characteristic 0 case with barely any work. Others carry over due to the work of Meinolf Geck and Jürgen Müller in [GM09] in which they proved *James's Conjecture* for Iwahori-Hecke algebras of exceptional type. It was first stated by Gordon James in [Jam90] in a slightly different context and basically says that the representation theory of an Iwahori-Hecke algebra does not depend on the characteristic, but only on the order of some parameter. Its proof by Geck and Müller for the exceptional algebras enables us to use our results in the characteristic 0 case to prove the analogous result in characteristic ℓ.

1. Basic Definitions and Results

This preliminary chapter will be used to lay a solid foundation to study Iwahori-Hecke algebras and their representation theory, in particular the theory of so-called *imprimitive representations* or *imprimitive modules*.
(Throughout this paper we will use the terms *module* and *representation* interchangeably. The connection between the two is studied in section 1.4.)
To define an Iwahori-Hecke algebra we first define Coxeter groups followed by some basic facts about these structures. After this, we define the corresponding parabolic sub-structures and the concept of an imprimitive module. Having established these concepts we present first results on the structure of imprimitive modules.
The second part of this chapter is concerned with the most important tool in the study of Iwahori-Hecke algebras, the concept of specialisation from a generic Iwahori-Hecke algebra. It is preceded by a short introduction into the language of representation theory and Groethendieck groups. The key result from the concept of specialisation is the decomposition map which we will study particularly in the context of imprimitive representations.
This chapter draws heavily from Chapters 4, 7 and 8 of [GP00].

1.1. Coxeter groups

Iwahori-Hecke algebra are algebras that are closely related to Coxeter groups. Therefore, we first introduce some notions about such groups.

Definition 1.1 A Coxeter group W is a group that has a presentation

$$W = \langle s \ \in S \mid (st)^{m_{st}}, \ s, \ t \in S \rangle$$

where the m_{st} are elements of $\mathbb{N} \cup \{\infty\}$ satisfying $m_{ss} = 1$ for all s and $m_{st} > 1$ whenever $s \neq t$. We call the tuple (W, S) a Coxeter system.

Remark 1.2 Suppose (W, S) is a finite Coxeter system, i.e. W is finite. Then we can represent the relations given by the m_{st}'s in the so-called *Coxeter graph*: Its vertex set is given by the generators $s \in S$, and two vertices s and t are connected if and only if m_{st} is greater than 2. If m_{st} is at least 4 the edge connecting s and t is labelled by m_{st}.
If the Coxeter graph of W is connected, we call W an *irreducible* Coxeter

group. Every finite Coxeter group is the direct product of finitely many irreducible Coxeter groups which correspond to the connected components of the Coxeter graph. The irreducible finite Coxeter groups are classified by the Coxeter graphs found in Table 1.1. In the cases of A_n, B_n and D_n there are exactly n vertices.
The groups $\{A_n \mid n \geq 1\}$, $\{B_n \mid n \geq 2\}$, $\{D_n \mid n \geq 4\}$, E_6, E_7, E_8, F_4 and G_2 are called *Weyl groups* or *crystallographic* Coxeter groups, while H_3, H_4 and $I_2(m)$ for $m = 5$ or $m \geq 7$ are called *non-crystallographic* Coxeter groups. An irreducible Coxeter group is crystallographic if and only if all m_{st}'s are either $1, 2, 3, 4$ or 6.
Another categorization of Coxeter groups can be made into the *classical* Coxeter groups $\{A_n \mid n \geq 1\}$, $\{B_n \mid n \geq 2\}$ and $\{D_n \mid n \geq 4\}$ in contrast to the *exceptional* Coxeter groups E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(m)$ for $m = 5$ or $m \geq 7$.

As for every group given by generators we can define a length function:

Definition 1.3 Let (W, S) be a Coxeter system. We define the *length function* $\ell : W \to \mathbb{N}_0$ sending each w to the length of a minimal expression of w in the generators, that is

$$\ell(w) = \min\{n \mid \exists\ s_1, \ldots, s_n \text{ such that } s_1 \cdots s_n = w\}.$$

In particular, $\ell(1)$ is 0.
We call $s_1 \cdots s_n$ a *reduced expression* of w if $w = s_1 \cdots s_n$ and $\ell(w) = n$.

Lemma 1.4 *For any $J \subseteq S$ the group $W_J := \langle s \mid s \in J \rangle \leq W$ is a Coxeter group in its own right. Hence, (W_J, J) is a Coxeter system and we call any such subgroup of W a* parabolic subgroup. *Its length function is the restriction of the length function of W to W_J.*

Proof See [GP00, 1.2.9, Proposition 1.2.10]. ∎

The following lemma is part of [GP00, Proposition 2.1.1].

Lemma 1.5 *Suppose (W, S) is a Coxeter system. Let J be a subset of S and define*

$$X_J := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$

This set has the following properties:
X_J is a set of coset representatives of W_J/W called the set of distinguished right coset representatives. This implies that for every $w \in W$ there exist unique $v \in W_J$ and $x \in X_J$ such that $w = vx$. For these elements we also have $\ell(w) = \ell(v) + \ell(x)$.

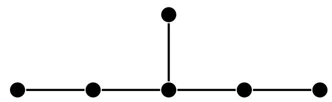
Table 1.1.: Coxeter graphs of irreducible Coxeter groups

A_n, $n \geq 1$

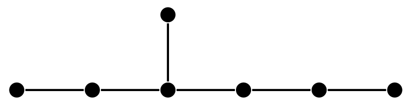
B_n, $n \geq 2$

D_n, $n \geq 4$

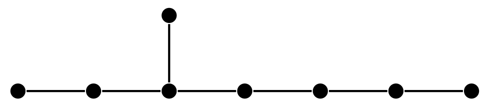
E_6



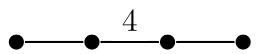
E_7



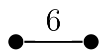
E_8



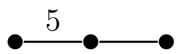
F_4



G_2



H_3



H_4

$I_2(m)$,
$m = 5$ or $m \geq 7$

1.2. Iwahori-Hecke algebras

We will now define the main object of interest in this thesis.

Definition 1.6 Suppose (W, S) is a Coxeter system. Let A be a commutative unitary ring and for every s in S let u_s be an element of A such that $u_s = u_t$ whenever s and t are conjugate in W. Let $H_A(W, S, (u_s \mid s \in S))$ be the quotient of the free associative A-algebra generated by $\{T_s \mid s \in S\}$ by the relations

- $T_s^2 = u_s \cdot 1 + (u_s - 1) \cdot T_s$ for every s in S (quadratic relations) and
- $\underbrace{T_sT_tT_s\cdots}_{m_{st}} = \underbrace{T_tT_sT_t\cdots}_{m_{st}}$ for every s and t in S (braid relations).

The resultant A-algebra is called the Iwahori-Hecke algebra of (W, S) over A with parameters $(u_s \mid s \in S)$.

Example 1.7 The Iwahori-Hecke algebra $H_A(W, S, (u_s := 1 \mid s\ inS))$ is naturally isomorphic to the group algebra $A[W]$ over A.

There are several observations to be made here. To do this we fix a Coxeter system (W, S) and an accompanying Iwahori-Hecke algebra $H := H_A(W, S, (u_s \mid s \in S)$ for some ring A with suitable elements u_s of A.

Definition 1.8 Let w be an element of W and $w = s_1 \cdots s_n$ a reduced expression of w. We define $T_w := T_{s_1} \cdots T_{s_n}$.

This definition yields several important results.

Lemma 1.9

- *T_w is well defined. In particular, T_w is independent of the choice of the reduced expression for w.*
- *T_1 is the neutral element for the multiplication in H.*
- *H is free as an A-module with a basis given by $\{T_w \mid w \in W\}$.*
- *If s is in S and w in W we have*

$$T_sT_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w) \\ u_sT_{sw} + (u_s - 1)T_w, & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Proof See [GP00, Lemma 4.4.3] and [GP00, Theorem 4.4.6]. ∎

1.3. Parabolic subalgebras and induction

The concept of parabolic subgroups caries over to the setting of Iwahori-Hecke algebras, see [GP00, 4.4.7]:

Definition 1.10 Suppose (W, S) is a Coxeter system. Let A be a commutative unitary ring and $H := H_A(W, S, (u_s \mid s \in S))$ an Iwahori-Hecke algebra over A for some parameters u_s such that $u_s = u_t$ whenever $s, t \in S$ are conjugate in W. Suppose that J is a proper subset of S and W_J the corresponding parabolic subgroup. The corresponding Iwahori-Hecke algebra $H_J := H_A(W_J, J, (u_s \mid s \in J))$ is called a *parabolic subalgebra* of H. There is a natural embedding $H_A(W_J, S, (u_s \mid s \in J) \hookrightarrow H$. This follows from the fact that both algebras are free as modules over A and that $\ell_{W_J}(v) = \ell_W(v)$ for all $v \in W_J$, i.e. the length function of W restricts to the length function of W_J.

Lemma 1.11 *Let $H_J \leq H$ be a parabolic subalgebra as above. Then H is a free (left-)-H_J-module with basis $\{T_x \mid x \in X_J\}$, where X_J is the set of distinguished right coset representatives of W_J in W.*

Proof Suppose w is in W. Then $w = w'x$ for a unique w' in W_J and x in X_J. Following Lemma 1.5 we know that $\ell(w) = \ell(w') + \ell(x)$ and therefore we also have $T_w = T_{w'}T_x$. By Lemma 1.9 this implies that H is isomorphic to $\bigoplus_{x \in X_J} H_J T_x$ as a left H_J-module. ∎

Similarly to the representation theory of finite groups there exists a concept of induction from parabolic subalgebras:

Definition 1.12 Suppose J is a proper subset of S and define the accompanying Iwahori-Hecke algebra by $H_J := H_A(W, S, (u_s \mid s \in S))$ regarded as a subalgebra of H. For any H_J-module M we define the *H-module induced from M* by

$$\mathrm{Ind}_J^S(M) := M \otimes_{H_J} H.$$

An H-module is called *imprimitive* if it is induced from a proper parabolic subalgebra.

The concept of induction satisfies a certain transitivity:

Lemma 1.13 *If $L \subseteq J \subseteq S$ is a chain of subsets of S and M is an H_L-module then the modules $\mathrm{Ind}_L^S(M)$ and $\mathrm{Ind}_J^S(\mathrm{Ind}_L^J(M))$ are isomorphic H-modules. That is, any H-module induced from an H_L-module is also induced from an H_J-module.*

Proof The isomorphism follows from the associativity of the tensor product. ∎

This lemma shows that it will be sufficient to consider maximal parabolic subalgebras to classify all imprimitive H-modules.

Together with the notion of the set of distinguished right coset representatives we get a first necessary condition on imprimitive modules:

Lemma 1.14 *Assume there exists a field $K \subseteq A$ and let $J \subseteq S$ and suppose that V is an H-module induced from an H_J-module V_J, that is $V = \mathrm{Ind}_J^S(V_J)$. The dimension of V is then given by the product of the subgroup index $[W : W_J]$ and the dimension of V_J, that is*

$$\dim_K(V) = [W : W_J] \dim_K(V_J).$$

Proof The field K embeds into H_J and therefore into H via $z \mapsto zT_1$. Let X_J be the set of distinguished right coset representatives for J. By Lemma 1.11 we know that H decomposes as the direct sum of $|X_J| = [W : W_J]$ many left H_J-modules, that is $\mathrm{Ind}_J^S(V_J) = V_J \otimes_{H_J} H \cong V_J \otimes_{H_J} \left(\bigoplus_{x \in X_J} H_J T_x \right)$. By [DF04, Theorem 17, Section 10.4] this is isomorphic as a K-vector space to $\bigoplus_{x \in X_J} \left(V_J \otimes_{H_J} (H_J T_x) \right)$.

For every v in W_J and x in X_J we have $\ell(vx) = \ell(v) + \ell(x)$ which translates to $T_v T_x = T_{vx}$ for the corresponding elements of H_J. The elements T_w for w in W are K-linearly independent, so this holds in particular for the pairwise distinct elements T_{vx} with v in W_J and some fixed x in X_J. Hence, hT_x for x in X_J and h in H_J is 0 if and only if h is 0. Therefore, $H_J T_x$ is isomorphic to H_J as an H_J-module which in turn implies that $V_J \otimes_{H_J} H_J T_x$ is isomorphic to V_J as an H_J-module. In particular, its K-dimension is exactly that of V_J and we have $\dim_K(V) = |X_J| \dim_K(V_J) = [W : W_J] \dim_K(V_J)$. ■

Lemma 1.11 has another important consequence for the structure of irreducible imprimitive modules:

Lemma 1.15 *Suppose A contains a field K and assume that $J \subsetneq S$ and V is an irreducible imprimitive H-module induced from an H_J-module V_J. Then V_J is irreducible.*

Proof Suppose V_J is not irreducible. Then there exists a non-zero H_J-submodule $M_J \leq V_J$ such that $M_J \neq V_J$. Let $\varphi : M_J \to V_J$ the natural embedding. Then φ is injective. Now $\varphi \otimes_{H_J} \mathrm{id}_H : M_J \otimes_{H_J} H \to V_J \otimes_{H_J} H$ is also injective because H is a free and therefore flat H_J-module. As H is a right H-module, this is a monomorphism of right H-modules. Because $M_J \otimes_{H_J} H$ and $V_J \otimes_{H_J} H$ are K-vector spaces of unequal, non-zero dimension we know that $\varphi \otimes_{H_J} \mathrm{id}_H$ is not surjective and that $M_J \otimes_{H_J} H$ is a proper non-zero H-submodule of $V_J \otimes_{H_J} H = V$ which renders V reducible. ■

In conclusion, we only need to check irreducible modules of maximal parabolic subalgebras to find all irreducible imprimitive modules.

1.4. Representations and characters

This thesis's goal is to study the module structure of Iwahori-Hecke algebras, that is to say their representation theory. The concept of a representation can be defined in a rather general setting, but we will restrict ourselves to a finite dimensional K-algebra H for some field K. Furthermore, in this section we only consider H-modules whose dimension as a K-vector space is finite unless explicitly stated otherwise.

Definition 1.16 Let V be an H-module. Then V affords a *representation* ρ_V which is a homomorphism $\rho_V : H \to \mathrm{End}_K(V)$. (This homomorphism defines the scalar multiplication of H on V, that is for $v \in V$ and $h \in H$ we have $v.h = (v)\rho_V(h)$ where . is the scalar multiplication of elements of H with elements of V.)
Now suppose V has dimension n as a K-vector space.
Choose an K-basis of V and view $\mathrm{End}_K(V)$ as $K^{n\times n}$, hence $\rho_V : H \to K^{n\times n}$. We call n the *degree of* ρ_V.
The function $\chi_V : H \to K : h \mapsto \mathrm{Tr}(\rho_V(h))$ mapping h to the trace of $\rho_V(h)$ is called the *character afforded by* V. It is a well defined K-linear map independent of the basis chosen for V.

Definition 1.17 Let V and V' be H-modules. We say ρ_V and $\rho_{V'}$ are *equivalent representations of* H if there exists an A-linear bijection $\varphi : V \to V'$ that is compatible with the representations: For all $v \in V$ and $h \in H$ we have

$$\varphi((v)\rho_V(h)) = (\varphi(v))\rho_{V'}(h).$$

Therefore, the representations ρ_V and $\rho_{V'}$ are equivalent if and only if V and V' are isomorphic as H-modules.

Lemma 1.18 *If V and V' are isomorphic H-modules they afford the same characters.*

Returning to our study of Iwahori-Hecke algebras and induced modules it will be useful to further examine the corresponding characters. As in the representation theory of finite groups the induction of modules carries over to the induction of the corresponding characters. To show this we will use techniques similar to those used in the proof of Lemma 1.14.

Lemma 1.19 *Let A be a commutative K-algebra, define an Iwahori-Hecke algebra $\mathcal{H}$ by $\mathcal{H} := H_A(W, S, (u_s \mid s \in S))$ and let $\mathcal{H}_J$ be a parabolic subalgebra for some $J \subseteq S$.*
Now let V_J be an $\mathcal{H}_J$-module of finite K-dimension. Then $\mathrm{Ind}_J^S(V_J)$ is an $\mathcal{H}$-module and we have seen in Lemma 1.14 that the dimension of $\mathrm{Ind}_J^S(V_J)$ is finite as the dimension of V_J is finite.
If χ_{V_J} is the character afforded by V_J, the character of $\mathrm{Ind}_J^S(V_J)$ will be denoted by $\mathrm{Ind}_J^S(\chi_{V_J})$. It can be computed as follows:

For $x, y \in X_J$ and w in W let $h^w(x,y)$ be the unique elements in $\mathcal{H}_J$ such that

$$T_x T_w = \sum_{y \in X_J} h^w(x,y) T_y.$$

We then have

$$\mathrm{Ind}(\chi_{V_J})(T_w) = \sum_{x \in X_J} \chi_{V_J}(h^w(x,x)).$$

Proof As we have see in the proof of Lemma 1.14 we have $\mathrm{Ind}_J^S(V_J) \cong \bigoplus_{x \in X_J} V_J$ as A-modules. Furthermore, if $B = (b_1, \ldots, b_n)$ is an A-basis of V_J, an A-basis of $\mathrm{Ind}_J^S(V_J)$ is given by $C := (b_1 \otimes T_x, \ldots, b_n \otimes T_x \mid x \in X_J)$. It remains to show the character formula: Let $\chi := \chi_{V_J}$. As $\mathcal{H} \cong \bigoplus_{x \in X_J} \mathcal{H}_J T_x$ we see that the $h^w(x,y)$ are well defined and unique. For some $b_i \otimes T_x$ in C and $w \in W$ we then have

$$\begin{aligned}(b_i \otimes T_x) T_w =& (b_i \otimes (T_x T_w)) \\ =& (b_i \otimes (\sum_{y \in X_J} h^w(x,y) T_y) \\ =& (\sum_{y \in X_J} b_i h^w(x,y)) \otimes T_y).\end{aligned}$$

To compute the character of T_w we only need to consider the coefficient of $b_i \otimes T_x$ in this product. As we have to sum over all $i = 1, \ldots, n$ this will contribute exactly $\chi_{V_J}(h^w(x,x))$.
To work this out precisely we define $\mathrm{coeff}(b_i \otimes T_x, z) \in A$ to be the coefficient at $(b_i \otimes T_x)$ in the linear combination of z for z in $\mathrm{Ind}_J^S(V_J)$. Analogously we define $\mathrm{coeff}(b_i, z')$ for $z' \in V_J$. This yields

$$\begin{aligned}\chi_{\mathrm{Ind}_J^S(V_J)}(T_w) =& \sum_{x \in X_J} \sum_{i=1}^{n} \mathrm{coeff}\,(b_i \otimes T_x, (b_i \otimes T_x) T_w) \\ =& \sum_{x \in X_J} \sum_{i=1}^{n} \mathrm{coeff}\left(b_i \otimes T_x, \sum_{y \in X_J} (b_i h^w(x,y) \otimes T_y)\right) \\ =& \sum_{x \in X_J} \sum_{i=1}^{n} \mathrm{coeff}\,(b_i \otimes T_x, (b_i\, h^w(x,x)) \otimes T_x) \\ =& \sum_{x \in X_J} \sum_{i=1}^{n} \mathrm{coeff}(b_i, b_i h^w(x,x)) \\ =& \sum_{x \in X_J} \chi(h^w(x,x)) \\ =& \chi\left(\sum_{x \in X_J} h^w(x,x)\right),\end{aligned}$$

where the last equality follows from the fact that χ is A-linear.
Note that we have used that $b_i z \otimes T_x$ is in $\langle b_1 \otimes T_x, \ldots, b_n \otimes T_x \rangle$ for every z in $\mathcal{H}_J$. ■

Remark 1.20 The previous Lemma yields an obvoius algorithm to compute the value of an induced character on T_w for some $w \in W$. It has been implemented by the author in GAP using CHEVIE.

When considering modules and their representations it will be convenient to do so using the notion of *Grothendieck groups*. Its definition can e.g. be found in [CR90, §16B].

Definition 1.21 The *Grothendieck group* of H is denoted by $R_0(H)$. It is defined as follows: Take the free abelian group generated by symbols (M), one for each isomorphism class of finitely generated H-modules M. Then consider the subgroup generated by short exact sequences: If M_1, M_2, and M are H-modules and $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence, then $(M) - (M_1) - (M_2)$ is one of the generators of this subgroup. The Grothendieck group of H then is defined as the factor of the free abelian group by this subgroup and we denote by $[M]$ the image of (M) under the canonical epimorphism.
Finally, we define $R_0^+(H)$ as the subset of $R_0(H)$ consisting of all elements $[M]$ where M is an H-module. This is a monoid with $[0]$ as its neutral element.

Remark 1.22 The algebra H is finite dimensional as a K-vector space and therefore the Grothendieck group is a free abelian group. A basis is given by the classes $[V]$ where V runs over the irreducible H-modules up to isomorphism. Furthermore, $[M] = [M']$ for two H-modules M and M' if and only if the two modules have the same composition factors counting multiplicities, see [CR90, Proposition 16.6].

Characters are well defined on elements of the Grothendieck group of the K-algebra H:

Lemma 1.23 *If V and V' are H-modules and $[V] = [V']$ in $R_0^+(H)$, then V and V' yield the same character.*

Proof If $[V]$ is equal to $[V']$ then V and V' have the same composition factors. The character of any H-lattice is the sum of the characters of all its composition factors, including multiplicities. This follows from the fact that the corresponding matrix representation can be taken to be of lower block triangular shape if we choose a basis corresponding to the decomposition series. Hence, the character of an H-lattice is independent of the order of the composition factors in a composition series and therefore V and V' yield the same character. ∎

The concept of a Grothendieck group is compatible with induction in Iwahori-Hecke algebras:

Lemma 1.24 *Let (W, S) be a finite Coxeter system and $u_s \in K$ for every $s \in S$ such that $u_s = u_t$ whenever s and t are conjugate in W.*
Let $\mathcal{H} := H_K(W, S, (u_s \mid s \in S))$ be the corresponding Iwahori-Hecke algebra and $\mathcal{H}_J$ a parabolic subalgebra for some proper subset J of S. Then the map

$$\mathrm{Ind}_J^S : R_0(\mathcal{H}_J) \to R_0(\mathcal{H}) : [V_J] \to [\mathrm{Ind}_J^S(V_J)]$$

is a well defined group homomorphism.

Proof Clearly, $\mathrm{Ind}_J^S(V_J)$ and $\mathrm{Ind}_J^S(V_J')$ are isomorphic $\mathcal{H}$-modules if V_J and V_J' are isomorphic $\mathcal{H}_J$-modules. This implies that Ind_J^S gives rise to a morphism $\widehat{\mathrm{Ind}_J^S}$ from the free abelian group generated by isomorphism classes of finitely generated $\mathcal{H}_J$-modules to $R_0(\mathcal{H})$. Since $\mathcal{H}$ is a free and therefore flat $\mathcal{H}_J$-module we know that Ind_J^S is an exact functor between the category of $\mathcal{H}_J$-modules and that of $\mathcal{H}$-modules and therefore it preserves short exact sequences. Hence, if $0 \to V_1 \to V \to V_2 \to 0$ is a short exact sequence of $\mathcal{H}_J$-modules, then $0 \to \mathrm{Ind}_J^S(V_1) \to \mathrm{Ind}_J^S(V) \to \mathrm{Ind}_J^S(V_2) \to 0$ is a short exact sequence of $\mathcal{H}$-modules, which implies that the kernel of $\widehat{\mathrm{Ind}_J^S}$ contains $(V) - (V_1) - (V_2)$. Since these are the defining relations for $R_0(H)$ it follows that Ind_J^S is well defined. ∎

Corollary 1.25 *Lemma 1.23 allows us to define characters generally on classes in the Grothendieck group rather than only on $R_0^+(H)$, since every element of $R_0(H)$ can be uniquely written as an integer linear combination in the classes of simple modules:*
If χ_V is the character afforded by an irreducible H-module V, the character of $\sum_V a_V[V]$ is defined as $\sum_V a_V \chi_V$, where the a_V are integer coefficients. By definition, this is compatible with induction in Iwahori-Hecke algebras.

1.5. Specialisation and decomposition maps

One of the most important tools for Iwahori-Hecke algebras is the concept of specialisation. It enables us to transfer acquired information about one Iwahori-Hecke algebra to another Iwahori-Hecke algebra for the same Coxeter group but possibly defined over another ring. Building on this concept we will define a *decomposition map* and study its properties. We follow Chapter 7 in [GP00].

Throughout this section we fix a finite Coxeter system (W, S), a commutative unitary ring A and an Iwahori-Hecke algebra $H := H_A(W, S, (u_s \mid s \in S))$ with parameters u_s over A.

Definition 1.26 If $\theta : A \to B$ is a ring homomorphism into a unitary commutative ring B, then the algebra $BH := H \otimes_A B$ is called the *specialisation of H via θ* where we regard B as an A-module via θ.

Lemma 1.27 *Suppose $\theta : A \to B$ is a ring homomorphism into a commutative unitary ring B. The specialisation BH of H via θ is naturally isomorphic to the Iwahori-Hecke algebra $H_B(W, S, (\theta(u_s) \mid s \in S))$, see [GP00, 8.1.2].*

Example 1.28 The algebra H is obtained by specialisation from an Iwahori-Hecke algebra over the ring $R := \mathbb{Z}[X_s \mid s \in S]$, where the X_s are indeterminates satisfying exactly the equations $X_s = X_t$ whenever s and t are conjugate in W. This can be seen as follows: There is a unique ring homomoprhism from $\mathbb{Z}$ to A, which we extend to R by mapping X_s to u_s. Thus, the algebra H is the specialisation of $H_R(W, S, (X_s \mid s \in S))$ via this ring homomorphism.

Example 1.29 If $\theta : A \to A : \ u_s \mapsto 1$ defines a ring homomorphism, then the specialisation H via θ is isomorphic to the group algebra AW.

The concepts of specialisation and induction are compatible:

Lemma 1.30 *Assume the setting of Definition 1.26. Let $J \subsetneq S$ be a proper subset of S and V_J an H_J-module. Then $\mathrm{Ind}_J^S(V_J) \otimes_A B$ is isomorphic as a BH-module to $\mathrm{Ind}_J^S(V_J \otimes_A B)$. We say that specialisation commutes with induction.*

Proof This is 9.1.5.c in [GP00]. ∎

For the remainder of this chapter we fix the following setting:
Suppose that A is an integral domain and K is a field containing A. The natural embedding $A \hookrightarrow K$ is a ring homomorphism. Denote by KH the Iwahori-Hecke algebra obtained by specialising via this map.
We will now study the representations of KH. To this end we define a generalisation of the concept of a splitting field for a group to serve us in the context of Iwahori-Hecke algebras .

Definition 1.31 Suppose B is a K-algebra and V an irreducible B-module. According to Schur the algebra E consisting of B-endomorphisms of V is a division ring over K. We call V a *split* module if the K-dimension of E is 1, that is $E \cong K$.
If all irreducible B-modules are split we call B itself *split* and K a *splitting field* for B.

Splitting fields will enable us to resort to irreducible Coxeter groups when studying irreducible imprimitive representations of KH:

Theorem 1.32 *Suppose $W = W_1 \times W_2$ for some non-trivial Coxeter groups W_1 and W_2 with corresponding sets of generators S_1 and S_2 which are subsets of S with $S = S_1 \dot{\cup} S_2$. In particular, W is not irreducible. Recall that*

$KH \cong H_K(W, S, (u_s \mid s \in S))$. If we define $H^i := H_K(W_i, S_i, (u_s \mid s \in S_i))$ for $i = 1, 2$ we obtain the following isomorphism of K-algebras:

$$KH \cong H^1 \otimes_K H^2$$

Suppose J' is a maximal subset of S. If the algebras H^1 and H^2 as well as the parabolic subalgebras $H^1_{J' \cap S_1}$ and $H^2_{J' \cap S_2}$ are split, there exists an irreducible KH-module induced from a $KH_{J'}$-module, if and only if there exists an irreducible H^i-module induced from an $H^i_{J' \cap S_i}$-module for $i = 1$ or $i = 2$.

Proof The isomorphism $KH \cong H^1 \otimes_K H^2$ can be found in [GP00, Exercise 8.4].
The statement on irreducible imprimitive modules is an analogue of a similar result on outer tensor products of modules of a group algebra. Our proof here is an adaptation of the proof of [Kar90, Theorem 6.2, Chapter 2], where this is indeed stated for group algebras.
Our first observation is that the irreducible modules of KH are exactly the tensor products of irreducible modules of H^1 and H^2. This follows from the fact K is a splitting field for H^1 and H^2 by [CR90, Theorem 10.38] in combination with the first Remark in [CR90, §7B].
Since J' is a maximal subset of S we can assume without loss of generality that $J' = J \dot{\cup} S_2$ for a maximal subset J of S_1.
Let $M_{J'}$ be a $KH_{J'}$-module. We will show that $\mathrm{Ind}^S_{J'}(M_{J'})$ is irreducible if and only if there exists an H^1_J-module M^1_J such that $\mathrm{Ind}^{S_1}_J(M^1_J)$ is irreducible. By Lemma 1.15 the module $\mathrm{Ind}^S_{J'}(M_{J'})$ can only be irreducible if $M_{J'}$ is irreducible.
Applying [CR90, Theorem 10.38] again we see that the irreducible modules of $KH_{J'}$ are exactly the tensor products of irreducible modules of H^1_J and $H^2_{S_2} = H^2$, since $KH_{J'}$ is isomorphic to $H^1_J \otimes_K H^2$.
Therefore, $M_{J'}$ is irreducible if and only if there exist an irreducible H^1_J-module M^1_J and an irreducible H^2-module M^2 such that $M_{J'}$ is isomorphic to the tensor product $M^1_J \otimes_K M^2$. Applying induction yields $\mathrm{Ind}^S_{J'}(M_{J'}) \cong (M^1_J \otimes_K M^2) \otimes_{KH_{J'}} (H^1 \otimes_K H^2) =: \mathfrak{A}$. We are done if we can show that this is isomorphic as a KH-module to $\mathrm{Ind}^S_J(M^1_J) \otimes_K M^2 = \left(M^1_J \otimes_{H^1_J} H^1\right) \otimes_K M^2 =: \mathfrak{B}$, since $\mathfrak{B}$ is irreducible if and only if $\mathrm{Ind}^S_J(M^1_J)$ is irreducible.
We use the direct sum decomposition of parabolic subalgebras to obtain two isomorphisms of K-vector spaces, where we use the commutativity of the tensor product with the direct sum, see [DF04, Theorem 17, Section 10.4]:

$$\mathfrak{A} \cong \bigoplus_{x \in X_J} \left(M^1_J \otimes_K M^2\right) \otimes_{KH_{J'}} KH_{J'} (T_x \otimes_K T_1)$$
$$\mathfrak{B} \cong \bigoplus_{x \in X_J} \left(M^1_J \otimes_{H^1_J} H^1_J T_x\right) \otimes_K M^2$$

For these isomorphisms note that the set X_J of distinguished right coset representatives of $(W_1)_J \leq W_1$ can be identified with a subset of $W = W_1 \times W_2$

by sending x to $(x, 1)$. The resulting set is clearly the set of distinguished right coset representatives of $W_{J'}$ in W, because the length of an element $(w_1,\ w_2) \in W_1 \times W_2$ is simply $\ell_{W_1}(w_1) + \ell_{W_2}(w_2)$, where ℓ_{W_i} is the length function of the Coxeter group W_i.
Suppose that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_r)$ are K-bases for M^1_J and M^2 respectively. These can be used to construct bases for the related structures:

- $(a_i \otimes_K b_j \mid i \in \{1, \ldots, n\},\ j \in \{1, \ldots, r\}\}$ is a K-basis for $M^1_J \otimes_K M^2$. Therefore, $((a_i \otimes_K b_j) \otimes_{KH_{J'}} (T_x \otimes_K T_1) \mid i \in \{1, \ldots, n\},\ j \in \{1, \ldots, r\},\ x \in X_J)$ is a K-basis for $\mathfrak{A}$.

- $(a_i \otimes_{H^1_J} T_x \mid i \in \{1, \ldots, n\},\ x \in X_J)$ is a K-basis for $M^1_J \otimes_{H^1_J} H^1$. Therefore, $((a_i \otimes_{H^1_J} T_x) \otimes_K b_j \mid i \in \{1, \ldots, n\},\ j \in \{1, \ldots, r\},\ x \in X_J)$ is a K-basis for $\mathfrak{B}$.

As in the proof of Lemma 1.14 this follows from the fact that $KH_{J'}T_x$ is isomorphic to $KH_{J'}$ and that $H^1_J T_x$ is isomorphic to H^1_J.
Clearly,

$$f : \mathfrak{A} \to \mathfrak{B} : (a_i \otimes_K b_j) \otimes_{KH_{J'}} (T_x \otimes_K T_1) \mapsto (a_i \otimes_{H^1_J} T_x) \otimes_K b_j$$

defines an isomorphism of K-vector spaces. We conclude the proof by showing that this map is compatible with the KH-structure of $\mathfrak{A}$ and $\mathfrak{B}$.
A K-basis for KH is given by $(T_{w_1} \otimes_K T_{w_2} \mid w_i \in W_i,\ i = 1, 2)$ and it suffices to consider the multiplication of elements of this basis with basis elements of $\mathfrak{A}$.
Now let w_1 be an element of W_1 and w_2 one of W_2. Furthermore, let $y \in X_J$ as well as $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r\}$. We want to show that

$$f\left[\Big((a_i \otimes_K b_j) \otimes_{KH_{J'}} (T_y \otimes_K T_1)\Big)\,(T_{w_1} \otimes_K T_{w_2})\right]$$
$$=$$
$$f\left[\Big((a_i \otimes_K b_j) \otimes_{KH_{J'}} (T_y \otimes_K T_1)\Big)\right](T_{w_1} \otimes_K T_{w_2}),$$

where the last term can be simplified to $\Big(a_i \otimes_{H^1_J} T_y T_{w_1}\Big) \otimes_K (b_j \otimes_{H^2} T_{w_2})$
For every v in W_J there exist c_v in K, x_v in X_J and g_v in $(W_1)_J$ such that $T_y T_{w_1} = \sum_v c_v T_{g_v} T_{x_v}$. These elements are defined by the decomposition of H^1 into the direct sum of left H^1_J-modules.

We get the following equations:

$$\begin{aligned}
& f\left[\Big((a_i \otimes_K b_j) \otimes_{KH_{J'}} (T_y \otimes_K T_1)\Big)(T_{w_1} \otimes_K T_{w_2})\right] \\
= \;& f\left[\Big((a_i \otimes_K b_j) \otimes_{KH_{J'}} (T_y T_{w_1} \otimes_K T_{w_2})\Big)\right] \\
= \;& f\left[\left((a_i \otimes_K b_j) \otimes_{KH_{J'}} \left(\left(\sum_v c_v T_{g_v} T_{x_v}\right) \otimes_K T_1 T_{w_2}\right)\right)\right] \\
= \;& f\left[\sum_v (a_i \otimes b_j) \otimes_{KH_{J'}} (c_v T_{g_v} T_{x_v} \otimes T_{w_2})\right] \\
= \;& \sum_v c_v f\left[(a_i T_{g_v} \otimes b_j T_{w_2}) \otimes_{KH_{J'}} (T_{x_v} \otimes T_1)\right] \\
= \;& \sum_v c_v \left(a_i T_{g_v} \otimes_{H_J^1} T_{x_v}\right) \otimes_K (b_j T_{w_2} \otimes_{H^2} T_1) \\
= \;& \left(a_i \otimes_{H_J^1} \sum_v c_v T_{g_v} T_{x_v}\right) \otimes_K (b_j \otimes_{H^2} T_{w_2}) \\
= \;& \left((a_i \otimes_{H_J^1} T_y T_{w_1}\right) \otimes_K (b_j \otimes_{H^2} T_{w_2}) \\
= \;& f\left[\Big((a_i \otimes_K b_j) \otimes_{KH_{J'}} T_y\Big)\right](T_{w_1} \otimes_K T_{w_2})
\end{aligned}$$

In conclusion, we have shown that f is indeed an KH-module isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. ■

Once an algebra split we need not concern ourselves with further field extensions. The analogous result for group algebras is [Isa76, Theorem 9.9]. To phrase it in a general way we first define a useful function:

Definition 1.33 Let $\mathfrak{p}_K : R_0^+(KH) \to \mathrm{Maps}(H, K[X])$ be the map sending $[V]$ to the map which sends a $h \in H$ to the minimal polynomial of $\varphi_V(h)$ where φ_V is a representation of KH afforded by V. Here we identify $h \in H$ with $h \otimes 1$ in KH.

Lemma 1.34 *Let $K \subseteq K'$ be a field extension and assume that KH is split. Then $K'H$ is also split and if V' is a $K'H$-module there exists a KH-module V realising V', that is $V \otimes_K K' \cong V'$. Moreover, there exists a canonical isomorphism*

$$d_K^{K'} : R_0(KH) \to R_0(K'H) : [V] \mapsto [V \otimes_K K']$$

mapping classes of irreducible modules to classes of irreducible modules. Furthermore, if $t_K^{K'} : \mathrm{Maps}(H, K[X]) \to \mathrm{Maps}(H, K'[X])$ is the canonical embedding, the following diagram commutes:

$$\begin{array}{ccc}
R_0^+(KH) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(H, K[X]) \\
\big\downarrow d_K^{K'} & & \big\downarrow t_K^{K'} \\
R_0^+(K'H) & \xrightarrow{\mathfrak{p}_{K'}} & \mathrm{Maps}(H, K'[X])
\end{array}$$

Proof See [GP00, Lemma 7.3.4]. ■

We note that for our study of imprimitive representations we can interchangeably consider different field extensions of K if K satisfies certain conditions:

Corollary 1.35 *Let $K \subseteq K'$ be a field extension. Suppose that KH is split as well as KH_J for every maximal subset J of S. A KH-module V is irreducible imprimitive if and only if $V \otimes_K K'$ is irreducible imprimitive.*
Since all modules of $K'H$ are realisable over K we see that there exists an irreducible imprimitive KH-module if and only if there exists one for $K'H$.

Proof Assume that V is irreducible and induced from V_J for some maximal subset $J \subseteq S$, that is $V = V_J \otimes_{KH_J} KH$. Because induction and specialisation commute, we know that

$$[V \otimes K'] = [\mathrm{Ind}_J^S(V_J) \otimes K'] = [\mathrm{Ind}_J^S(V_J \otimes K')].$$

Because V is irreducible and $d_K^{K'}$ sends classes of irreducibles to classes of irreducible modules, $V \otimes K'$ too, is irreducible. It is therefore isomorphic to $\mathrm{Ind}_J(V_J \otimes K')$ and hence imprimitive.

Conversely, assume that $V' := V \otimes K'$ is irreducible and imprimitive, induced from a $K'H$-module M_J for some maximal $J \subseteq S$. Then V is irreducible as $d_K^{K'}([V]) = [V']$ and V' is irreducible.
We have to show that V is imprimitive. For this, we consider Lemma 1.34 for KH_J and $K'H_J$. Because KH_J is split, there is a bijective map $d' : R_0(KH_J) \to R_0(K'H_J)$. Due to this map's surjectivity, there exists a KH_J-module N_J such that $[N_J \otimes K'] = [M_J]$. Because M_J induces to $V \otimes K'$ which is irreducible, M_J, too, is irreducible according to Lemma 1.15. Hence, $N_J \otimes K'$ is isomorphic to M_J and therefore $[V \otimes K'] = [\mathrm{Ind}_J^S(N_J) \otimes K']$, where we once again use the commutativity of induction and specialisation. From the injectivity of $d_K^{K'}$ it follows that $[V] = [\mathrm{Ind}_J^S(N_J)]$ and because V is irreducible this shows that V is isomorphic to $\mathrm{Ind}_J^S(N_J)$, so V is imprimitive. ■

The aforementioned *decomposition map* is a generalisation of the map $d_K^{K'}$. For this, we need some facts about valuation rings which can be found in 7.3.5 in [GP00].

Remark 1.36 Let K be a field and A a subring of K. A subring $\mathcal{O}$ of K is called a *valuation ring* if for every $0 \neq x$ in K either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$ holds. Such a ring is local, i.e. its Jacobian radical $J(\mathcal{O})$ is a maximal ideal. The following holds:

V1) If $I \subseteq A$ is a prime ideal there exists a valuation ring $\mathcal{O} \subseteq K$ containing A such that $J(\mathcal{O}) \cap A = I$.

V2) The intersection of all valuation rings $\mathcal{O} \subseteq K$ with $A \subseteq \mathcal{O}$ is the integral closure of A in K. Each valuation ring itself is therefore integrally closed in K.

Lemma 1.37 *If $\mathcal{O}$ is a valuation ring in K and V is a KH-module affording a representation $\rho : KH \to M_n(K)$, the K-basis for V can be chosen such that for every h in H every entry of $\rho(h)$ lies in $\mathcal{O}$. More precisely, there exists an $\mathcal{O}H$-module $\tilde{V}$ such that $V \cong K\tilde{V}$, where $K\tilde{V}$ is defined as $\tilde{V} \otimes_{\mathcal{O}} K$.*

Proof See [GP00, 7.3.5]. ■

Using this fact yields information about the images of $\mathfrak{p}_K$. Recall that A is an integral domain contained in K.

Corollary 1.38 *Let A^* be the integral closure of A in K. Then for every $[V] \in R_0^+(KH)$ and every $h \in KH$ we have*

$$\mathfrak{p}_K([V])(h) \in A^*[X],$$

which follows from Lemma 1.37 and the fact that the intersection of all valuation rings lying above A is the integral closure of A. Hence, we can redefine $\mathfrak{p}_K$ such that its co-domain is $\mathrm{Maps}(H, A^*[X])$.

We further refine our setting to define the so-called decomposition map:
Let A be an integral domain in a field K in which it is integrally closed. The map $\theta : A \to L$ is a ring homomorphism into a field L such that L is the field of fractions of $\theta(A)$. Therefore, $\ker(\theta)$ is a prime ideal of A and there exists a valuation ring $\mathcal{O}$ in K containing A such that $J(\mathcal{O}) \cap A = \ker(\theta)$. Let $\mathcal{O}$ be such a valuation ring and $k := \mathcal{O}/J(\mathcal{O})$ its field of residues with $\pi : \mathcal{O} \to k$ defined as the natural epimorphism. As before, let H be an Iwahori-Hecke algebra over A. The specialisations via the inclusion $A \hookrightarrow K$ and via θ will be denoted by KH and LH respectively. We assume that both KH and LH are split.

Lemma 1.39 *We can regard L as a subfield of k. The setting is then visualized in the following diagram:*

$$\begin{array}{ccccc} A & \subseteq & \mathcal{O} & \subseteq & K \\ \theta\big\downarrow & & \pi\big\downarrow & & \\ L & \subseteq & k & & \end{array}$$

Proof We know that $J(\mathcal{O}) \cap A = \ker(\theta)$. Therefore, $\ker(\pi|_A) = \ker(\theta)$. By the homomorphism theorem, $\theta(A)$ is isomorphic to $\pi(A) \subseteq k$. Since L is the field of fractions of $\theta(A)$ and $\theta(A)$ is a subset of k, we know that L, too, is included in k. ∎

Remark 1.40 Because LH is split, so is kH. By Lemma 1.34 this implies that $R_0^+(LH)$ and $R_0^+(kH)$ are isomorphic as monoids, so we will identity the two from now on.

We are now able to define the decomposition map corresponding θ. For this let $t_\theta : \mathrm{Maps}(H, A[X]) \to \mathrm{Maps}(H, L[X])$ be the natural map derived from $\theta : A \to L$ by applying θ to every coefficient of a polynomial in $A[X]$.

Theorem 1.41 (Decomposition Map) *a) If V is a KH-module, then there exists an $\mathcal{O}H$-module $\tilde{V}$ which is finitely generated and free over $\mathcal{O}$ realising V. This induces a well defined map $d_\theta : R_0^+(KH) \to R_0^+(LH)$ satisfying $d_\theta([K\tilde{V}]) = [k\tilde{V}]$, where $k\tilde{V} := \tilde{V} \otimes_{\mathcal{O}} k$ is regarded as an element of $R_0^+(LH)$ as in Remark 1.40.*

b) The diagram

$$\begin{array}{ccc} R_0^+(KH) & \xrightarrow{\mathfrak{p}_K} & Maps(H, A[X]) \\ \big\downarrow d_\theta & & \big\downarrow t_\theta \\ R_0^+(LH) & \xrightarrow{\mathfrak{p}_L} & Maps(H, L[X]) \end{array}$$

commutes.

c) d_θ is uniquely defined by the commutativity of this diagram and independent of the choice of $\mathcal{O}$.

d) The matrix corresponding to d_θ with respect to bases of $R_0^+(KH)$ and $R_0^+(LH)$ given by the classes of irreducibles modules is called the Decomposition matrix of θ *and denoted by D_θ. All its entries are non-negative integers.*

Proof See [GP00, Theorem 7.4.3]. ∎

The decomposition map carries information about KH-modules over to LH-modules:

Lemma 1.42 *a) Let χ be a character of KH afforded by a KH-module V. This restricts to a character $\dot{\chi} : H \to A$. This defines an L-linear map $\dot{\chi}^L : LH \to L$ by sending $h \otimes 1$ to $\theta(\dot{\chi}(h))$. This $\dot{\chi}^L$ is the character of $d_\theta([V])$. We call it the* specialisation of χ.

Using the decomposition matrix $D_\theta = (d_{V,V'})$ we obtain a character decomposition

$$\dot{\chi}^L = \sum_{V'} d_{V,V'}\ \varphi_{V'},$$

where V' runs over all simple LH modules (up to isomorphism) and $\varphi_{V'}$ is the character afforded by V'.

b) Let V be a KH-module and $d_\theta([V]) = [V']$. Then we have

$$\dim_K(V) = \dim_L(V').$$

Proof The first part of this proof can be found in [GP00, Remark 7.4.4]. We first show that $\dot{\chi}$ is well defined, particularly that $\chi(h)$ is in A for all h in H. For this consider $\mathfrak{p}_K([V])(h)$ which is the characteristic polynomial of a representation of h afforded by V. As seen in Corollary 1.38 we have $\mathfrak{p}_K([V])(h) \in A[X]$, that is all coefficients of this characteristic polynomial lie in A. Since $\chi(h)$ is a coefficient in said polynomial, $\dot{\chi}$ is well defined.
The fact that $\dot{\chi}^L$ is the character corresponding to $d_\theta([V])$ follows directly from the commutativity of the diagram in Theorem 1.41, using the fact that character values are coefficients in characteristic polynomials.
The dimension equality is shown similarly: The degree of $\mathfrak{p}_K([V])$ is the K-dimension of V and $\mathfrak{p}_L([V'])$ is the L-dimension of V'. Since characteristic polynomials are monic and $\theta(1) = 1$, the claim now follows from the diagram's commutativity. ∎

Corollary 1.43 *The decomposition map has a natural extension to the Grothendieck group $R_0(KH)$: Every element of the Grothendieck group can be uniquely written as an integer linear combination in the classes of irreducible modules. We define the extension of d_θ by $d_\theta(\sum_V a_V\ [V]) := \sum_V a_V d_\theta([V]) \in R_0(LH)$, where V runs over the irreducible KH-modules up to isomorphism and the a_V are some integers.*
We want to express this in terms of characters. Let h be an element of H and denote by χ_M and $\varphi_{M'}$ the characters of classes M and M' in $R_0(KH)$ and $R_0(LH)$ respectively. This gives us the following equation:

$$\begin{aligned}
\varphi_{\left(d_\theta(\sum_V a_V\ [V])\right)}(h \otimes_A 1) &= \varphi_{\left(\sum_V a_V d_\theta([V])\right)}(h \otimes_A 1) \\
&= \sum_V a_V\ \varphi_{(d_\theta([V]))}(h \otimes_A 1) \\
&= \sum_V a_V\ \theta\left(\dot{\chi}_V(h)\right) \\
&= \theta\left(\sum_V a_V \dot{\chi}_V(h)\right) \\
&= \theta\left(\dot{\chi}_{\left(\sum_V a_V [V]\right)}\ (h)\right)
\end{aligned}$$

If LH is semisimple, the decomposition map is trivial:

Theorem 1.44 (Tits' Deformation Theorem) *As before let us assume that KH and LH are split. Furthermore, assume that LH is semisimple. Then the algebra KH is also semisimple and the decomposition map d_θ is an isomorphism which preserves isomorphism classes of simple modules. In particular, the map* $\mathrm{Irr}(KH) \to \mathrm{Irr}(LH) : \chi \mapsto \dot{\chi}^L$ *is a bijection.*

Proof [GP00, Theorem 7.4.6] ∎

We consider trivial decomposition maps in the context of imprimitive representations:

Theorem 1.45 *Assume that both KH and LH are split and that the decomposition map d_θ is a bijection sending classes of irreducible modules to classes of irreducible modules. Then the following holds:*

a) If KH has an imprimitive irreducible representation so does LH.

b) Assume additionally that J is a proper subset of S, the parabolic subalgebras KH_J and LH_J are split, and the corresponding decomposition map $d_{\theta,J} : R_0^+(KH_J) \to R_0^+(LH_J)$ is also bijective sending classes of irreducible modules to classes of irreducible moduels. Under these conditions, if LH has an irreducible imprimitive representation induced from an LH_J-module, then KH has an irreducible imprimitive representation induced from a KH_J-module.

Proof Recall that $\mathcal{O}$ is a valuation ring in K with $A \subseteq \mathcal{O} \subseteq K$ and $J(\mathcal{O}) \cap A = \ker(\theta)$.
To prove a), let V be an irreducible imprimitive KH-module induced from an irreducible KH_J-module V_J. Then there exists an $\mathcal{O}H$-lattice V' such that $V \cong V' \otimes_{\mathcal{O}} K$. Similarly, there exists an $\mathcal{O}H_J$-lattice V'_J realizing V_J. Using the commutativity of specialisation and induction we get the chain following of equations:

$$\begin{aligned}
[V' \otimes_{\mathcal{O}} L] &= d_\theta([V' \otimes_{\mathcal{O}} K]) \\
&= d_\theta([V]) \\
&= d_\theta([\mathrm{Ind}_J^S(V_J)]) \\
&= d_\theta([\mathrm{Ind}_J^S(V'_J \otimes_{\mathcal{O}} K)]) \\
&= d_\theta([\mathrm{Ind}_J^S(V'_J) \otimes_{\mathcal{O}} K]) \\
&= [\mathrm{Ind}_J^S(V'_J) \otimes_{\mathcal{O}} L] \\
&= [\mathrm{Ind}_J^S(V'_J \otimes_{\mathcal{O}} L)].
\end{aligned}$$

$V' \otimes_{\mathcal{O}} L$ is irreducible because d_θ sends classes of irreducible modules of classes of irreducible modules. Hence, our array of equation gives us an irreducible module $M := V' \otimes_{\mathcal{O}} L$ with $M \cong \mathrm{Ind}_J^S(V'_J \otimes_{\mathcal{O}} L)$ and therefore M is an irreducible imprimitive representation of LH proving a).

Now we prove b), which is the converse of a) under some additional assumptions. Let M be an irreducible imprimitive representation of LH induced from an LH_J-representation M_J. Since $d_{\theta,J}$ is bijective, there exists an $\mathcal{O}H_J$-lattice N_J such that $[N_J \otimes_{\mathcal{O}} L] = [M_J]$ (Choose an $\mathcal{O}H_J$-lattice affording a representative of the class $d_{\theta,J}^{-1}([M_J])$). Since M_J is irreducible according to Lemma 1.15, this implies $M_J \cong N_J \otimes_{\mathcal{O}} L$. Hence, we see that

$$\begin{aligned} d_\theta^{-1}([M]) &= d_\theta^{-1}([\mathrm{Ind}_J^S(M_J)]) \\ &= d_\theta^{-1}([\mathrm{Ind}_J^S(N_J) \otimes_{\mathcal{O}} L]) \\ &= [\mathrm{Ind}_J^S(N_J) \otimes_{\mathcal{O}} K)] \\ &= [\mathrm{Ind}_J^S(N_J \otimes_{\mathcal{O}} K)]. \end{aligned}$$

Therefore, if $[V] = d_\theta^{-1}([M])$, then V is irreducible because M is irreducible and therefore V is imprimitive, as it is isomorphic to $\mathrm{Ind}_J^S(N \otimes_{\mathcal{O}} K)$. ∎

Remark 1.46 The conditions on the decomposition maps in the last theorem are for example satisfied if LH and LH_J are semisimple. This follows directly from Tits' Deformation Theorem.

Under similar but weaker conditions we can also show that the decomposition map commutes with induction:

Lemma 1.47 *Additionally to the initial conditions in Theorem 1.41 suppose that J is a proper subset of S and that KH_J and LH_J are split. Then there exists a second well-defined decomposition map $d_{\theta,J} : R_0(KH_J) \to R_0(LH_J)$. Suppose $M = \sum_V a_V\ [V]$ is an element of $R_0(KH_J)$, where the Vs are irreducible KH_J-modules and the a_V are integer coefficients. Then*

$$\mathrm{Ind}_J^S\left(d_{\theta,J}(M)\right) = d_\theta\left(\mathrm{Ind}_J^S(M)\right).$$

Proof Let $\mathcal{O}$ be a valuation ring as in Theorem 1.41. For every KH_J-module V there exists an $\mathcal{O}H_J$-lattices V' affording V, i.e. $V' \otimes_{\mathcal{O}} K \cong V$. Note that this implies that $\mathrm{Ind}_J^S(V')$ is an $\mathcal{O}H$-lattice. Hence, we get the following equations:

$$
\begin{aligned}
\mathrm{Ind}_J^S\left(d_{\theta,J}(M)\right) &= \mathrm{Ind}_J^S\left(d_{\theta,J}\left(\sum_V a_V\ [V]\right)\right) \\
&= \mathrm{Ind}_J^S\left(\sum_V a_V\ d_{\theta,J}([V'\otimes_{\mathcal{O}} K])\right) \\
&= \mathrm{Ind}_J^S\left(\sum_V a_V\ [V'\otimes_{\mathcal{O}} L]\right) \\
&= \sum_V a_V\ \left[\mathrm{Ind}_J^S(V'\otimes_{\mathcal{O}} L)\right] \\
&= \sum_V a_V\ \left[\mathrm{Ind}_J^S(V')\otimes_{\mathcal{O}} L\right] \\
&= \sum_V a_V\ d_\theta\left(\left[\mathrm{Ind}_J^S(V')\otimes_{\mathcal{O}} K\right]\right) \\
&= d_\theta\left(\sum_V a_V\ \left[\mathrm{Ind}_J^S(V'\otimes_{\mathcal{O}} K)\right]\right) \\
&= d_\theta\left(\mathrm{Ind}_J^S\left(\sum_V a_V\ [V]\right)\right) \\
&= d_\theta\left(\mathrm{Ind}_J^S(M)\right) \qquad \blacksquare
\end{aligned}
$$

2. Generic Iwahori-Hecke Algebras and Their Specialisations

We want to study the representation theory of Iwahori-Hecke algebras. To this end we will study the so called generic Iwahori-Hecke algebras and their specialisations. This chapter will be structured in two parts: The first part contains results on more or less arbitrary parameters while the second part covers the case that all parameters are equal.

2.1. Representation theory of generic Iwahori-Hecke algebras

Let (W, S) be a Coxeter system and W finite. We will be studying so called generic Iwahori-Hecke algebras of W.

Definition 2.1 Let R be a subring of $\mathbb{C}$. Let v_s be an indeterminate over R for every s in S such that $v_s = v_t$ whenever s and t are conjugate in W. (Note that the v_s are not necessarily distinct indeterminates.) The *generic Iwahori-Hecke algebra over R with parameters v_s* is the Iwahori-Hecke algebra $H_A(W, S, (u_s := v_s^2 \mid s \in S))$ where we set $A := R\,[v_s^{\pm 1} | s \in S]$. If $v_s = v_t$ for conjugate s and t is the only relation between the elements v_s, we call this algebra the *generic multi-parameter Iwahori-Hecke algebra over R*. If all v_s are equal we call this the *equal parameter case*.

Remark 2.2 It should be clear that this is a rather natural setting for specialisation: Take R to be the integers and consider the generic multi-parameter algebra over $\mathbb{Z}$. Given any other commutative ring B with elements $x_s \in B^*$ and $x_s = x_t$ whenever s and t are conjugate, there exists a unique ring homomorphism $A \to B$ sending v_s to x_s. Hence, the Iwahori-Hecke algebra $H_B(W, S, (x_s^2 \mid s \in S))$ is obtained as a specialisation of the generic multi-parameter algebra over $\mathbb{Z}$.

Remark 2.3 Note that we will always be in the equal parameter case if all generators $s \in S$ are conjugate in W. In particular, this will occur whenever

W is of type $A_n,\ D_n,\ E_6,\ E_7,\ E_8,\ H_3,\ H_4$ or $I_2(m)$ for odd m. This follows from [GP00, Exercise 1.2 *b*)].

While it is everything but obvious why we would ask the parameters to be squares of elements of A this will become clearer in the next theorem. It is due to the work of several mathematicians over more than a decade. Involved were amongst others Benson, Curtis, Lusztig, Alvis, Digne and Michel. This version is [GP00, Theorem 9.3.5].

Theorem 2.4 *Let K_0 be a splitting field of W of characteristic 0 and R the ring of integers therein. Furthermore, let $(v_s \mid s \in S)$ be some indeterminates over K_0 with $v_s = v_t$ whenever s and t are conjugate in W. Now set*

$$K := K_0(v_s|\ s \in S).$$

Then the Iwahori-Hecke algebra $H_K(W, S, (u_s = v_s^2 \mid s \in S))$ is split. Moreover, if $(T_w \mid w \in W)$ is its standard basis and χ is an irreducible character of the Iwahori-Hecke algebra we have $\chi(T_w) \in R\,[v_s|s \in S]$.

Note that we first consider a splitting field only for the group W. Obviously, the complex numbers are a splitting field of characteristic 0 for every finite Coxeter group, but it will be very useful later on to consider smaller splitting fields. The following result is due to the work of Benard, Benson, Curtis and Grove.

Lemma 2.5 *Let W be an irreducible Coxeter group. Then $\mathbb{Q}$ is a splitting field for W unless W is of type H_3, H_4 or $I_2(m)$. For H_3 and H_4 a splitting field is given by $\mathbb{Q}\left(\sqrt{5}\right)$. Finally, $\mathbb{Q}\left(\cos(2\pi/m)\right)$ is a splitting field for $I_2(m)$. In particular, for every finite Coxeter group there exists an algebraic number field K_0 such that K_0W is split.*
Note that the splitting fields given here are also splitting fields for the maximal parabolic subgroups of W.

Proof This can be found in [GP00, Theorem 6.3.8]. ■

Remark 2.6 If K is a splitting field of characteristic 0 for a finite Coxeter group W, then we have access to the full information about its irreducible characters via its ordinary character table. These are known for all finite Coxeter groups and accessible e.g. via the GAP part of CHEVIE.

Our next goal will be to apply our knowledge of decomposition maps to the generic Iwahori-Hecke algebras. To do this we have to consider integrally closed rings. The following result is mostly taken from exercises in Chapter 5 of [AM69].

Lemma 2.7 *Let F be a field and R an integrally closed subring of F. Then the following holds:*

a) $R[x]$ is integrally closed in $F(x)$

b) $R[x^{\pm 1}]$ is integrally closed in $F(x)$.

c) $R[x_1, \ldots, x_n]$ is integrally closed in $F(x_1, \ldots, x_n)$.

d) $R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is integrally closed in $F(x_1, \ldots, x_n)$.

Proof

1. $F[x]$ is integrally closed in $F(x)$ as it is a factorial domain. Therefore, we only have to show that $R[x]$ is integrally closed in $F[x]$. We first prove the following: If f and g are monic polynomials in $F[x]$ and the product fg is in $R[x]$ then f and g are already in $R[x]$.
   Consider some splitting field $F \subseteq L$ of fg. In this field, there exist elements α_i such that $fg = \prod_i (x - \alpha_i)$. The α_is are integral over R as fg is in $R[x]$. The coefficients of f and g are polynomials in the α_i and therefore also integral over R. But since f and g are elements of $F[x]$, their coefficients lie in F and therefore in R, as R is integrally closed. This proves, that f and g are in $R[x]$.
   Now consider an element f of $F[x]$ that is integral over $R[x]$. By definition there exist g_i's in $R[x]$ and some m in $\mathbb{N}$ such that

$$f^m + f^{m-1} g_1 + \cdots + g_m = 0.$$

   To apply our first result, we need monic polynomials. To this end, let $t > 0$ be a natural number such that t is greater than m and also greater than the largest degree of both f and every g_i and set $f_1 := f - x^t$. Plugging this into the equation yields

$$(f_1 + x^t)^m + \cdots + g_m = 0$$

   which after some reordering becomes

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0,$$

   where $h_m = (x^t)^m + g_1 (x^t)^{m-1} + \cdots + g_m \in R[x]$. Subtracting h_m on both sides gives us

$$f_1 (f_1^{m-1} + \cdots + h_{m-1}) = -h_m.$$

   The right-hand side is in $R[x]$ and both factors on the left-hand side are monic polynomials in $F[x]$ by our choice of t. Hence, f_1 is in $R[x]$ and therefore so is f.

2. This follows directly from the first part: Assume $f \in F(x)$ is integral over $R[x^{\pm 1}]$. Then there are $g_i \in R[x^{\pm 1}]$ such that

$$f^m + f^{m-1} g_1 + \cdots + g_m = 0.$$

Choose $t > 0$ such that $x^t g_i \in R[x]$ for every i and multiply the equation with x^{tm}. We obtain

$$(fx^t)^m + (fx^t)^{m-1} g_1 x^t + \cdots + g_m x^{tm} = 0.$$

The coefficients of the powers of fx^t all lie in $R[x]$ and therefore fx^t is integral over $R[x]$. As $R[x]$ is integrally closed in $F(x)$ this shows that fx^t is in $R[x]$ and thus $f \in R[x^{\pm 1}]$.

3. This is proved using a simple induction argument: $R[x_1]$ is integrally closed in $F(x_1)$, therefore, by the first part, $R[x_1][x_2] = R[x_1, x_2]$ is integrally closed in $F(x_1)(x_2) = F(x_1, x_2)$.

4. Completely analogous. ■

From now on we will tacitly assume that the indeterminates v_s satisfy at least the condition that $v_s = v_t$ if s and t are conjugate in W, whenever talking about indeterminates v_s for s in S. Furthermore, we fix a field $K_0 \subseteq \mathbb{C}$ such that K_0W is split. More precisely, we choose a field extension of the splitting field in Lemma 2.5.

Theorem 2.8 *Denote the ring of Laurent polynomials in indeterminates v_s over K_0 by $B := K_0 [v_s^{\pm 1} | s \in S]$. Let K be its field of fractions. We consider a generic Iwahori-Hecke algebra over K_0 and define it as*

$$H := H_B(W, S, (v_s^2 \mid s \in S)).$$

Furthermore, we define a ring homomorphism $\theta : B \to K_0$ by $v_s \mapsto 1$. Then the specialised Iwahori-Hecke algebra K_0H is isomorphic to the group algebra K_0W and there exists a well defined decomposition map

$$d_\theta : R_0^+(KH) \to R_0^+(K_0W).$$

KH is split semisimple and d_θ is an isomorphism sending classes of irreducible modules to classes of irreducible modules.

Proof The proof will mostly rely on Tits' Deformation Theorem and we only have to assure ourselves that we are indeed in a suitable setting:
Because K_0 has characteristic 0 and it is a splitting field of W, we know that K_0W is split semisimple. From Lemma 2.7 we know that B is integrally closed in K and by Theorem 2.4 the algebra KH is split. Hence, d_θ is well defined as K_0H and KH satisfy all the hypotheses. The claim then follows using Tits' Deformation Theorem 1.44. ■

Corollary 2.9 *Let K and H be as in 2.8. Furthermore, suppose that W is of exceptional type, i.e. $W \in \{F_4,\ G_2,\ H_3,\ H_4,\ E_6,\ E_7,\ E_8,\ I_2(m) \mid m \geq 5, m \neq 6\}$. Then there exists no imprimitive irreducible representation of KH.*

Proof For any maximal parabolic subgroup W_J of W we know that KH_J, too, is split semisimple by Lemma 2.5, Theorem 2.4, and Theorem 2.8. We apply Theorem 1.45 to the specialisation $\theta : v_s \mapsto 1$, which tells us that it is sufficient to study the irreducible imprimitive representations of K_0W.
For the groups $\{F_4,\ G_2,\ H_3,\ H_4,\ E_6,\ E_7,\ E_8\}$ we use the induction tables available in CHEVIE to see that K_0W has no irreducible imprimitive representation and apply Theorem 1.45.
For the groups of type $I_2(m)$ this follows from the fact that all irreducible representations of $K_0I_2(m)$ have degree 1 or 2, but the subgroup index of any proper parabolic subgroup is m. Since the dimension of an induced representation is divisible by the parabolic subgroup's index we see that $K_0I_2(m)$, too, has no irreducible imprimitive representation, so we can apply Theorem 1.45 once more. ∎

Remark 2.10 Since K_0W is split, there exists a bijection between $\mathrm{Irr}(K_0W)$ and $\mathrm{Irr}(\mathbb{C}\,W)$ because all complex representations of W are realisable over K_0, see [Isa76, Theorem 9.9]. Applying the previous Theorem we obtain a bijection between $\mathrm{Irr}(\mathbb{C}\,W)$ and $\mathrm{Irr}(H_K(W, S, (u_s := v_s^2 | s \in S))$, that is a generic Iwahori-Hecke algebra over K_0. This bijection preserves dimensions.

Similar to group representation theory, there exists the idea of a character table for Iwahori-Hecke algebras. The following definition is given in [GP00, Definition 8.2.9].

Definition 2.11 Let K_0 be a field of characteristic 0 such that K_0W is split and set $K := K_0\,(v_s \mid s \in S)$ for some indeterminates v_s. Suppose that $A \subseteq K$ is a subring that is integrally closed in K containing $v_s^{\pm 1}$ for every s in S. Consider the Iwahori-Hecke algebra $H := H_A(W, S, (u_s := v_s^2 \mid s \in S))$ and its specialisation KH. Let $\mathrm{Cl}(W)$ be the conjugacy classes of W and for each $C \in \mathrm{Cl}(W)$ fix an element w_C such that w_C is of minimal length in the class C. Furthermore, let $\mathrm{Irr}(KH)$ be the set of all characters of irreducible representations of H up to isomorphism.
The matrix

$$X(H) := (\chi(T_{w_c}))_{\chi \in \mathrm{Irr}(KH),\ C \in \mathrm{Cl}(W)}$$

has entries in A by Corollary 1.38 and is called the *character table of the Iwahori-Hecke algebra H*. It is independent of the choice of the w_C.
The character table $X(H)$ has full rank, which follows from Theorem 2.8 and the fact that the character table of a complex group algebra has full rank.
If $\theta : A \to L$ is a ring homomorphism into a field L which is the field of fractions of $\theta(A)$ and LH is split, we define the *specialised character table*

$$\theta(X(H)) := (\theta(\chi(T_{w_c})))_{\chi \in \mathrm{Irr}(H),\ C \in \mathrm{Cl}(W)},$$

which encodes the information about the specialised characters of KH, see Lemma 1.42. Clearly, we have $\mathrm{rank}(\theta(X(H))) \leq \mathrm{rank}(X(H))$

Remark 2.12 The character table fully defines the irreducible characters of KH: For every $w \in W$ there exist well defined elements $f_{w,C} \in \mathbb{Z}[u_s^{\pm 1}]$ such that

$$\chi(T_w) = \sum_{C \in \mathrm{Cl}(W)} f_{w,C}\chi(T_{w_c}).$$

See [GP00, Theorem 8.2.3].

2.2. Schur elements of Iwahori-Hecke algebras

We now want to discuss the extent of structure that is transferred when specialising a generic Iwahori-Hecke algebra. Using Tits' Deformation Theorem we know that basically the whole structure is carried over if the specialised algebra is semisimple. This section introduces a criterion to check for semisimplicity.

We assume that (W, S) is a finite Coxeter system. Let K_0 be field of characteristic 0 such that K_0W is split and R an integrally closed subring of K_0. Set $A := R[v_s^{\pm 1} | s \in S]$, let K be its field of fractions and $H := H_A(W, S, (u_s := v_s^2))$ a generic Iwahori-Hecke algebra over R. As usual, let KH denote the specialised Iwahori-Hecke algebra over K.

The results in this section can mostly be found in [GP00, Chapter 7]. They can be applied because Iwahori-Hecke algebras are *symmetric algebras*:

Lemma 2.13 *Define an A-linear map $\tau : H \to A$ by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $w \neq 1$. Then H is a symmetric algebra with symmetrizing trace τ, that is $\tau(h_1h_2) = \tau(h_2h_1)$ for all $h_1, h_2 \in H$ and the bilinear form defined by*

$$H \times H \to A\colon (h_1, h_2) \mapsto \tau(h_1, h_2),$$

is non-degenerate, i.e. the determinant of a corresponding Gram matrix is a unit in A.
The value of τ on the product of two basis elements is

$$\tau(T_wT_{w'}) = \begin{cases} u_w & \text{if } w^{-1} = w', \\ 0 & \text{if } w^{-1} \neq w', \end{cases}$$

where $u_w \in A$ is defined by $u_w := u_{s_1} \cdots u_{s_n}$ whenever $w = s_1 \cdots s_n$ is a reduced expression for w. These elements are well defined by Matsumoto's Theorem, see [GP00, Theorem 1.2.2].
The unique dual basis to $\{T_w \mid w \in W\}$ is given by $\{T_w^\vee \mid w \in W\}$ with $T_w^\vee = u_w^{-1}T_{w^{-1}}$.

Proof See [GP00, Proposition 8.1.1]. ∎

Remark 2.14 The natural extension of τ to any specialisation of H is a symmetrizing trace for the specialised algebra. To simplify notation, this function will also be denoted by τ.

The existence of this symmetrizing trace has several useful consequences for the representation theory of H. Maybe the most important aspect is the concept of *Schur elements*. Recall that we have already seen that KH is split semisimple.

Lemma 2.15 *For an irreducible KH-module V let χ_V denote the character afforded by that module. There exists a unique decomposition*

$$\tau = \sum_V \overline{c_V} \chi_V,$$

where V runs over the irreducible KH-module up to isomorphism. The $\overline{c_V}$ are non-zero elements in K and therefore we can define $c_V := \overline{c_V}^{-1}$ for every irreducible KH-module V. We call c_V the Schur element *of V.*

Proof See [GP00, Theorem 7.2.6]. ∎

Similarly to classical characters of groups the characters of Iwahori-Hecke algebras satisfy a certain orthogonality relation in which the Schur elements play a crucial role:

Lemma 2.16 *Let V and V' be irreducible KH-module with characters χ_V and $\chi_{V'}$. Note that V and V' are split because KH is. Then the following relation holds:*

$$\sum_{w \in W} \chi_V(T_w)\chi_{V'}(T_w^\vee) = \begin{cases} c_V \dim_K(V) & \text{if } \chi_V = \chi_{V'} \\ 0 & \text{otherwise} \end{cases}$$

If we define the index representation $\operatorname{ind} : H \to A \colon T_w \to u_w$ *with u_w as in Lemma 2.13 and use the linearity of characters this translates to*

$$\sum_{w \in W} \operatorname{ind}(T_w)^{-1} \chi_V(T_w)\chi_{V'}(T_{w^{-1}}) = \begin{cases} c_V \chi_V(T_1) & \text{if } \chi_V = \chi_{V'}, \\ 0 & \text{otherwise.} \end{cases}$$

The Schur elements all lie in A. More precisely, if w_0 is the unique longest element of W we know that $\operatorname{ind}(T_{w_0})c_V$ *is an element of $R[v_s \mid s \in S]$.*

Proof See [GP00, Corollary 7.2.4, Proposition 7.3.9, Theorem 9.3.5]. ∎

Example 2.17 The orthogonality relation can be used to compute the Schur element for the index representation. Setting V and V' to modules affording the index representation we compute its Schur element as

$$P_W := c_{\text{ind}} = \sum_{w \in W} \operatorname{ind}(T_w) \in \mathbb{N}_0 \left[v_s^2 \mid s \in S \right].$$

We denote it by P_W and call it the *Poincaré polynomial of H*.

The fact that the Schur elements lie in A gives us the opportunity to study their image under a specialisation. This will give us a semisimplicity criterion. Recall that semisimplicity of the specialised algebra implies that the study of module structure of LH can be reduced to that of KH by using Tits' Deformation Theorem.

Theorem 2.18 *Let L be a field and $\theta : A \to L$ a ring homomorphism such that L is the field of fractions of $\theta(A)$. Suppose LH is split. Then LH is semisimple if and only if $\theta(c_V) \neq 0$ for all irreducible KH-modules V.*

Proof See [GP00, Theorem 7.4.7]. ■

2.3. Generic equal-parameter Iwahori-Hecke algebras

In the remainder of this chapter we will study generic one-parameter Iwahori-Hecke algebras and their specialisations. Our most important tool to study specialisation is certainly the decomposition map. One of the conditions necessary to obtain such a map was that the specialised algebra is split. Hence, our first goal is to characterize a setting in which we can be certain that this is the case. The proper setting for this is a so called L_0-good ring. This definition is taken from [GJ11, Table 3.1].

Definition 2.19 Let (W, S) be a finite Coxeter system. A ring $R \subseteq \mathbb{C}$ is called *L_0-good for W* if it is L_0-good for every irreducible component of W. For irreducible Coxeter groups the criteria for being L_0-good are listed in Table 2.1.

We give some examples for irreducible Coxeter groups:

Example 2.20 Suppose W is an irreducible Coxeter group. If W is a Weyl group, let P be the set of primes that need to be invertible in a ring R

Table 2.1.: Conditions of R to be L_0-good for irreducible Coxeter groups

Irreducible Coxeter Group	Conditions on R
A_n, B_n	none
D_n, G_2	$2 \in R^*$
F_4, E_6, E_7	$2, 3 \in R^*$
E_8	$2, 3, 5 \in R^*$
$I_2(m)$, $6 \neq m \geq 5$	$m \in R^*$ and $2\cos(\frac{2\pi}{m}) \in R$
H_3	$2, 5 \in R^*$ and $2\cos(\frac{2\pi}{5}) \in R$
H_4	$2, 3, 5 \in R^*$ and $2\cos(\frac{2\pi}{5}) \in R$

such that it is L_0-good for W. Then $\mathbb{Z}[1/p \mid p \in P]$ is L_0-good for W and integrally closed in its field of fractions.
Suppose that W is non-crystallographic. Note that $\cos(2\pi/m)$ is in R if a primitive m'th root of unity ζ_m is in R. Hence, $\mathbb{Z}[\zeta_5, 1/2, 1/5]$ is L_0-good for H_3 and $\mathbb{Z}[\zeta_5, 1/2, 1/3, 1/5]$ is L_0-good for H_4. Finally, $\mathbb{Z}[\zeta_m, 1/m]$ is L_0-good for $I_2(m)$. All these rings are integrally closed in their respective field of fractions.

Lemma 2.21 *If R is L_0-good for a finite Coxeter group W, it is also L_0-good for all parabolic subgroups of W.*

Proof Using a simple induction argument we only have to show this for maximal parabolic subgroups of irreducible Coxeter groups and these are easily checked using Table 2.1. ∎

Remark 2.22 Comparing the definition of an L_0-good ring with the splitting fields of Coxeter groups given in Lemma 2.5 we see that the field of fractions of an L_0-good ring for W is a splitting field for W.

An L_0-good ring is basically all we need to obtain split specialised Iwahori-Hecke algebras:

Lemma 2.23 *Let (W, S) be a finite Coxeter system. Assume $R \subseteq \mathbb{C}$ to be L_0-good for W and integrally closed in its field of fractions. Set $A := R[v^{\pm 1}]$ to be the ring of Laurent polynomials over R. The generic Iwahori-Hecke algebra over R then is $H := H_A(W, S, (u_s := u := v^2 \mid s \in S))$. Now let K be the field of fractions of A and suppose that $\theta : A \to L$ is a ring homomorphism such that L is the field of fractions of $\theta(A)$. Then the following holds:*

a) KH is split semisimple.

b) The specialisation LH is split.

c) There is a well defined decomposition map $d_\theta : R_0^+(KH) \to R_0^+(LH)$.

d) The natural extension of d_θ to the full Grothendieck groups is surjective. In particular, every irreducible LH module is a constituent of at least one KH-module's specialisation and the columns of the corresponding decomposition matrix D_θ are linearly independent.

Proof Note that c) follows from the combination of a) and b) which in turn follow from Theorem 2.8 and [GJ11, Lemma 3.1.13] respectively. Finally, d) is part of [GJ11, Theorem 3.1.14]. ∎

The hypotheses of Lemma 2.23 have another important consequence:

Lemma 2.24 *Suppose the setting of Lemma 2.23 and recall the definition of the specialised character table $\theta(X(H))$ from Definition 2.11. The specialised character table has full rank, i.e.*

$$|\operatorname{Irr}(LH)| = \operatorname{rank}(\theta(X(H))).$$

Proof See [GJ11, Proposition 3.4.11]. ∎

There are some immediate consequences of this lemma which will turn out to be very helpful in the study of irreducible imprimitive representations.

Corollary 2.25 *As every irreducible LH-module is constituent of at least one irreducible KH-module's specialisation, the dimension of an irreducible LH-module is bound by the maximum dimension of an irreducible KH-module, since d_θ preserves dimensions. The dimensions of irreducible KH-modules are exactly those of irreducible $\mathbb{C}W$-modules by Remark 2.10. Hence, if V is an irreducible LH-module the following holds:*

$$\dim_L(V) \leq d_W^{\max} := \max\{\dim_{\mathbb{C}}(M) \mid M \in \operatorname{Irr}(\mathbb{C}W)\}.$$

We summarize our results on dimension of irreducible imprimitive modules: If V is an irreducible imprimitive LH-module induced from an (irreducible, see Lemma 1.15) LH_J-module V_J we know that

1. $\dim_L(V) \leq d_W^{\max}$,
2. $\dim_L(V)$ *is divisible by the index* $[W : W_J]$ *and*
3. $\dim_L(V_J) = \frac{\dim_L(V)}{[W:W_J]}$

by Lemma 1.15.

Corollary 2.26 *The number of irreducible LH-modules can be computed as the rank of the specialised character table, see Lemma 2.24.*
Using the fact that the decomposition map's extension to the full Grothendieck group is surjective we can compute the character of any LH-module M: We compute a linear combination of classes of irreducible KH-modules such that its image under d_θ is exactly the class of M. The character afforded by M is then given by this linear combination, if we replace the classes of modules by the specialisation of their character, see Corollary 1.43.

The decomposition map can also be used to compute the dimension of all irreducible LH-modules as follows:
Let $r := |\operatorname{Irr}(KH)| = |\operatorname{Irr}(\mathbb{C}W)|$ and set

$$v := \left(\deg_K(\chi)_{\chi\in\operatorname{Irr}(KH)}\right) = \left(\deg_{\mathbb{C}}(\chi)_{\chi\in\operatorname{Irr}(\mathbb{C}W)}\right) \in \mathbb{N}^{r\times 1},$$

that is the dimension vector of irreducible KH-modules. Assume that the rows of D_θ are ordered in the same way as the entries of v. Then the dimensions of irreducible LH-modules are given by the entries of the unique solution vector x of the equation

$$D_\theta \; x = v.$$

Tits' Deformation Theorem and our results building on it give us a range of options to study semisimple Iwahori-Hecke algebras. It is therefore paramount to have an easily checked criterion whether or not a specialisation is semisimple. Such a criterion will be studied now.
We restrict ourselves to Weyl groups, but an analogous observation can be made for non-crystallographic Coxeter groups.

Similarly to a ring being L_0-good there exists the concept of so called *good primes* for a Weyl group. In fact, both concepts describe the same idea and depend on divisors of the leading coefficients of Schur elements.

Definition 2.27 A prime ℓ is called *bad* for a finite Coxeter group W if it is bad for at least one of its irreducible components. It is called *bad* for an irreducible Coxeter group W if and only if it is one of the cases listed here:

Type of W	Bad primes
A_n	–
B_n, C_n, D_n	2
G_2, F_4, E_6, E_7	2, 3
E_8	2, 3, 5

Primes that are not bad for W are called *good.*

We fix a finite Coxeter system (W, S) for a Weyl group W. Note that in this case $\mathbb{Q}$ is a splitting field for W. Let $A := \mathbb{Z}\left[v^{\pm 1}\right]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in v and $H := H_A(W, S, (u := v^2))$ the corresponding generic Iwahori-Hecke algebra over $\mathbb{Z}$. The ring A of Laurent polynomials is integrally closed in K, the field of fractions of A. Finally, suppose that $\theta : A \to L$ is a ring homomorphism such that L is the field of fractions of $\theta(A)$. We will now develop a strong criterion to see whether or not $LH \cong H_L(W, S, (\theta(v)^2))$ is semisimple.

Lemma 2.28 *Let $P_W := \sum_{w \in W} \operatorname{ind}(T_w)$ be the Poincaré polynomial of W and V an irreducible KH-module with Schur element $c_V \in \mathbb{Z}\left[v^{\pm 1}\right]$. Then c_V divides P_W in $\mathbb{Q}\left[v^{\pm 1}\right]$. Even stronger,*

$$D_V := \frac{P_W}{c_V} = \frac{1}{f_V} g_V \in \mathbb{Q}\left[v^2\right] = \mathbb{Q}[u],$$

where f_V is a positive integer and g_V is a monic polynomial in $\mathbb{Z}[u]$. All prime divisors of f_V are bad for W.
D_V is called the generic degree *of V.*

Proof First note that c_V is in $\mathbb{Z}\left[v^{\pm 1}\right]$ by Lemma 2.16.
The generic degree D_V is in $\mathbb{Q}[u]$ by [GP00, Corollary 9.3.6]. This corollary also contains the factorisation into $1/f_V$ and g_V.
Finally, the statement about prime divisors of f_V can be deduced from the explicit formulas for generic degrees found in [Car93, p. 446-453, p. 480-488]. ∎

This leads us to a strong result on the semisimplicity of Iwahori-Hecke algebras of which a first version was proved by Gyoja and Uno in [GU89]. The proof used here can also be found e.g. in [GM09].

Lemma 2.29 *Suppose the characteristic of L is either 0 or a good prime for L. Then LH is semisimple if and only if $\theta(P_W) \neq 0$.*

Proof Recall from Theorem 2.18 that LH is semisimple if and only if $\theta(c_V) \neq 0$ for all irreducible KH-modules V. Hence, LH cannot be semisimple unless $\theta(P_W) \neq 0$ since P_w itself is a Schur element of the irreducible index representation.
Conversely, suppose that $\theta(P_W) \neq 0$. We have to show that LH is semisimple. Let V be an irreducible KH-module with Schur element c_V. By Lemma 2.28 there exists a positive integer f_V and a monic polynomial $g_V \in \mathbb{Z}[u]$ such that $c_V g_V = f_V P_W$. We apply θ to both sides and see that $\theta(c_V)\theta(g_V) = \theta(f_V)\theta(P_W)$. As the characteristic of L is either 0 or good for W, we know that $\theta(f_V)$ is not 0, since every prime divisor of f_V is a bad prime for W and therefore not equal to the characteristic of L, which is the only prime in $\mathbb{Z}$ possibly mapped to 0 under θ. Hence, the right-hand side of our equation and therefore both sides are not 0. This implies that $\theta(c_V) \neq 0$ and because V was chosen arbitrarily we now know that LH is semisimple by Theorem 2.18. ∎

Corollary 2.30 *Suppose the characteristic of L is either 0 or a good prime for L. If LH is semisimple, so is LH_J for every subset J of S.*

Proof Suppose J is a of S. By Lemma 2.29 we are done if we can show that P_{W_J} divides P_W. To this end, let X_J the set of distinguished right coset representatives of W_J in W. Recall that for every w in W there exist unique elements x_w in X_J and w' in W_J such that $w = w'x_w$ and $\ell(w) = \ell(w') + \ell(x_w)$. Applying this to the Poincaré polynomial yields the following equations:

$$\begin{aligned} P_W &= \sum_{w \in W} \operatorname{ind}(T_w) \\ &= \sum_{w \in W} v^{2\ell(w)} \\ &= \sum_{w' \in W_J} v^{2\ell(w')} \sum_{x \in X_J} v^{2\ell(x)} \\ &= P_{W_J} \sum_{x \in X_J} v^{2\ell(x)}. \end{aligned}$$

■

The semisimplicity result is particularly helpful because Poincaré polynomials have very nice factorizations:

Lemma 2.31 *The Poincaré polynomial P_W has a factorization into cyclotomic polynomials. In particular, if the characteristic of L is either 0 or a good prime for W the algebra LH can only* not *be semisimple if $\theta(v)$ is a root of unity. The only orders of $\theta(v)$ for which LH is semisimple can be read off the Poincaré polynomial's factorization.*

Proof If W is of classical type, the Poincaŕe polynomial can be found in [GP00, 10.5.1]. For W of exceptional type they are listed in [GP00, Appendix E]. In both cases we see immediately that a factorization into cyclotomic polynomials exists.
Now we know that LH is semisimple if and only if $\theta(P_W)$ is not 0 so let us assume that $\theta(P_W)$ is 0. Suppose that $P_W = \prod_{t_i} \Phi_{t_i}$ is a factorization of P_W in $\mathbb{Z}[u]$ where Φ_t is the t'th cyclotomic polynomial. In particular, Φ_t is a monic polynomial in $\mathbb{Z}[u = v^2]$. We apply θ to P_W and get $\prod_{t_i} \theta\left(\Phi_{t_i}\right)$. Since $\theta(P_W)$ is 0 we know that $\theta\left(\Phi_{t_i}\right)$ is 0 for some t_i. Let t be such a t_i. We know that Φ_t divides $u^t - 1$ in $\mathbb{Z}[u]$ and therefore $\theta\left(\Phi_t\right)$ divides $\theta(u^t - 1) = \theta(u)^t - 1$. Since $\theta\left(\Phi_t\right)$ is 0 this implies that $\theta(u)^t - 1$, too, is 0. Hence, we have $0 = \theta(u)^t - 1 = \theta(v)^{2t} - 1$ and $\theta(v)$ is a root of unity. ■

3. Imprimitive Representations of Iwahori-Hecke Algebras of Exceptional Type in Characteristic 0

Our aim for this chapter is to prove the following main result:

Theorem 3.1 *Let (W, S) be an irreducible Coxeter system of exceptional type. If W is a Weyl group, let L be any field of characteristic 0. If W is of non-crystallographic type H_3 or H_4, let L be a field extension of $\mathbb{Q}(\zeta_5)$. Finally, if W is of type $I_2(m)$ with $m = 5$ or $m > 7$ let L be a field extending $\mathbb{Q}(\zeta_m)$. Then the following holds:*
If q is an element of L^*, then the Iwahori-Hecke algebra $H_L(W, S, (u_s := q^2 \mid s \in S))$ has no *irreducible imprimitive representation.*

We will prove this theorem separately for non-crystallographical groups and Weyl groups.

3.1. Imprimitive representations of Iwahori-Hecke algebras of non-crystallographic type

In this section we prove Theorem 3.1 for Iwahori-Hecke algebras of non-crystallographic type.
Let (W, S) be a finite Coxeter system and W an irreducible Coxeter group of non-crystallographic type, that is W is either H_3, H_4 or $I_2(m)$ for some m that is either 5 or greater than 6. Following Example 2.20 we define the following rings: Let

- $R_{H_3} := \mathbb{Z}\left[\zeta_5, \frac{1}{2}, \frac{1}{5}\right]$,

- $R_{H_4} := \mathbb{Z}\left[\zeta_5, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right]$, and

- $R_{I_2(m)} := \mathbb{Z}\left[\zeta_m, \frac{1}{m}\right]$.

Each R_W is integrally closed in its field of fractions. We define the ring of Laurent polynomials over R_W as $A_W := R_W\left[v^{\pm 1}\right]$ for some indeterminate v and the corresponding generic Iwahori-Hecke algebra of W by $H := H_{A_W}(W, S, (u_s := u := v^2))$. Setting K_W as the field of fractions of A_W we obtain the Iwahori-Hecke algebra $K_W H$ by extending scalars to K_W. By Lemma 2.7 the ring A_W is integrally closed in K_W.
We now prove the main result for non-crystallographic Coxeter groups:

Lemma 3.2 (Theorem 3.1 for non-crystallographic groups) *Suppose L is a field extension of $\mathbb{Q}(\zeta_5)$ if W is of type H_3 or H_4 and that L is a field extension of $\mathbb{Q}(\zeta_m)$ if W is of type $I_2(m)$. If q is an invertible element of L, then the Iwahori-Hecke algebra $H_L(W, S, (q^2))$ has no irreducible imprimitive representation.*

Proof Denote the field of fractions of $R_W[q]$ by L'. By our assumptions on L it is a subfield of L. We apply Corollary 1.35 to see that it is sufficient for us to prove the claim for L'. To apply this Corollary, note that since $R_W[q]$ is L_0-good for W it is also L_0-good for all parabolic subgroups by Lemma 2.21 and therefore the parabolic subalgebras over L', too, are split. From now on we can therefore assume that $L = L'$.

We define a ring homomorphism $\theta : A_W \to L$ by sending v to q. Then $H_L(W, S, (q^2))$ is isomorphic to the specialisation of the generic algebra H over A via θ. It is therefore sufficient to consider this specialisation LH. Since L is the field of fraction of $\theta(A)$ we are in the setting of Lemma 2.23. Hence, the corresponding decomposition map is surjective and the dimension arguments from Corollary 2.25 apply.
We list the maximal character degrees $d_W^{\max}$ and compare them to the indices of maximal parabolic subgroups of W:

W	$d_W^{\max}$	Subgroup indices $[W : W_J]$ for $W_J \leq W$ maximal parabolic subgroup
H_3	5	12, 20, 30
H_4	48	120, 720, 1200, 600
$I_2(m)$	2	m, m

Here the values for H_3 and H_4 have simply been checked using the GAP part of CHEVIE ([GHL+96]). For $I_2(m)$ note that the maximal parabolic subgroups are generated by one of the two generators of $I_2(m)$ and therefore have index m.
In all cases, even the smallest index of a maximal parabolic subgroup is larger than $d_W^{\max}$. By Corollary 2.25 this implies that there are no irreducible imprimitive LH-modules: Their dimension would have to be simultaneously divisible by (and therefore at least as large as) a subgroup index of a maximal parabolic subgroup and also at most $d_W^{\max}$, which is a contradiction. ∎

3.2. Imprimitive representations of Iwahori-Hecke algebras of exceptional Weyl groups

In this section we prove Theorem 3.1 for Iwahori-Hecke algebras of exceptional Weyl groups.
Let (W, S) be a finite Coxeter system and W in $\{E_6, E_7, E_8, F_4, I_2(6) \cong G_2\}$. We note that $\mathbb{Q}$ is a splitting field for W.
Suppose L is a field of characteristic 0 and q an invertible element therein. Then we want to show that $H_L(W, S, (q^2))$ has no irreducible imprimitive representations. To do this, let $A := \mathbb{Q}[v^{\pm 1}]$ be the ring of Laurent polynomials over $\mathbb{Q}$ and $K := \mathbb{Q}(v)$ its field of fractions. We denote the generic Iwahori-Hecke algebra over $\mathbb{Q}$ by $H := H_A(W, S, (v^2))$. By extending scalars to K we obtain a split semisimple algebra KH by Theorem 2.8. Now we define a ring homomorphism $A \to L$ by sending v to q. If we set L' to be the field of fractions of $\theta(L)$ we know that the specialisation $L'H$ is split by Lemma 2.23, since $\mathbb{Q}$ is clearly L_0-good for every Weyl group. This also holds for all parabolic subalgebras of $L'H$ and by Theorem 1.45 we know that LH has an irreducible imprimitive representation if and only if $L'H$ has one. Hence, we will just assume that L is indeed the field of fractions of $\theta(A)$ from now on. This is the setting of Lemma 2.23, so the lemma and all its corollaries apply in this situation.
The groups that are not of type E_n for some n can now be dealt with very easily:

Lemma 3.3 *If W is of type F_4 or G_2, there are no irreducible imprimitive representations of LH.*

Proof We compare the maximal character degree $d_W^{\max}$ to the indices of maximal parabolic subgroups:

W	$d_W^{\max}$	Subgroup indices $[W : W_J]$ for $W_J \leq W$ maximal parabolic subgroup
G_2	2	6, 6
F_4	16	24, 24, 96, 96

These numbers can be computed using the GAP part of CHEVIE. In both cases, d_W^{max} is clearly smaller than the indices of all maximal parabolic subgroups. We apply Corollary 2.25 and see that this already contradicts the possibility of LH having an irreducible imprimitive representation: Its dimension would be simultaneously at most $d_W^{\max}$ and also divisible by $[W : W_J]$ for some maximal subset J of S. ∎

It remains for us to prove Theorem 3.1 for the groups E_6, E_7 and E_8, so for the rest of this chapter suppose that W is one of these groups. We will have to go into greater detail than for the other groups. Therefore, we give their Coxeter graphs and fix names for the generators in Table 3.1. Since we are interested in imprimitive representations we also give the maximal parabolic subgroups with their subgroup index and decomposition into irreducible Coxeter components, which we derive directly from the graphs. The indices have been computed using CHEVIE.

We copy the proof of Lemma 3.3 to drastically reduce the number of cases we have to consider:

Lemma 3.4

- *Suppose W is of type E_6 and J is either $\{1,2,4,5,6\}$, $\{1,2,3,5,6\}$, or $\{1,2,3,4,6\}$. Then there is no irreducible representation of LH induced from a representation of LH_J.*
- *Suppose W is of type E_7 and J is either $\{1,3,4,5,6,7\}$, $\{1,2,4,5,6,7\}$, $\{1,2,3,5,6,7\}$, $\{1,2,3,4,6,7\}$, or $\{1,2,3,4,5,7\}$. Then there is no irreducible representation of LH induced from a representation of LH_J.*
- *Suppose W is of type E_8 and J is either $\{1,3,4,5,6,7,8\}$, $\{1,2,4,5,6,7,8\}$, $\{1,2,3,5,6,7,8\}$, $\{1,2,3,4,6,7,8\}$, or $\{1,2,3,4\ 5,7,8\}$. Then there is no irreducible representation of LH induced from a representation of LH_J.*

Proof The maximal character degrees are $d_{E_6}^{\max} = 90$, $d_{E_7}^{\max} = 512$ and $d_{E_8}^{\max} = 7168$. We check Table 3.1 to see that in all cases listed in this Lemma we have $d_W^{\max} < [W : W_J]$. The claim follows with Corollary 2.25. ■

Recall that the semisimple case has already been solved in Corollary 2.9:

Lemma 3.5 *If LH is semisimple, there is no irreducible imprimitive representation of LH.*

Proof If LH is semisimple, so is LH_J for any maximal subset J of S by Corollary 2.30. We apply Theorem 1.45 to see that LH has an irreducible imprimitive representation if and only if KH has one. But the generic algebra KH has no irreducible imprimitive representation by Corollary 2.9.

Hence, the last remaining cases are non-semisimple Iwahori-Hecke algebras LH and possible induction from maximal parabolic subalgebras of LH that are not already taken care of in Lemma 3.4.
Recall that it is sufficient to study the Poincaré polynomials image under θ to see whether LH is semisimple, see Lemma 2.29.

Table 3.1.: Graphs and Maximal Parabolic Subgroups of E_n

<table>
<tr><th>W</th><th>$d^W_{\max}$</th><th>Coxeter Graph</th><th>J</th><th>$W_J \cong$</th><th>$[W : W_J]$</th></tr>
<tr><td rowspan="6">E_6</td><td rowspan="6">90</td><td rowspan="6">6
|
5
|
2 − 4
|
3
|
1</td><td>$\{2,3,4,5,6\}$</td><td>D_5</td><td>27</td></tr>
<tr><td>$\{1,3,4,5,6\}$</td><td>A_5</td><td>72</td></tr>
<tr><td>$\{1,2,4,5,6\}$</td><td>$A_1 \times A_4$</td><td>216</td></tr>
<tr><td>$\{1,2,3,5,6\}$</td><td>$A_2 \times A_1 \times A2$</td><td>720</td></tr>
<tr><td>$\{1,2,3,4,6\}$</td><td>$A_4 \times A_1$</td><td>216</td></tr>
<tr><td>$\{1,2,3,4,5\}$</td><td>D_5</td><td>27</td></tr>
<tr><td rowspan="7">E_7</td><td rowspan="7">512</td><td rowspan="7">7
|
6
|
5
|
2 − 4
|
3
|
1</td><td>$\{2,3,4,5,6,7\}$</td><td>D_6</td><td>126</td></tr>
<tr><td>$\{1,3,4,5,6,7\}$</td><td>A_6</td><td>576</td></tr>
<tr><td>$\{1,2,4,5,6,7\}$</td><td>$A_1 \times A_5$</td><td>2016</td></tr>
<tr><td>$\{1,2,3,5,6,7\}$</td><td>$A_2 \times A_1 \times A_3$</td><td>10080</td></tr>
<tr><td>$\{1,2,3,4,6,7\}$</td><td>$A_4 \times A_2$</td><td>4032</td></tr>
<tr><td>$\{1,2,3,4,5,7\}$</td><td>$D_4 \times A_1$</td><td>756</td></tr>
<tr><td>$\{1,2,3,4,5,6\}$</td><td>E_6</td><td>56</td></tr>
<tr><td rowspan="8">E_8</td><td rowspan="8">7168</td><td rowspan="8">8
|
7
|
6
|
5
|
2 − 4
|
3
|
1</td><td>$\{2,3,4,5,6,7,8\}$</td><td>D_7</td><td>2160</td></tr>
<tr><td>$\{1,3,4,5,6,7,8\}$</td><td>A_7</td><td>17280</td></tr>
<tr><td>$\{1,2,4,5,6,7,8\}$</td><td>$A_1 \times A_6$</td><td>69120</td></tr>
<tr><td>$\{1,2,3,5,6,7,8\}$</td><td>$A_2 \times A_1 \times A_4$</td><td>483840</td></tr>
<tr><td>$\{1,2,3,4,6,7,8\}$</td><td>$A_4 \times A_3$</td><td>241920</td></tr>
<tr><td>$\{1,2,3,4,5,7,8\}$</td><td>$D_5 \times A_2$</td><td>60480</td></tr>
<tr><td>$\{1,2,3,4,5,6,8\}$</td><td>$E_6 \times A_1$</td><td>6720</td></tr>
<tr><td>$\{1,2,3,4,5,6,7\}$</td><td>E_7</td><td>240</td></tr>
</table>

Lemma 3.6 *For a natural number d let Φ_d be the d'th cyclotomic polynomial over $\mathbb{Z}$ in the indeterminate $u = v^2$. Then the Poincaré polynomials of E_6, E_7 and E_8 are given by*

$$\begin{aligned} P_{E_6} &= \Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}, \\ P_{E_7} &= \Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}, \\ P_{E_8} &= \Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}. \end{aligned}$$

Proof The polynomials can be found in [GP00, Appendix F]. ■

Corollary 3.7 *LH is non-semisimple if and only if we are in one of the following cases:*

a) $W = E_6$ and q^2 is a primitive root of unity of order e and $e \in \{2, 3, 4, 5, 6, 8, 9, 12\}$.

b) $W = E_7$ and q^2 is a primitive root of unity of order e and $e \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$.

c) $W = E_8$ and q^2 is a primitive root of unity of order e and $e \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}$.

Proof LH is non-semisimple if and only if $\theta(P_W)$ is 0. The ring homomorphism θ is the identity on $\mathbb{Q}$ and P_W is a polynomial in $u = v^2$. Hence, $\theta(P_W(u)) = P_W(\theta(v)^2) = P_W(q^2)$. Furthermore, $P_W(q^2)$ is 0 if and only if $\Phi_e(q^2)$ is 0 for some cyclotomic polynomial Φ_e dividing P_W. Thus, q^2 is a primitive e'th root of unity. The values for e can be read off the factorisation of P_W. ■

Remark 3.8 The decomposition matrices D_θ for all non-semisimple specialisations of LH for $W = E_n$ for n in $\{6, 7, 8\}$ can be found in [GJ11, Chapter 7]. In combination with the degrees of irreducible representations of KH, which are exactly those of $\mathbb{C}W$ and which can be computed in CHEVIE, we are then able to apply Corollary 2.26 to compute the degrees of all irreducible representations of LH. We expand on that in Appendix A and list the degrees of all irreducible LH-modules for W of type E_6, E_7 or E_8.

Having successfully computed the degrees of all irreducible LH-modules for all possible values of q, we once again apply the dimension restrictions of Corollary 2.25. We make a small change in notation to state more clearly which group we are currently considering:

Definition 3.9 In our current setting, we write $LH(E_6)$ to indicate that we assume the group W to be of type E_6. Analogously, we define $LH(W)$ for any other W. Furthermore, we extend this notation to the parabolic subalgebras and denote e.g. by $LH_J(A_5) \leq LH(E_6)$ the parabolic subalgebra of $LH(E_6)$ that is of type A_5. If not explicitly stated otherwise we will assume that J is the corresponding subset of S given in Table 3.1. The parabolic subgroup's isomorphism type will define J uniquely unless its type is D_5 in E_6. This will not cause any problems.

Lemma 3.10 *There is no irreducible imprimitive representation of $LH(E_6)$ induced from $LH_J(A_5)$.*
There exists no irreducible imprimitie representation of $LH(E_8)$ induced from either $LH_J(D_7)$ or $LH_J(E_6 \times A_1)$.

Proof Suppose that V is an irreducible LH-module induced from some LH_J-module V_J, with LH and LH_J as specified here. Then $\dim_L(V) = \dim_L(V_J)\,[W : W_J]$. As we know the dimensions of all irreducible LH-modules by Remark 3.8, we easily check that no LH-module of suitable dimension exists using the tables in Appendix A. ∎

Note that this result works for every possible e, where q^2 is a primitive e'th root of unity.
This leaves us with only a small number of cases given in the table below. We have seen that for every *other* combination of maximal parabolic subgroup $W_J \leq W$, where $W \cong E_n$, and e as specified in Corollary 3.7 there exists no irreducible imprimitive representation of LH of type W induced from the Iwahori-Hecke algebra LH_J of type W_J.
Remaining cases:

W	$e \in$	$W_J \cong$	$[W : W_J]$
E_6	$\{2, 3, 4, 5, 6, 8, 9, 12\}$	D_5	27
	$\{2, 3, 4, 5, 6, 8, 9, 12\}$	D_5	27
E_7	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$	E_6	56
	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$	D_6	126
E_8	$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}$	E_7	240

The technique used in the proof of Lemma 3.10 can be used again to reduce the possibilities for e we have to consider.

Lemma 3.11
- *There exists no irreducible imprimitive representation of $LH(E_6)$ for e in $\{2, 4, 5, 8\}$ that is induced from a module of a parabolic subalgebra of type D_5.*
- *There exists no irreducible imprimitive representation of $LH(E_7)$ for $e = 3$ induced from a module of the parabolic subalgebra $LH_J(E_6)$.*
- *There exists no irreducible imprimitive representation of $LH(E_7)$ for e in*
  $\{3, 4, 5, 6, 10\}$ induced from a module the parabolic subalgebr $LH_J(D_6)$

- *There exists no irreducible imprimitive representation of $LH(E_8)$ for e in $\{2, 3, 6, 18, 20\}$ induced from a module of the parabolic subalgebra $LH_J(E_7)$.*

Proof The idea is the same as in the proof of 3.10: An LH-module V can only be irreducible and induced from an LH_J-module V, if its dimension is divisible by $[W : W_J]$. Even then can it only be induced, if there is an irreducible LH_J-module of dimension $\dim(V)/[W : W_J]$. As it turns out, the first criterion already rules out most cases in this lemma, see tables in Appendix A. Only for $E_7 \leq E_8$ and $e \in \{18, 20\}$ there do indeed exist irreducible $LH(E_8)$-modules whose dimension is divisible by the subgroup index, but since we also know the dimensions of all irreducible $LH_J(E_7)$-modules we easily check that none of them has the corresponding dimension. ∎

The remaining cases fall into two categories: Either we are looking at $D_{n-1} \leq E_n$ or at $E_{n-1} \leq E_n$. We handle both cases separately, starting with the latter.

Lemma 3.12 *There is no irreducible $LH(E_7)$-module induced from an $LH_J(E_6)$-module. Similarly, there is no irreducible $LH(E_8)$-module induced from an $LH_J(E_7)$-module.*

Proof Recall that q^2 is a primitive root of unity of order e and that the only cases we still need to check are

- $e \in \{2, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$ for $LH(E_7)$ and
- $e \in \{4, 5, 7, 8, 9, 10, 12, 14, 15, 24, 30\}$ for $LH(E_8)$,

as all other cases have been taken care of.
We prove the lemma using characters.
Suppose M is an irreducible $LH(E_n)$-module (with n either 7 or 8), whose dimension is divisible by the subgroup index $[E_n : E_{n-1}]$ and there exists an irreducible $LH_J(E_{n-1})$-module M_J, whose dimension is equal to $\dim(M)/[E_n : E_{n-1}]$. These are necessary conditions on M to be irreducible and imprimitive, see Corollary 2.25. We want to show, that whenever these necessary conditions are met, we can find an element h of H on which the characters of M and $\mathrm{Ind}_J^S(M_J)$ take different values. This will be sufficient to show that M and $\mathrm{Ind}_J^S(M_J)$ are not isomorphic.
Since the decomposition map $d_\theta : R_0(KH) \to R_0(LH)$ is surjective, there exist integer coefficients a_V for every irreducible KH-module V such that $\sum_V a_V\ d_\theta([V]) = [M]$. Similarly, the decomposition map $d_{\theta,J} : R_0(KH_J) \to R_0(LH_J)$ for the parabolic subalgebras is surjective and there exist integer coefficients b_N for every irreducible KH_J-module N such that $[M_J]$ is equal to $\sum_N b_N\ d_{\theta,J}([N])$. Since both $LH(E_n)$ and $LH_J(E_{n-1})$ are of type E_m

for some m, we can obtain the decomposition maps d_θ and $d_{\theta,J}$ from the tables in [GJ11] and [GP00]. Therefore, we are able to actually compute suitable coefficients a_V and b_N as solutions of simple linear equations. They can be found in Appendix B for all M and M_J satisfying the dimension restrictions.
We now translate the linear combinations of classes in the Grothendieck groups to values of the corresponding characters on an element h of H. For any irreducible KH-module V let χ_V be the character afforded by it. Similarly, the character afforded by an irreducible KH_J-module N will be denoted by χ_N. The character afforded by M is $\varphi_M(h \otimes_A 1) = \theta\left(\sum_V a_V \dot{\chi}_V(h)\right)$. The character of $\mathrm{Ind}_J^S(M_J)$ on the other hand is given as follows: We know that $[M_J] = \sum_N b_N \; d_{\theta,J}([N])$ and therefore also $\left[\mathrm{Ind}_J^S(M_J)\right] = \sum_N b_N \; \mathrm{Ind}_J^S\left(d_{\theta,J}\left([N]\right)\right)$. By Lemma 1.47 we know that $\mathrm{Ind}_J^S\left(d_{\theta,J}\left([N]\right)\right) = d_\theta\left(\mathrm{Ind}_J^S(N)\right)$. Finally, we translate this to characters and obtain

$$\varphi_{[\mathrm{Ind}_J^S(M_J)]}(h \otimes_A 1) = \theta\left(\sum_N b_N \; \dot{\chi}_{\left(\mathrm{Ind}_J^S(N)\right)}(h)\right).$$

Following this observation, we will compute characters and induced characters in KH, i.e. in the generic case, and then apply θ to see whether the two characters are indeed not equal on some element.
The tables in Appendix B list the values obtained in this manner on some suitably chosen elements of $LH(E_7)$ and $LH(E_8)$. For each e considered in this proof we assume that q^2 is a primitive e'th root of unity and then list all $LH(E_n)$-modules M and $LH_J(E_{n-1})$-modules M_J that satisfy the necessary dimension conditions. One checks easily that

$$\theta\left(\sum_V a_V \dot{\chi}_V(h_n)\right) \neq \theta\left(\sum_N b_N \; \dot{\chi}_{\mathrm{Ind}_J^S(N)}(h_n)\right)$$

for the elements h_n specified in Appendix B in every such case. Hence, we know that M and $\mathrm{Ind}_J^S(M_J)$ are not isomorphic and therefore M_J does not induce M. ∎

Having taken care of $LH_J(E_{n-1})$ in $LH(E_n)$ we are now left with $LH_J(D_{n-1})$ in $LH(E_n)$.
First we consider only $LH_J(D_6) \leq LH(E_7)$ for $e = 2$ as this case differs from all the others which will all be handled simultaneously afterwards. First, we need a rather general result:

Lemma 3.13 *Suppose (W,S) is a Coxeter system, F is a field and u an element of F. Let $H' := H_F(W,S,(u))$ be the Iwahori-Hecke algebra over F with parameter u. If $\rho : H' \to F$ is a one-dimensional representation of H', then $\rho(T_s) \in \{-1,\ u\}$ for every $s \in S$. Conversely, both $T_s \mapsto -1$ for all s and $T_s \mapsto u$ for all s extend to well-defined representations.*

Proof Let s be an element of S and set $z := \rho(T_s)$. Since ρ is a representation we know that the images of the T_s satisfy the braid and the quadratic relations of generators of Iwahori-Hecke algebras . Therefore, we know that $z^2 = u + (u-1)z$ which can easily be rearranged to the equation $(z+1)(z-u) = 0$ showing that z is either -1 or u.
To see the converse, note that $T_s \mapsto -1$ and $T_s \mapsto u$ both satisfy the quadratic and the braid relations. Hence, they give rise to well-defined representations of H'. ■

Lemma 3.14 *If q^2 is a primitive root of unity of order 2, that is $q^2 = -1$, there is no irreducible $LH(E_7)$-module which is induced from an $LH_J(D_6)$-module*

Proof There exists a unique $LH(E_7)$-module M (up to isomorphism) whose dimension is divisible by $[E_7 : D_6]$. This can be checked using the tables in Appendix A. Its class in the Grothendieck group can be expressed as $[M] = d_\theta([315'_a]) - d_\theta([189'_b])$ where $315'_a$ and $189'_b$ are irreducible $KH(E_7)$-modules denoted by their Frame name (this concept is explained in the introduction to Appendix B). This follows from the decomposition tables in [GJ11, Section 7.4]. The dimension of M is 126 and therefore equal to $[E_7 : D_6]$, so if M were induced from an $LH_J(D_6)$-module M_J then M_J would have to have dimension 1.
By Lemma 3.13 we know that there exists exactly one one-dimensional $LH_J(D_6)$-module since the parameter of $LH_J(D_6)$ is itself equal to -1. The representation and therefore also the character of this irreducible one-dimensional representation is given by $\rho : LH_J(D_6) \to L : T_s \mapsto -1$. We copy the technique of Lemma 3.12.
Let $w := s_7s_6s_7s_5s_6s_7s_4s_5s_6s_7s_2s_4s_5s_6s_7s_3s_4s_5s_6s_7s_2s_4s_5s_6s_3s_4s_5s_2s_4s_3$ where the s_i correspond to the vertices in the Coxeter graph in Table 3.1. Then the character of M on T_w is 126 but the character induced from ρ is 78. Hence, M is not the module induced from ρ and as M was the only irreducible $LH(E_7)$ module whose dimension was divisible by $[E_7 : D_6]$ we have proved the claim. ■

Finally, we come to the remaining cases of $LH_J(D_5)$ in $LH(E_6)$ and $LH_J(D_6)$ in $LH_J(E_7)$. By making an easy observation we see a good way to prove Theorem 3.1 for these cases:

Remark 3.15 For $LH(E_6)$ consider $e \in \{3, 6, 9, 12\}$, for $LH(E_7)$ consider $e \in \{7, 8, 9, 12, 14, 18\}$. These are all cases not yet considered. From the tables in Appendix A we know that there are only few irreducible $LH(E_n)$-modules whose dimension is divisible by $[E_n : D_{n-1}]$ and that every such $LH(E_n)$-module has dimension 3 $[E_n : D_{n-1}]$. By Corollary 2.25 we are therefore done if we can show that $LH_J(D_5)$ and $LH_J(D_6)$ have no irreducible modules of dimension 3.

We take the route suggested in this Remark and prove that for W of type D_5 or D_6 and q^2 a primitive e'th root of unity as specified in Remark 3.15 the algebra $LH(D_n)$ has no irreducible representation of rank 3. In fact, these are special cases of a more general result:

Lemma 3.16 *Suppose (W, S) is a Coxeter system of type D_5 or D_6. Let F be a field whose characteristic is not 2 and q an element of F^*. Let $H_F(W, S, (q^2))$ be the Iwahori-Hecke algebra of W over F with parameter q^2. Then the following holds:*

a) If W is of type D_6 and q^2 is a primitive root of unity of order e where e is greater than 6 then there exists no irreducible representation of $H_F(W, S, (q^2))$ of dimension 3.

b) If W is of type D_5 and q^2 is a primitive root of unity of order e where e is greater than 5 then there exists no irreducible representation of $H_F(W, S, (q^2))$ of dimension 3.

c) If W is of type D_5 and q^2 is a primitive root of unity of order 3 and the characteristic of F is neither 2 nor 3 then there exists no irreducible representation of $H_F(W, S, (q^2))$ of dimension 3.

The proof of part c) is far more messy than that of the other two parts so we start with them to acclimatize ourselves with the proper techniques.

Proof of a) and b) in Lemma 3.16 We do this for $W = D_5$. For $W = D_6$ it works completely analogously.
Let $B := \mathbb{Z}\left[\frac{1}{2}, v^{\pm 1}\right]$ be the ring of Laurent polynomials in v over $\mathbb{Z}\left[\frac{1}{2}\right]$ and $\mathcal{H} := H_B(W, S, (u := v^2))$ the corresponding generic Iwahori-Hecke algebra. Define a ring homomorphism $\eta : B \to F$ by sending v to q. Note that B is integrally closed in its field of fractions K and that η is well defined because the characteristic of F is not 2.
Let $F' \leq F$ be the field of fractions of $\eta(B)$. By Lemma 2.23 we know that F' is a splitting field of the specialised algebra $F'\mathcal{H}$. The algebra $H_F(W, S, (q^2))$ is isomorphic to $(F'\mathcal{H}) \otimes_{F'} F$, the specialisation of $F'\mathcal{H}$ obtained by extending scalars to F, and by Lemma 1.34 we see that it suffices to study the irreducible representations of $F'\mathcal{H}$ since it is split. Hence, we assume $F = F'$ from now on and study the representations of $F\mathcal{H}$.
The Coxeter graph of D_5 is given by $\begin{array}{c} 1 - 3 - 4 - 5 \\ \;\;\;\;| \;\;\;\;\;\;\;\;\;\; \\ \;\;\;\;2 \;\;\;\;\;\;\;\;\;\; \end{array}$ where the i'th vertex corresponds to a generator s_i in S. Clearly, the subset $I := \{s_1, s_3, s_4, s_5\} \subseteq S$ generates a parabolic subgroup of type A_4 in D_5. This gives us a parabolic subalgebra $F\mathcal{H}_I$ of type A_4 in $F\mathcal{H}$. We claim that $F\mathcal{H}_I$ is split semisimple and that all irreducible representations of $F\mathcal{H}_I$ have a dimension that is either 1 or at least 4.

The Iwahori-Hecke algebra $F\mathcal{H}_I$ is split by Lemma 2.23 as it is a specialisation of the Iwahori-Hecke algebra $\mathcal{H}_I$ which is defined over the L_0-good ring B. It is semisimple by Lemma 2.29: The Poincaré polynomial of $\mathcal{H}_I$ is $P_{A_4}(u) = \Phi_2^2(u)\Phi_3(u)\Phi_4(u)\Phi_5(u)$, see [GP00, 10.5.1], and $F\mathcal{H}_I$ can only not be semisimple if $\eta(P_{A_4})$ is 0. Assuming this to be the case and using the fact that the Φ_t's give a factorization of P_{A_4} in $\mathbb{Z}[u = v^2]$ we see that $\eta(\Phi_t(u))$ is 0 for some t in $\{2, 3, 4, 5\}$. But $\eta(\Phi_t(u))$ divides $\eta(u)^t - 1 = (q^2)^t - 1$. Since q^2 is a primitive e'th root of unity and e is greater than 5 this is a contradiction.
Therefore, $F\mathcal{H}_I$ is split semisimple and by Tits' Deformation Theorem 1.44 we know that the dimension of irreducible $F\mathcal{H}_I$-modules are exactly those of irreducible $K\mathcal{H}_I$-modules, where $K := \mathbb{Q}(v)$ is the field of fractions of B. The dimensions of irreducible $K\mathcal{H}_I$-modules are exactly those of irreducible $\mathbb{C}A_4$-modules, see Remark 2.10. We use the character table in GAP to see that these are either 1 or at least 4.

Now suppose that $\rho : F\mathcal{H} \to F^{3\times 3}$ is an arbitrary 3-dimensional representation of $F\mathcal{H}$. The restriction $\rho_I := \rho|_{F\mathcal{H}_I}$ to the parabolic subalgebra $F\mathcal{H}_I$ is a 3-dimensional representation of $F\mathcal{H}_I$. Because the dimension is 3 and all irreducible $F\mathcal{H}_I$-modules whose dimension is at most 3 have dimension 1, the composition factors of ρ_I all have dimension 1. Since $F\mathcal{H}_I$ is semisimple we know that ρ_I is the direct sum of three 1-dimensional representations and without loss of generality we can assume that $\rho_I(T_{s_i}) = \rho(T_{s_i})$ is a diagonal matrix for every s_i in I. The entries on the diagonal correspond to 1-dimensional representations of $F\mathcal{H}_I$. Let $\delta : F\mathcal{H}_I \to F$ be such a 1-dimensional representation. Since the T_{s_i} satisfy the braid relations and δ is a representation, the images $\delta(T_{s_i})$, too, will satisfy the braid relations. Also, since the T_{s_i}'s are invertible in $F\mathcal{H}_I$ this will also be true for their images under δ. For example we have

$$\delta(T_{s_5})\delta(T_{s_4})\delta(T_{s_5}) = \delta(T_{s_4})\delta(T_{s_5})\delta(T_{s_4}).$$

By multiplying both sides of this equation with $\delta(T_{s_4})^{-1}\delta(T_{s_5})^{-1}$ we see that $\delta(T_{s_5})$ and $\delta(T_{s_4})$ are equal. We apply this argument to every two neighbouring vertices in the Coxeter graph of A_4 to see that δ maps all T_{s_i} for s_i in I to the same invertible element of F.
Therefore, ρ maps all the T_{s_i} for s_i in I to the same invertible diagonal matrix, say $\rho(T_{s_i}) = \Delta := \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix}$ for invertible elements α_j of F.
The only information needed to fully define ρ is the image of the basis element T_{s_2}. Suppose this is image is some matrix $M := \rho(T_{s_2})$. Then M has to satisfy the following equations, which are just the quadratic and the braid relations expressed in M:

1) $M^2 = q^2 1 + (q^2 - 1)M$

2) $\rho(T_{s_i})M = M\rho(T_{s_i})$ for i in $\{1,4,5\}$, that is $\Delta M = M\Delta$.

3) $\rho(T_{s_3})M\rho(T_{s_3}) = M\rho(T_{s_3})M$, that is $\Delta M \Delta = M\Delta M$.

We combine the equation in 2) with that in 3) and obtain the equation

$$M\Delta\Delta = M\Delta M.$$

As images of invertible elements of $F\mathcal{H}_I$ under a representation, both M and Δ are invertible and therefore so is $M\Delta$. We multiply both sides of the equation with $(M\Delta)^{-1}$ from the left and see that Δ equals M.
In conclusion, the basis elements T_{s_i} for s_i in S are all mapped to the same diagonal matrix Δ. Since every basis element T_w for $w \in D_5$ can be expressed as a product in the T_{s_i} (see Lemma 1.9) we see that every standard basis element and therefore every element of $F\mathcal{H}$ is mapped to a diagonal matrix. Clearly, this implies that the submodule generated by $\begin{pmatrix}1 & 0 & 0\end{pmatrix}$, i.e. $\begin{pmatrix}1 & 0 & 0\end{pmatrix}\rho(F\mathcal{H})$, is a proper 1-dimensional submodule of the module affording ρ and therefore ρ is reducible. ∎

To prove c) in Lemma 3.16 we will use a similar technique. However, this case is more difficult as the Iwahori-Hecke algebra of type A_4 is no longer semisimple. This is particularly unhelpful because we can no longer just look up the dimensions of its irreducible modules in an ordinary character table using Tits' Deformation Theorem. Instead, we compute these degrees using the representations for the generic case available in [Mic15], the theory of specialisation and finally *Norton's irreducibility criterion.* The latter is a powerful result on irreducibility of modules whose applications include the famous MEATAXE algorithm(see [Par84]).
The following version is Theorem 1.3.3 in [LP10], where this is done for left-modules.

Lemma 3.17 (Norton's Irreducibility Criterion) *Let F be some field and R an F-algebra. Furthermore, let V be an R-module that has finite dimension over F. If $\ker_V(b) \neq 0$ for some $b \in R$ then V is irreducible if and only if*

a) $V = v\ R$ for all $v \in \ker_V(b)$ with $v \neq 0$ and

b) $V^* = R\ x$ for some $x \in \ker_{V^*}(b)$,

where V^* is the dual space of V.

Clearly, the first condition is most easily shown if the element b has one-dimensional kernel.
We use this theorem to prove a result on the irreducible modules of an Iwahori-Hecke algebra of type A_4 with a primitive third root of unity as its

parameter. The fact that this lemma can be stated for a nearly arbitrary field is basically an adaptation of the proof of Proposition 5.5 in [Gec92].

Lemma 3.18 *Let* (A_4, S) *be a Coxeter system of type* A_4 *and* F *a field whose characteristic is not* 2*. Suppose* q *is an element of* F *such that* q^2 *is a primitive third root of unity. Then there are exactly* 5 *irreducible* $H_F(A_4, S, (q^2))$*-modules up to isomorphism and their dimensions are* 1, 1, 4, 4, *and* 6*.*

Proof Let $B := \mathbb{Z}[v^{\pm 1}]$ the ring of Laurent polynomials in v over $\mathbb{Z}$ and let $\mathcal{H} := H_B(A_4, S, (u = v^2))$ the corresponding generic Iwahori-Hecke algebra. The Iwahori-Hecke algebra $H_F(A_4, S, (q^2))$ is a specialisation of $\mathcal{H}$ via $\eta : B \to F : v \mapsto q$. As in the proof of a) and b) of Lemma 3.16 we can assume without loss of generality that F is the field of fractions of $\eta(B)$ to show the results on irreducible characters. As before denote by K the field of fractions of B.
By Corollary 2.26 it is sufficient to study the rank of the specialised character table to find the number of irreducible $F\mathcal{H}$-modules. Since the generic character table is available in CHEVIE we just specialise it and find that its rank and therefore the number of irreducible $F\mathcal{H}$-modules is 5.
Using the character table in CHEVIE once more we see that the irreducible KH-modules have dimensions 1, 1, 4, 4, 5, 5, and 6. We will show that the specialisations of irreducible representations not of rank 5 are all irreducible and non-isomorphic. Then we are done because we will have found 5 non-isomorphic irreducible representations of $F\mathcal{H}$ of suitable rank.
We have seen in the proof of the first parts of Lemma 3.16 that any 1-dimensional representation of $F\mathcal{H}$ maps all T_s for s in S to the same image. In combination with Lemma 3.13 we see that there are exactly two 1-dimensional representations of $F\mathcal{H}$ up to isomorphism, namely $T_s \mapsto q^2$ and $T_s \mapsto -1$, which are non-isomorphic since $-1 \neq q^2$.
It remains to show that the irreducible KH-modules of dimension 4 and 6 remain irreducible and non-isomorphic under specialisation. We use the development version of CHEVIE ([Mic15]) to obtain two non-isomorphic irreducible 4-dimensional representation and one irreducible 6-dimensional representation of $K\mathcal{H}$. In CHEVIE they are labelled by 2111, 41 and 311 respectively, since they correspond to partitions of 5. For ease of notation we will refer to them as ρ_4, ρ_4' and ρ_6 respectively. They are each given as four 4×4 or four 6×6-matrices, respectively. These matrices are the images of the basis elements $T_{s_1}, \ldots, T_{s_4}$, where the generators s_i in S correspond to the vertices in the Coxeter graph $1 - 2 - 3 - 4$ of A_4. All these matrices only have entries in $\mathbb{Z}[u]$. Matrices for the specialised representations can therefore be obtained by simply applying η to all entries and we will denote them by $\eta(\rho_4)$, $\eta(\rho_4')$ and $\eta(\rho_6)$. We want to show that these representations are irreducible, that is to say the natural modules of $\eta(\rho_4(F\mathcal{H}))$, $\eta(\rho_4'(F\mathcal{H}))$ and $\eta(\rho_6(F\mathcal{H}))$ are irreducible. To this end we use Norton's Irreducibility Criterion:

- For ρ_4 let $b := \eta\left(\rho_4\left(T_{s_3}T_{s_1}T_{s_2} + T_{s_2}Ts2T_{s_1}T_{s_1}\right)\right)$. This matrix has co-rank 1 since its determinant is 0 but the matrix obtained by removing the fourth row and the fourth column has determinant $-2q^4 \neq 0$. Its kernel is the vector space generated by $(0,0,0,1)$ and the kernel of its transpose is generated by $(1, -q^4, q^2, q^2)^{\mathrm{Tr}}$. Using Norton's Irreducibility Criterion with b we see that $\eta(\rho_4)$ is indeed irreducible.

- For ρ_4' let $b := \eta\left(\rho_4'\left(T_{s_1}T_{s_3}T_{s_4} + T_{s_2}T_{s_3}\right)\right)$. This matrix has co-rank 1 since its determinant is 0 but the matrix obtained by removing the fourth row and fourth column has determinant $-q^2 \neq 0$. Its kernel is the vector space generated by $(1, -1, 0, -q^2)$ and the kernel of its transpose is generated by $(3, -2q^2-4q^4, -q^2-2q^4, -q^2-2q^4)^{\mathrm{Tr}}$. Using Norton's Irreducibility Criterion with b we see that $\eta(\rho_4)$ is indeed irreducible.

- For ρ_6 let $b := \eta\left(\rho_6\left(T_{s_4}T_{s_3}T_{s_4} + T_{s_4}T_{s_1}T_{s_2} + T_{s_2}T_{s_1} + T_{s_4}T_{s_3}\right)\right)$. This matrix has co-rank 1 since its determinant is 0 but the matrix obtained by removing the first row and the fifth column has determinant $-4 \neq 0$. Its kernel is the vector space generated by $(0, 1, -1, q^2, -q^2, 0)$ and the kernel of its transpose is generated by $(4, -q^2-4q^4, -q^2-4q^4, 2q^2-2q^4, 2q^2-2q^4, q^2-q^4)^{\mathrm{Tr}}$. Using Norton's Irreducibility Criterion with b we see that $\eta(\rho_6)$ is indeed irreducible.

The only thing left to show is that the two irreducible specialised representations of rank four are not isomorphic. This is done as follows. Consider the unique longest element w_0 in A_4. Using CHEVIE we check that $\eta(\rho_4(T_{w_0}^2)) = \mathrm{Diag}(q^4, q^4, q^4)$ and $\eta(\rho_4'(T_{w_0}^2)) = \mathrm{Diag}(1,1,1)$. Hence, there exists no isomorphism between $\eta(\rho_4))$ and $\eta(\rho_4')$ as scalars are left invariant by any isomorphism. ∎

Remark 3.19 It is no coincidence that the images of $T_{w_0}^2$ are scalar matrices. It is generally true that $T_{w_0}^2$ is central in an Iwahori-Hecke algebra if w_0 is the unique longest word in the Coxeter group (see [GP00, Theorem 9.2.2]) and by Schur's Lemma we know that central elements are mapped to scalar matrices under irreducible representations. The map sending a central element to this scalar is called the *central character* afforded by that representation. The values of central characters afforded by the irreducible representations of $KH(A_4)$ on $T_{w_0}^2$ are available in CHEVIE, which makes it easy to compute the specialised representations $\eta(\rho_4)$ and $\eta(\rho_4')$ on this element.

We continue our study of representations of an Iwahori-Hecke algebra of type A_4 which we will eventually use to prove our statement on representations of an Iwahori-Hecke algebra of type D_5.

Proposition 3.20 *Let (A_4, S) be a Coxeter system of type A_4 and F a field whose characteristic is not 2. Suppose q is an element of F such that*

q^2 is a primitive third root of unity. If $\rho : H_F(A_4, S, (q^2)) \to F^{3\times 3}$ is a 3-dimensional representation of $H_F(A_4, S, (q^2))$ then $\rho(T_s) = \rho(T_t)$ for all s and t in S.

Proof Suppose $T_{s_i} \mapsto M_i \in F^{3\times 3}$ defines a representation of $H_F(A_4, S, (q^2))$. By Lemma 3.18 we know that all composition factors of such a representation are of rank 1. Choosing a suitable basis we can therefore assume that the M_i's are lower triangular matrices where the diagonal corresponds to the 1-dimensional composition factors. We have seen already that any representation of rank 1 assumes equal values on all the T_{s_i}'s and that the assumed values are either -1 or q^2 Hence, we can further assume that the M_i's are lower triangular matrices all of which have the same diagonal, say $M_i = \begin{pmatrix} \alpha_1 & & \\ m_{i,1} & \alpha_2 & \\ m_{i,2} & m_{i,3} & \alpha_3 \end{pmatrix}$ for some elements $\alpha_j \in \{-1, q^2\}$. We want to show that $M_i = M_{i'}$ for $i,\ i' \in \{1,\ 2,\ 3,\ 4\}$, that is $m_{i,k} = m_{i',k}$ for all $i,\ i' \in \{1,\ 2,\ 3,\ 4\}$ and $k \in \{1,\ 2,\ 3\}$. This will follow from the relations between these elements that arise from the fact that $T_{s_i} \to M_i$ is a representation of an Iwahori-Hecke algebra. These relations are obtained from the quadratic and the braid relations, respectively, and yield the following equations:

1. $M_i^2 - q^2\mathbb{I} - (q^2-1)M_i = 0$ for all i.

2. $M_iM_j - M_jM_i = 0$ whenever $|i-j|$ is greater than 1.

3. $M_iM_jM_i - M_jM_iM_j = 0$ whenever $|i-j|$ equals 1.

Here, $\mathbb{I}$ denotes the 3×3 identity matrix.
These relations suffice to prove that $m_{i,k} = m_{i',k}$ for all $i,\ i' \in \{1,\ 2,\ 3,\ 4\}$ and $k \in \{1,\ 2,\ 3\}$: To see this, define the ring $R := \mathbb{Z}[\hat{q}^2, \hat{m}_{i,k} \mid i \in \{1,2,3,4\},\ k \in \{1,2,3\}]$, where the $\hat{m}_{i,k}$'s and $\hat{q}^2$ all pairwise distinct indeterminates. Clearly, $\lambda : R \to F : \hat{m}_{i,k} \mapsto m_{i,k}, \hat{q}^2 \mapsto q^2$ defines a ring homomorphism.
Now we define $\hat{M}_i$ as the matrix obtained from M_i by replacing every nonzero entry with the hatted version and consider the matrices

1. $\hat{M}_i^2 - \hat{q}^2\mathbb{I} - (\hat{q}^2-1)\hat{M}_i$ for all i,

2. $\hat{M}_i\hat{M}_j - \hat{M}_j\hat{M}_i$, whenever $|i-j|$ is greater than 1, and

3. $\hat{M}_i\hat{M}_j\hat{M}_i - \hat{M}_j\hat{M}_i\hat{M}_j$, whenever $|i-j|$ equals 1.

From the equations in the M_i's it follows that all entries of these matrices lie in the kernel of λ.
Hence, if we define $\mathfrak{L} \lhd R$ as the ideal generated by the entries of these matrices, then $\mathfrak{L}$ is contained in the kernel of λ. Using MAGMA ([BCP97])

we check whether $\hat{m}_{i,k} - \hat{m}_{j,k}$ is in $\mathfrak{L}$ for all i, j in $\{1,2,3,4\}$ and k in $\{1,2,3\}$. This is indeed the case and therefore $\hat{m}_{i,k} - \hat{m}_{j,k}$ is an element of the kernel of λ, that is $m_{i,k} - m_{j,k}$ equals 0 for all i, j in $\{1,2,3,4\}$ and k in $\{1,2,3\}$. Therefore, the matrices M_i are all equal. ∎

Now we are finally fully prepared to prove the last case of Lemma 3.16.

Proof of Part c) of Lemma 3.16 Define B, $\mathcal{H}$, K and I as in the proof of Parts a) and b). Suppose that $F\mathcal{H} \to F^{3\times 3}$ is a representation. Then the restriction $\rho_I := \rho|_{F\mathcal{H}_I}$ is a representation of the Iwahori-Hecke algebra $F\mathcal{H}_I$ of type A_4. By Proposition 3.20 we know that the elements T_{s_i} are all mapped to the same element, say $\Delta := \rho(T_{s_1})$. Now we argue as in the proof of the first two parts and the claim follows. ∎

Remark 3.21 At least for the case considered in this chapter, that is the characteristic 0 case, there is another nice way to see that $LH(D_5)$ and $LH(D_6)$ have no irreducible representation of degree 3 for all relevant possibilities for e. The parameter of $LH(D_n)$ is some e'th root of unity ζ_e. In all these cases the Poincaré polynomial of $KH(D_n)$ is divisible at most once by $u - \zeta_e$. Following [Gec92] this implies that the structure of the decomposition map is rather simple: Either an irreducible $KH(D_n)$-module stays irreducible under specialisation or its decomposition structure into irreducible $LH(D_n)$-modules is determined by so-called *Brauer trees for* Iwahori-Hecke algebras. These were defined in [Gec92] and have a very simple structure which enables us to compute the degrees of irreducible $LH(D_n)$-modules. Even without knowing the complete Brauer trees we are able to make sufficient statements about the possible dimensions of irreducible $LH(D_n)$-modules as the dimensions are solutions of small sets of linear equations. However, we do at least need to know which KH-modules belong to the same Brauer tree. This can be done using central characters and the fact that $T_{w_0}^2$, where w_0 is the unique longest word in W, is a central element of KH. This information is available in CHEVIE.

4. Imprimitive Representations of Iwahori-Hecke Algebras in Characteristic $\ell > 0$

In this chapter we want to study what happens if the field over which we are constructing Iwahori-Hecke algebras is of positive characteristic ℓ. Our goal is to prove an analogue of Theorem 3.1. As before we do this separately for non-crystallographic Coxeter groups and Weyl groups. The first are handled similarly to the characteristic 0 case. For the latter we do in fact need our earlier results for the characteristic 0 case. They will be transferred to the case of positive characteristic by *James's conjecture* which was proven for exceptional Weyl groups by Geck and Müller in [GM09].

4.1. Imprimitive representations of Iwahori-Hecke algebras of non-crystallographic type

Let (W, S) be a Coxeter system where W is a irreducible Coxeter group of non-crystallographic type, that is $W = H_3$, $W = H_4$ or $W = I_2(m)$ for $m = 5$ or $m \geq 7$. We prove the following analogue of Theorem 3.1:

Theorem 4.1 *Suppose L is a field of positive characteristic satisfying the following conditions depending on the type of W:*

a) If W is of type H_3, the characteristic of L should not be 2 or 5. Furthermore, L should contain a primitive fifth root of unity.

b) If W if of type H_4, the characteristic of L should not be 2, 3 or 5. Furthermore, L should contain a primitive fifth root of unity.

c) If W is of type $I_2(m)$, the characteristic of L should not divide m. Furthermore, L should contain a primitive m'th root of unity.

Now let q be an invertible element of L. Then the Iwahori-Hecke algebra $H_L(W, S, (q^2))$ has no irreducible imprimitive representation.

Proof As for characteristic 0 we define the following rings depending on the type of W:

a) $R_{H_3} := \mathbb{Z}\left[\zeta_5, \frac{1}{2}, \frac{1}{5}\right]$,

b) $R_{H_4} := \mathbb{Z}\left[\zeta_5, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right]$, and

c) $R_{I_2(m)} := \mathbb{Z}\left[\zeta_m, \frac{1}{m}\right]$,

where ζ_k is a primitive k'th root of unity. In every case we see that R_W is L_0-good for W. Now let $A_W := R_W\left[v^{\pm 1}\right]$ be the ring of Laurent polynomials in v over R_W and define the corresponding generic Iwahori-Hecke algebra $H := H_{A_W}(W, S, (v^2))$. Note that A_W is integrally closed in K_W, its field of fractions. By the conditions on L there exists a ring homomorphism from R_W to L. We extend this to a ring homomorphism $\theta : A_W \to L$ by sending v to q. The algebra $H_L(W, S, (q^2))$ is naturally isomorphic to the algebra obtained by specialisation of H via θ. By Lemma 2.23 we know that the field of fractions of $\theta(A)$ which we denote by L' is a splitting field for the algebra $L'H$ so by Corollary 1.35 we see that we only have to consider $L'H$. Hence, we just assume that L is the field of fractions of $\theta(A)$ from now on.

We are now in the setting of Lemma 2.23 and therefore its Corollary 2.25 holds. The rest of the proof works exactly as that of Lemma 3.2 where this was done for characteristic 0. ∎

4.2. Imprimitive representations of Iwahori-Hecke algebras of exceptional Weyl groups

Let (W, S) be a Coxeter system where W is an exceptional Weyl group, that is $W = F_4$, $W = G_2$, $W = E_6$, $W = E_7$ or $W = E_7$. We prove the following analogue of Thereom 3.1:

Theorem 4.2 *Let L be a field of characteristic $\ell > 0$ and let q be an invertible element in L such that q has order $2e$ in L^* and q is not 1. Suppose $e\ell$ does not divide a degree of W (see Table 4.1) and the characteristic ℓ is a good prime for W. Then the Iwahori-Hecke algebra $H_L(W, S, (q^2))$ has no irreducible imprimitive representation.*

Table 4.1.: Degrees of exceptional Weyl groups

Group	Degrees
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

The proof is a bit more complicated than that for non-crystallographic groups and requires some deep results on the decomposition map due to Geck and Müller. The key idea is that we get information about the decomposition map for a field of positive characteristic by factoring it over that for a root of unity. Our proof here follows the one in [Gec98].

Lemma 4.3 *Let R be an L_0-good ring for W as in Example 2.20 and $A := R\left[v^{\pm 1}\right]$ the ring of Laurent polynomials in v over R. Then A is integrally closed in K, its field of fractions. Let $H := H_A(W, S, (u := v^2))$ be the generic Iwahori-Hecke algebra over R and suppose $\theta_L : A \to L$ is a ring homomorphism such that L is the field of fractions of $\theta_L(A)$ and $\theta_L(v)$ is a primitive $2e$'th root of unity for some integer e. Then there exists a primitive $2e$'th root of unity ζ in $\mathbb{C}$ such that θ_L factorizes over $R\left[\zeta\right]$ by sending v to ζ, i.e. there exist homomorphisms θ_e and $\theta_{e,L}$ such that the following diagram commutes:*

$$\begin{array}{ccc} R[v^{\pm 1}] & \xrightarrow{\theta_L} & L \\ & \searrow^{\theta_e} \quad {}^{\theta_{e,L}}\nearrow & \\ & R[\zeta] & \end{array}$$

This induces a corresponding commutative diagram for decomposition maps by setting $k := \mathbb{Q}(\zeta)$.

$$\begin{array}{ccc} R_0(KH) & \xrightarrow{d_{\theta_L}} & R_0(LH) \\ & \searrow^{d_{\theta_e}} \quad {}^{d_{\theta_{e,L}}}\nearrow & \\ & R_0(kH) & \end{array}$$

For convenience we define $d_L := d_{\theta_L}$, $d_e := d_{\theta_e}$ and $d_{e,L} := d_{\theta_{e,L}}$. In terms of the corresponding decomposition matrices the diagram can be expressed as $D_L = D_{e,L} D_e$.

Proof The factorisation of θ_L is clear as $\Phi_{2e}(v)$ is obviously in the kernel of θ_L. We have to show the factorisation of decomposition maps. Let us first

convince ourselves that all decomposition maps are well defined: First note that A is integrally closed in K, as is $R[\zeta]$ in k. Clearly, k is the field of fractions of $\theta_e(A)$ and L is the field of fractions of $\theta_L(A)$ by the hypothesis, therefore it is also the field of fractions of $\theta_{e,L}(R[\zeta])$ because θ_e is surjective. By Lemma 2.23 we know that all three Iwahori-Hecke algebras KH, LH and kH are split and we obtain three well-defined decomposition maps.
The decomposition maps d_e and $d_{e,L}$ yield the following two commutative diagrams:

$$\begin{array}{ccc} R_0^+(KH) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(H, R[v][X]) \\ \Big\downarrow{\scriptstyle d_e} & & \Big\downarrow{\scriptstyle t_{\theta_e}} \\ R_0^+(kH) & \xrightarrow{\mathfrak{p}_k} & \mathrm{Maps}(H, k[X]) \end{array} \quad \text{and} \quad \begin{array}{ccc} R_0^+(kH) & \xrightarrow{\mathfrak{p}_k} & \mathrm{Maps}(H, R[\zeta][X]) \\ \Big\downarrow{\scriptstyle d_{e,L}} & & \Big\downarrow{\scriptstyle t_{\theta_{e,L}}} \\ R_0^+(LH) & \xrightarrow{\mathfrak{p}_L} & \mathrm{Maps}(H, L[X]) \end{array}$$

Since $R[\zeta]$ is integrally closed in k we apply Corollary 1.38 to replace the lower row of the first diagram and we get

$$\begin{array}{ccc} R_0^+(KH) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(H, R[v][X]) \\ \Big\downarrow{\scriptstyle d_e} & & \Big\downarrow{\scriptstyle t_{\theta_e}} \\ R_0^+(kH) & \xrightarrow{\mathfrak{p}_k} & \mathrm{Maps}(H, R[\zeta][X]) \end{array}$$

We see that the lower row of the diagram for d_e and the upper row for $d_{e,L}$ coincide. By joining the two diagrams we get a new commutative diagram

$$\begin{array}{ccc} R_0^+(KH) & \xrightarrow{\mathfrak{p}_K} & \mathrm{Maps}(H, R[v][X]) \\ \Big\downarrow{\scriptstyle d_e \circ d_{e,L}} & & \Big\downarrow{\scriptstyle t_{\theta_e} \circ t_{\theta_{e,L}}} \\ R_0^+(LH) & \xrightarrow{\mathfrak{p}_L} & \mathrm{Maps}(H, L[X]) \end{array}$$

We know that $t_{\theta_e} \circ t_{\theta_{e,L}}$ is just t_{θ_L} because $\theta_e \circ \theta_{e,L}$ equals θ_L. Therefore, this diagram is exactly the defining commuting diagram for the decomposition map d_L. By Theorem 1.41 the decomposition map is the unique map for which this diagram commutes and therefore we know that $d_L = d_{e,L} \circ d_e$. The statement on decomposition matrices follows from the definition of decomposition matrices. ■

Remark 4.4 From now on we will assume that we are in the setting of Lemma 4.3. This is no real restriction on our way to prove Theorem 4.2: If the characteristic of L is indeed a good prime for W, the subfield $L' := \mathbb{F}_\ell(q)$ of L satisfies the conditions in Lemma 4.3 and the Iwahori-Hecke algebra $H_{L'}(W, S, (q^2))$ is already split. Once again, we use Corollary 1.35 to see that it is sufficient to study this algebra's imprimitive representations to study those of $H_L(W, S, (q^2))$ and we will work with $L = \mathbb{F}_\ell(q)$ from now on.

Our next tool is a powerful result by Geck and Müller from 2009 which proves a conjecture for Iwahori-Hecke algebras of exceptional Weyl groups, first stated by James in 1989 for Iwahori-Hecke algebras of type A_n; see [GM09, Theorem 3.10] and [Jam90].

Lemma 4.5 (James's conjecture) *Suppose the setting of Lemma 4.3. If $e\ell$ does not divide a degree of W then $d_{e,L}$ is trivial, i.e. it is a bijection sending classes of irreducible modules to classes of irreducible modules.*

This result is particularly powerful because we have already studied the irreducible imprimitive representations of kH in depth in Chapter 3. We see some immediate consequences:

Corollary 4.6 *The degrees of irreducible LH-modules are exactly those of irreducible kH-modules.*

This observation will already be nearly sufficient to prove Theorem 4.2:

Proof of Theorem 4.2 By Corollary 4.6 we can copy all results from Section 2 of Chapter 3, whose only line of argumentation was the degrees of kH-modules and their divisibility and apply them to LH-modules. This proves a huge chunk of cases and the only ones we still need to consider are the following:

a) Parabolic subalgeras of type E_{n-1} of $LH(E_n)$ for n either 7 or 8

b) Parabolic subalgebras of type D_{n-1} of $LH(E_n)$ for n either 6 or 7

We start by proving a). Suppose $LH_J(E_{n-1})$ is a parabolic subalgebra of $LH(E_n)$ for n either 7 or 8. Here, both $LH_J(E_{n-1})$ and $LH(E_n)$ are Iwahori-Hecke algebras of exceptional Weyl groups. Hence, we can apply James's conjecture to both of them to see that the decomposition maps $R_0(kH(E_n)) \to R_0(LH(E_n))$ and $R_0(kH_J(E_{n-1})) \to R_0(LH_J(E_n))$ are trivial. In Chapter 3 we have seen that $kH_J(E_{n-1})$ induces no irreducible representation of $kH(E_n)$ and because the decomposition maps are trivial we can use Theorem 1.45 to see that the same holds for $LH_J(E_{n-1})$ and $LH(E_n)$. We go on to prove b). Suppose $LH_J(D_{n-1})$ is a parabolic subalgebra of $LH(E_n)$ for n either 6 or 7. The decomposition map $R_0(kH(E_n) \to R_0(LH(E_n))$ is trivial by James's conjecture. In Chapter 3 the question of irreducible imprimitive modules has been solved for most possibilities of e using only divisibility arguments, which we can copy by Corollary 4.6. We are left with

- e in $\{3, 6, 9, 12\}$ for $LH(E_6)$ and
- e in $\{2, 7, 8, 9, 12, 14, 18\}$ for $LH(E_7)$.

For $e = 2$ we know that the only dimension of an irreducible $LH(E_7)$-module divisible by $[E_7 : D_6]$ is equal to $[E_7 : D_6] = 126$ and there exists a unique module of this dimension up to isomorphism, see Lemma 3.14.

Hence, this module can only be induced from a 1-dimensional module whose representation has to be $T_s \mapsto -1$ by Lemma 3.13. We copy the proof of Lemma 3.14 and compute the character of the irreducible representation of degree 126 and the character of the representation induced from $T_s \mapsto -1$ on an element T_w of LH. We choose $w := s_1s_2s_3s_4$. The characters take the values 11 and 38 respectively. These values can only be equal if L has characteristic 3, but ℓ is good for E_7 and so the two characters take different values on T_w. Therefore, the representations are not isomorphic and the 126-dimensional module is not imprimitive. Since it was the only irreducible $LH(E_7)$-module whose dimension was divisible by the subgroup index, there exists no irreducible representation of $LH(E_7)$ for $e = 2$.
For the remaining cases we point to Remark 3.15 telling us that we only have to show that $LH_J(D_{n-1})$ has no irreducible representation of degree 3. We have proved this already in 3.16 and therefore we are done. ∎

Remark 4.7 The conditions on ℓ and e in Theorem 4.2 will always be satisfied if ℓ is greater than 7 or if ℓ is a good prime for W and e is greater than 3.

A. Degrees of Irreducible Representations of $LH(E_6)$, $LH(E_7)$ and $LH(E_8)$

We give the dimensions of irreducible modules of some Iwahori-Hecke algebras in this chapter:
Let (W, S) be a Coxeter system and W of type E_6, E_7 or E_8. Suppose $L := \mathbb{Q}(\zeta)$ where ζ is a primitive root of unity such that ζ^2 has order e for some e. Then we know that the Iwahori-Hecke algebra $LH := H_L(W, S, (\zeta^2))$ is split. It is a specialisation from the generic Iwahori-Hecke algebra $H := H_{\mathbb{Q}[v]}(W, S, (v^2))$ via $\theta : v \mapsto \zeta$ and for $K := \mathbb{Q}(v)$ there exists a well defined decomposition map $d_\theta : R_0(KH) \to R_0(LH)$. This map is trivial for most values of e in which case the dimensions of irreducible LH-modules are exactly those of irreducible KH-modules which in turn are the dimensions of irreducible $\mathbb{C}W$-modules. We will give these dimensions which can be found in the ordinary character tables in CHEVIE but we will also give these dimensions for every other value of e:
Corollary 2.26 tells us that we can compute the dimensions of all irreducible LH-modules if we know the decomposition matrix D_θ and the dimensions of irreducible KH-modules. The latter are easily found in the ordinary character tables while the former can be found in Chapter 7 of [GJ11]. However, the decomposition matrices can be arranged to have block-diagonal form and only some of these blocks are given in [GJ11]. Here, Geck and Jacon omit so-called blocks of defect 1 and 0. Nevertheless, this will not be problematic:
A block on the diagonal of D_θ corresponds in a natural way to a set of irreducible representations of KH, namely the representations indexing the rows of said block (recall that the rows of D_θ correspond to irreducible KH-modules whereas the columns correspond to irreducible LH-modules). For any diagonal block of D_θ we call the corresponding subset of irreducible KH-modules also a block.
The Appendix F of [GP00] lists all so-called blocks of defect 1 of irreducible KH-modules. Here, the modules within a block are given in an order such that the corresponding decomposition map (which is the corresponding block on the diagonal of D_θ) is an $m \times (m-1)$-matrix M where m is the number of irreducible KH-modules in that block, see [GP00, Remark F.1]. The entries $M_{i,i}$ and $M_{i+1,i}$ are all 1. All other entries are 0. This is

Table A.1.: Degrees of irreducible representations of $LH(E_6)$

e	Degrees of irreducible representations
1	1, 1, 10, 6, 6, 20, 15, 15, 15, 15, 20, 20, 24, 24, 30, 30, 60, 80, 90, 60, 60, 64, 64, 81, 81
2	1, 6, 10, 14, 14, 46, 64, 80
3	1, 1, 5, 5, 10, 10, 14, 14, 25, 25, 81, 81, 90
4	1, 1, 6, 6, 8, 15, 15, 20, 20, 20, 24, 24, 30, 30, 40, 60, 60, 64, 64
5	1, 6, 10, 15, 15, 15, 15, 20, 20, 20, 23, 30, 30, 58, 60, 60, 60, 80, 90
6	1, 1, 6, 6, 10, 11, 11, 13, 13, 14, 14, 15, 15, 20, 32, 64, 64, 81, 81, 90
8	1, 1, 6, 6, 10, 15, 15, 15, 15, 20, 20, 20, 24, 24, 29, 29, 52, 60, 60, 60, 64, 64, 80, 90
9	1, 1, 6, 6, 10, 15, 15, 15, 15, 19, 19, 20, 24, 24, 30, 30, 45, 45, 60, 60, 60, 80, 81, 81
12	1, 1, 5, 5, 10, 10, 10, 15, 15, 20, 20, 24, 24, 30, 30, 60, 60, 60, 64, 64, 80, 81, 81, 90

due to the theory of Brauer trees of Iwahori-Hecke algebras introduced by Geck in [Gec92]. See [GJ11, Theorem 3.3.13] for a conclusive summary.

Finally, every irreducible KH-module not appearing in a block in the tables in [GJ11] or [GP00] has *defect* 0. Its corresponding block of D_θ is trivial, i.e. it is just the 1×1 matrix containing 1. In other words, if V is an irreducible KH-module in a block of defect 0, then the specialisation of V via θ is irreducible.

In fact, V constitutes a block of defect 0 if and only if its Schur element c_V is *not* mapped to 0 under θ. As Schur elements of $KH(E_n)$ are readily available in CHEVIE, this provides an easy way to find all defect-0 modules. Combining the tables in Chapter 7 of [GJ11], the blocks of defect 1 in the Appendix of [GP00] with their known decomposition maps and the trivial decomposition maps for blocks of defect 0 we get the full decomposition matrices D_θ. As described in Corollary 2.26 we can use them to compute the degrees of all irreducible LH-modules. The results can be found in the following tables. By $e = 1$ we denote the case that LH is semisimple: If $\zeta^2 = 1$ then LH is naturally isomorphic to the group algebra $\mathbb{Q}W$, which is semisimple.

Table A.2.: Degrees of irreducible representations of $LH(E_7)$

e	Degrees of irreducible representations
1	1, 1, 7, 7, 15, 15, 21, 21, 21, 21, 27, 27, 35, 35, 35, 35, 56, 56, 70, 70, 84, 84, 105, 105, 105, 105, 105, 105, 120, 120, 168, 168, 189, 189, 189, 189, 189, 189, 210, 210, 210, 210, 216, 216, 280, 280, 280, 280, 315, 315, 336, 336, 378, 378, 405, 405, 420, 420, 512, 512
2	1, 6, 14, 14, 56, 56, 64, 78, 126, 216, 512, 512
3	1, 1, 7, 7, 14, 14, 21, 21, 27, 34, 34, 35, 35, 49, 49, 91, 91, 98, 98, 189, 189, 189, 196, 196, 315, 315, 405, 405
4	1, 1, 7, 7, 8, 8, 21, 21, 21, 21, 27, 27, 35, 35, 48, 48, 56, 56, 84, 84, 84, 84, 105, 105, 120, 120, 147, 147, 154, 154, 168, 168, 168, 168, 280, 280, 420, 420, 512, 512
5	1, 7, 15, 15, 21, 21, 27, 35, 35, 35, 35, 56, 70, 70, 83, 105, 105, 105, 105, 105, 105, 120, 120, 133, 141, 168, 168, 210, 210, 210, 210, 280, 280, 280, 280, 315, 315, 371, 405, 405, 420, 420
6	1, 1, 7, 7, 13, 13, 14, 14, 21, 21, 27, 27, 27, 35, 35, 42, 43, 43, 56, 56, 64, 64, 77, 77, 90, 92, 92, 162, 189, 189, 189, 216, 216, 216, 272, 280, 280, 512, 512
7	1, 7, 7, 15, 21, 21, 21, 21, 26, 35, 35, 35, 35, 56, 56, 70, 70, 84, 84, 94, 105, 105, 105, 105, 105, 105, 168, 168, 189, 189, 189, 189, 189, 189, 201, 210, 210, 210, 210, 280, 280, 280, 280, 311, 315, 315, 336, 336, 378, 378, 420, 420
8	1, 7, 15, 15, 21, 21, 21, 27, 29, 35, 35, 35, 35, 70, 70, 76, 84, 84, 104, 105, 105, 112, 113, 168, 168, 168, 189, 189, 210, 210, 210, 210, 280, 280, 315, 315, 336, 336, 378, 378, 405, 405, 420, 420, 512, 512
9	1, 7, 15, 15, 21, 21, 21, 21, 27, 27, 34, 35, 35, 49, 70, 70, 84, 84, 105, 105, 105, 105, 105, 105, 120, 120, 168, 168, 189, 189, 189, 189, 189, 189, 210, 210, 210, 210, 216, 216, 246, 266, 280, 280, 336, 336, 378, 378, 405, 405, 420, 420
10	1, 7, 7, 15, 15, 20, 20, 20, 21, 21, 35, 35, 35, 56, 56, 70, 70, 84, 84, 105, 105, 105, 105, 105, 105, 120, 120, 148, 148, 154, 169, 189, 189, 210, 210, 210, 210, 216, 216, 230, 251, 280, 280, 280, 280, 315, 315, 336, 336, 512, 512
12	1, 7, 7, 15, 15, 21, 21, 21, 27, 27, 35, 35, 35, 35, 55, 70, 70, 84, 84, 99, 105, 105, 105, 105, 105, 105, 155, 168, 168, 181, 189, 189, 189, 189, 189, 189, 210, 210, 216, 216, 280, 280, 315, 315, 378, 378, 405, 405, 420, 420, 512, 512
14	1, 1, 7, 7, 15, 15, 21, 21, 21, 21, 26, 26, 35, 35, 35, 35, 56, 56, 70, 70, 79, 79, 84, 84, 105, 105, 105, 105, 110, 120, 120, 168, 168, 189, 189, 189, 189, 210, 210, 210, 210, 216, 216, 280, 280, 280, 280, 315, 315, 336, 336, 378, 378, 405, 405, 420, 420, 512, 512
18	1, 1, 6, 6, 15, 15, 15, 15, 20, 21, 21, 27, 27, 35, 35, 56, 56, 70, 70, 84, 84, 105, 105, 105, 105, 105, 105, 120, 120, 168, 168, 189, 189, 189, 189, 189, 189, 210, 210, 210, 210, 216, 216, 280, 280, 280, 280, 315, 315, 336, 336, 378, 378, 405, 405, 420, 420, 512, 512

Table A.3.: Degrees of irreducible representations of $LH(E_8)$

e	Degrees of irreducible representations
1	1, 1, 28, 28, 35, 35, 70, 50, 50, 84, 84, 168, 175, 175, 210, 210, 420, 300, 300, 350, 350, 525, 525, 567, 567, 1134, 700, 700, 700, 700, 1400, 840, 840, 1680, 972, 972, 1050, 1050, 2100, 1344, 1344, 2688, 1400, 1400, 1575, 1575, 3150, 2100, 2100, 4200, 2240, 2240, 4480, 2268, 2268, 4536, 2835, 2835, 5670, 3200, 3200, 4096, 4096, 4200, 4200, 6075, 6075, 8, 8, 56, 56, 112, 112, 160, 160, 448, 400, 400, 448, 448, 560, 560, 1344, 840, 840, 1008, 1008, 2016, 1296, 1296, 1400, 1400, 1400, 1400, 2400, 2400, 2800, 2800, 5600, 3240, 3240, 3360, 3360, 7168, 4096, 4096, 4200, 4200, 4536, 4536, 5600, 5600
2	1, 8, 27, 42, 48, 112, 126, 160, 202, 246, 288, 378, 651, 792, 1056, 1184, 1863, 2016, 2688, 4096, 4480, 7168
3	1, 1, 8, 8, 28, 28, 35, 35, 48, 48, 56, 56, 70, 104, 104, 147, 147, 322, 322, 384, 384, 448, 497, 497, 518, 518, 567, 848, 848, 972, 972, 1008, 1008, 1036, 1134, 1225, 1225, 1296, 1575, 1575, 1896, 1896, 2268, 3240, 4536, 5670, 6075, 6075
4	1, 1, 8, 8, 16, 28, 28, 34, 34, 56, 66, 70, 77, 77, 84, 84, 96, 160, 160, 168, 168, 176, 176, 280, 280, 300, 300, 336, 448, 448, 448, 512, 512, 560, 560, 616, 616, 774, 784, 832, 832, 946, 946, 998, 1008, 1302, 1400, 1400, 1400, 1400, 1400, 1652, 1652, 1654, 1680, 1848, 2360, 2370, 2400, 2400, 4096, 4096, 4096, 4096, 4200, 4536, 5600, 5600, 7168
5	1, 1, 28, 28, 35, 50, 50, 70, 83, 83, 160, 166, 175, 175, 210, 300, 300, 350, 350, 400, 400, 420, 525, 525, 539, 539, 560, 680, 700, 700, 700, 700, 722, 722, 805, 1050, 1050, 1078, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 1729, 1729, 2030, 2100, 2100, 2100, 2400, 2400, 2680, 2800, 2800, 3150, 3200, 3200, 4200, 4200, 4200, 4200, 4200, 4480, 5600, 5600, 5600, 5670, 6075, 6075
6	1, 1, 8, 8, 28, 28, 35, 35, 40, 40, 41, 41, 56, 56, 70, 85, 85, 86, 112, 112, 160, 160, 210, 210, 225, 225, 259, 259, 266, 266, 279, 279, 288, 288, 448, 489, 489, 567, 567, 567, 660, 660, 768, 972, 1008, 1008, 1036, 1072, 1072, 1296, 1296, 1400, 1863, 1906, 2016, 2128, 2128, 2268, 2268, 2673, 2688, 2800, 2800, 3150, 4096, 4096, 4096, 4096, 5600, 6075, 6075
7	1, 8, 28, 28, 35, 35, 50, 56, 56, 70, 84, 84, 112, 112, 152, 168, 175, 175, 210, 210, 299, 350, 350, 400, 420, 448, 448, 448, 525, 525, 560, 560, 567, 567, 700, 700, 700, 700, 840, 840, 840, 840, 922, 1008, 1008, 1050, 1050, 1134, 1144, 1256, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 2016, 2100, 2100, 2100, 2240, 2240, 2268, 2268, 2278, 2688, 2800, 2800, 2835, 2835, 2840, 3150, 3360, 3360, 3797, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5600, 5600, 5670, 7168

Table A.4.: Degrees of irreducible representations of $LH(E_8)$(continued)

e	Degrees of irreducible representations
8	1, 1, 8, 28, 34, 34, 50, 50, 56, 56, 70, 84, 112, 152, 160, 160, 168, 174, 174, 210, 210, 300, 316, 350, 350, 373, 373, 384, 420, 448, 448, 448, 560, 560, 588, 832, 840, 840, 840, 840, 992, 992, 1008, 1008, 1042, 1042, 1050, 1050, 1134, 1268, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1668, 1680, 1896, 1968, 2100, 2232, 2304, 2400, 2400, 2688, 3150, 3200, 3200, 3360, 3360, 3516, 4096, 4096, 4096, 4096, 4200, 4200, 4200, 4480, 4536, 5600, 5600, 5670
9	1, 8, 8, 28, 35, 35, 49, 56, 56, 70, 84, 84, 112, 160, 168, 175, 175, 210, 210, 300, 300, 350, 350, 392, 392, 420, 448, 448, 448, 448, 525, 525, 567, 567, 651, 665, 665, 840, 840, 840, 840, 848, 972, 972, 1008, 1008, 1050, 1050, 1134, 1296, 1296, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1547, 1575, 1575, 1680, 1952, 2100, 2100, 2100, 2268, 2268, 2400, 2400, 2549, 2688, 2835, 2835, 3240, 3240, 3360, 3360, 3648, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5670, 6075, 6075, 7168
10	1, 1, 8, 8, 28, 28, 35, 50, 56, 56, 70, 75, 75, 112, 112, 160, 160, 168, 175, 175, 210, 210, 300, 300, 350, 350, 372, 372, 400, 400, 420, 448, 449, 449, 502, 525, 525, 525, 531, 531, 700, 700, 700, 700, 786, 786, 840, 840, 840, 840, 897, 897, 1050, 1050, 1134, 1296, 1296, 1344, 1344, 1344, 1350, 1400, 1400, 1485, 1575, 1575, 1680, 2016, 2100, 2100, 2100, 2240, 2240, 2400, 2400, 2406, 2688, 2715, 2800, 2800, 3200, 3200, 3360, 3360, 4096, 4096, 4096, 4096, 4200, 4200, 4536, 5600, 5600, 5600, 5670, 6075, 6075, 7168
12	1, 1, 8, 28, 28, 35, 35, 50, 50, 56, 56, 70, 76, 76, 84, 84, 99, 99, 132, 132, 168, 168, 168, 175, 175, 349, 349, 350, 350, 420, 448, 448, 449, 449, 552, 567, 567, 616, 616, 651, 651, 672, 672, 700, 700, 792, 840, 840, 972, 972, 974, 974, 1008, 1008, 1134, 1202, 1296, 1296, 1386, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 1792, 2016, 2100, 2100, 2268, 2268, 2400, 2400, 2408, 2835, 2835, 3150, 3240, 3240, 3584, 3584, 4096, 4096, 4096, 4096, 4200, 4480, 4536, 4536, 5600, 5600, 5600, 5670, 6075, 6075
14	1, 8, 28, 28, 35, 35, 50, 50, 56, 56, 70, 84, 84, 112, 112, 160, 160, 168, 175, 175, 202, 300, 300, 350, 350, 400, 400, 420, 448, 448, 448, 525, 525, 560, 560, 567, 567, 699, 700, 700, 840, 840, 840, 840, 972, 972, 1008, 1008, 1050, 1050, 1134, 1296, 1296, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 2016, 2066, 2100, 2100, 2100, 2240, 2240, 2400, 2400, 2541, 2688, 2800, 2800, 2835, 2835, 3150, 3200, 3200, 3360, 3360, 3534, 4096, 4096, 4096, 4096, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5670, 7168

Table A.5.: Degrees of irreducible representations of $LH(E_8)$ (continued)

e	Degrees of irreducible representations
15	1, 1, 8, 8, 28, 28, 35, 35, 50, 50, 56, 56, 70, 83, 83, 104, 104, 160, 160, 168, 175, 175, 210, 210, 300, 300, 350, 350, 400, 400, 420, 448, 448, 448, 525, 525, 560, 560, 567, 567, 700, 700, 700, 700, 840, 840, 840, 840, 972, 972, 1008, 1008, 1050, 1050, 1134, 1261, 1261, 1296, 1296, 1296, 1296, 1344, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 2016, 2100, 2100, 2100, 2240, 2240, 2268, 2268, 2400, 2400, 2688, 2800, 2800, 2800, 2800, 2835, 2835, 2835, 2835, 3150, 3200, 3200, 3240, 3240, 3360, 3360, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5600, 6075, 6075, 7168
18	1, 8, 8, 27, 27, 28, 28, 50, 50, 56, 56, 70, 84, 112, 112, 160, 160, 168, 175, 175, 209, 273, 273, 350, 350, 400, 400, 420, 448, 448, 448, 476, 525, 525, 567, 567, 567, 567, 700, 700, 700, 700, 799, 840, 840, 972, 972, 1050, 1050, 1099, 1296, 1296, 1301, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1680, 2016, 2100, 2240, 2240, 2268, 2268, 2688, 2800, 2800, 2835, 2835, 3150, 3200, 3200, 3240, 3240, 3360, 3360, 4096, 4096, 4096, 4096, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5600, 5600, 5670, 6075, 6075, 7168
20	1, 1, 8, 8, 28, 28, 35, 35, 50, 50, 56, 56, 70, 84, 84, 111, 111, 160, 160, 168, 175, 175, 210, 210, 300, 300, 350, 350, 400, 400, 420, 448, 448, 448, 456, 456, 525, 525, 560, 560, 700, 700, 700, 700, 840, 840, 840, 840, 840, 840, 972, 972, 1008, 1008, 1050, 1050, 1134, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 2016, 2100, 2100, 2100, 2240, 2240, 2268, 2268, 2400, 2400, 2688, 2800, 2800, 2835, 2835, 3150, 3200, 3200, 3240, 3240, 3360, 3360, 4096, 4096, 4096, 4096, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5600, 5600, 5670, 6075, 6075, 7168

Table A.6.: Degrees of irreducible representations of $LH(E_8)$ (continued)

e	Degrees of irreducible representations
24	1, 1, 8, 8, 28, 28, 34, 34, 50, 50, 56, 56, 70, 84, 84, 112, 112, 126, 126, 168, 175, 175, 210, 210, 224, 224, 300, 300, 400, 400, 420, 448, 448, 525, 525, 560, 560, 567, 567, 700, 700, 700, 700, 840, 840, 840, 840, 972, 972, 1008, 1008, 1050, 1050, 1134, 1296, 1296, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 2016, 2100, 2100, 2100, 2240, 2240, 2268, 2268, 2400, 2400, 2688, 2800, 2800, 2835, 2835, 3150, 3200, 3200, 3240, 3240, 3360, 3360, 4096, 4096, 4096, 4096, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5600, 5600, 5670, 6075, 6075, 7168
30	1, 1, 7, 7, 21, 21, 35, 35, 35, 35, 50, 50, 84, 84, 112, 112, 160, 160, 168, 175, 175, 210, 210, 300, 300, 350, 350, 400, 400, 420, 448, 448, 448, 525, 525, 560, 560, 567, 567, 700, 700, 700, 700, 840, 840, 840, 840, 972, 972, 1008, 1008, 1050, 1050, 1134, 1296, 1296, 1344, 1344, 1344, 1400, 1400, 1400, 1400, 1400, 1400, 1400, 1575, 1575, 1680, 2016, 2100, 2100, 2100, 2240, 2240, 2268, 2268, 2400, 2400, 2688, 2800, 2800, 2835, 2835, 3150, 3200, 3200, 3240, 3240, 3360, 3360, 4096, 4096, 4096, 4096, 4200, 4200, 4200, 4200, 4200, 4480, 4536, 4536, 4536, 5600, 5600, 5600, 5670, 6075, 6075, 7168

B. Some Irreducible Representations of $LH(E_6)$, $LH(E_7)$ and $LH(E_8)$ and Their Characters on Certain Elements

Assume the setting of Section 3.2. In Lemma 3.12, we denote by L a field of characteristic 0 and LH the Iwahori-Hecke algebra $H_L(W, S, (q^2))$ for some invertible element q of L. We know by Corollary 3.7 that only a few values for q are of interest, namely those where q^2 is some primitive root of unity of order e, where e is specified in said Corollary.
The tables in this chapter contain the following information: Whenever there exists an irreducible $LH(E_n)$ module M and an irreducible $LH(E_{n-1})$-module M_J such that $\dim(M_J)\ [E_n : E_{n-1}] = \dim(M)$, both modules are given in the tables for E_n and E_{n-1} respectively, including their dimensions. More precisely, we give the corresponding classes in the Grothendieck groups as integer linear combinations $[M] = \sum_V a_V\ d_\theta([V])$ and $[M_J] = \sum_N b_N\ d_{\theta,J}([N])$, where V and N run over irreducible KH and KH_J-modules up to isomorphism, respectively. We use a shorthand for this and write $M = \sum_V a_V\ \theta(V)$ and $M_J = \sum_N b_N\ \theta(N)$. These linear combinations can be deduced from the decomposition maps. For more details on the decomposition map refer to the introduction of Appendix A.
The irreducible modules up to isomorphism of both KH and KH_J are in bijection with those of the group algebras $\mathbb{C}\,E_n$ and $\mathbb{C}\,E_{n-1}$, which can be referred to by their *Frame name*, see [Lus84]. By identifying the irreducible KH-modules with the irreducible $\mathbb{C}\,E_n$-modules, they, too, will be referred to by these names.

Besides the integer linear combinations we give some information on character values: Let $w_7 := s_7s_5s_6s_2 \in E_7$ and $w_8 := s_2s_3s_4s_2s_3s_4s_5s_4s_2s_3s_4s_5 \in E_8$ where s_i corresponds to the i'th vertex in the corresponding Coxeter graph in Table 3.1. (These are shortest representatives of conjugacy classes as they are found in CHEVIE. In this program, w_7 is in the tenth conjugacy class of E_7 and w_8 in the third conjugacy class of E_8. These elements are available using the `.classtext` command.)

These group elements have corresponding standard basis elements $h_7 := T_{w_7} \in H(E_7)$ and $h_8 := T_{w_8} \in H(E_8)$.
We want to compare the values $\theta\left(\sum_V a_V\ \dot{\chi}_V(h_n)\right)$ and $\theta\left(\sum_N b_N\ \dot{\chi}_{\mathrm{Ind}_{J(N)}^S}(h_n)\right)$ where θ sends v to q. As it turns out, the characters we are considering all take values in $\mathbb{Z}[v^2]$ on the elements h_n. Restricted to this ring, θ is just a ring homomorphism sending the indeterminate v^2 to a primitive e'th root of unity, namely q^2. Hence, we compute $\sum_V a_V\ \dot{\chi}_V(h_n)$ and $\sum_N b_N\ \dot{\chi}_{\mathrm{Ind}_{J(N)}^S}(h_n)$ modulo $\Phi_e(v^2)$, where Φ_e is the e'th cyclotomic polynomial to compute the values $\theta\left(\sum_V a_V\ \dot{\chi}_V(h_n)\right)$ and $\theta\left(\sum_N b_N\ \dot{\chi}_{\mathrm{Ind}_{J(N)}^S}(h_n)\right)$.

Table B.1.: Irreducible representations M_J of $LH_J(E_6)$ whose degree is that of an irreducible $LH(E_7)$-representation divided by $[E_7 : E_6] = 56$

e	$[M_J] = \sum_N b_N \ d_{\theta,J}([N])$	$\dim(q)$	$\theta\left(\sum_N b_N \ \dot{\chi}_{\mathrm{Ind}_J^S(N)}(h_7)\right)$
2	$\theta(1'_p)$	1	32
4	$\theta(1_p)$	1	$16q^2 + 14$
	$\theta(1'_p)$	1	$-16q^2 + 14$
5	$\theta(1_p)$	1	$-31q^6 - 14q^4 - 15q^2 - 15$
	$\theta(1'_p)$	1	$q^4 - 16q^2 + 15$
6	$\theta(1_p)$	1	$-14q^2 + 15$
	$\theta(1'_p)$	1	$-15q^2 + 14$
7	$\theta(1_p)$	1	$15q^8 - 16q^6 + q^4$
	$\theta(1'_p)$	1	$q^4 - 16q^2 + 15$
	$\theta(6_p)$	6	$46q^8 - 74q^6 + 30q^4 - 2q^2$
	$\theta(6'_p)$	6	$-2q^6 + 30q^4 - 74q^2 + 46$
8	$\theta(6_p)$	6	$-74q^6 + 30q^4 - 2q^2 - 46$
	$\theta(6'_p)$	6	$-2q^6 + 30q^4 - 74q^2 + 46$
9	$\theta(6_p)$	6	$46q^8 - 74q^6 + 30q^4 - 2q^2$
	$\theta(6'_p)$	6	$-2q^6 + 30q^4 - 74q^2 + 46$
10	$\theta(1_p)$	1	$-q^6 - 14q^4 + 15q^2 - 15$
	$\theta(1'_p)$	1	$q^4 - 16q^2 + 15$
	$\theta(6_p)$	6	$-28q^6 - 16q^4 + 44q^2 - 46$
	$\theta(6'_p)$	6	$-2q^6 + 30q^4 - 74q^2 + 46$
12	$\theta(6_p) - \theta(1_p)$	5	$-58q^6 + 60q^4 - 2q^2 - 31$
	$\theta(6'_p) - \theta(1'_p)$	5	$-2q^6 + 29q^4 - 58q^2 + 31$
14	$\theta(1_p)$	1	$15q^8 - 16q^6 + q^4$
	$\theta(1'_p)$	1	$q^4 - 16q^2 + 15$
	$\theta(6_p)$	6	$46q^8 - 74q^6 + 30q^4 - 2q^2$
	$\theta(6'_p)$	6	$-2q^6 + 30q^4 - 74q^2 + 46$
18	$\theta(1_p)$	1	$15q^8 - 16q^6 + q^4$
	$\theta(1'_p)$	1	$q^4 - 16q^2 + 15$
	$\theta(6_p)$	6	$46q^8 - 74q^6 + 30q^4 - 2q^2$
	$\theta(6'_p)$	6	$-2q^6 + 30q^4 - 74q^2 + 46$

Table B.2.: Irreducible representations M of $LH(E_7)$ whose degree is divisible by $[E_7 : E_6] = 56$ for which there exists an irreducible representation of $LH_J(E_6)$ whose degree is exactly that quotient

e	$[M] = \sum_V a_V \ d_\theta([V])$	$\dim(V)$	$\theta\left(\sum_V a_V \ \dot{\chi}_V(h_7)\right)$
2	$\theta(56'_a)$	56	26
	$\theta(189'_b) - \theta(105'_a) - \theta(35_b) + \theta(7'_a)$	56	14
4	$\theta(56_a)$	56	$11q^2 + 5$
	$\theta(56'_a)$	56	$-11q^2 + 5$
5	$\theta(56_a)$	56	$5q^4 - 11q^2 + 10$
	$\theta(56'_a)$	56	$-21q^6 - 5q^4 - 10q^2 - 10$
6	$\theta(56_a)$	56	$-5q^2 + 6$
	$\theta(56'_a)$	56	$-6q^2 + 5$
7	$\theta(56_a)$	56	$4q^8 - 22q^6 + 48q^4 - 36q^2 + 6$
	$\theta(56'_a)$	56	$6q^8 - 36q^6 + 48q^4 - 22q^2 + 4$
	$\theta(336_a)$	336	$5q^4 - 11q^2 + 10$
	$\theta(336'_a)$	336	$10q^8 - 11q^6 + 5q^4$
8	$\theta(336_a)$	336	$-22q^6 + 48q^4 - 36q^2 + 2$
	$\theta(336'_a)$	336	$-36q^6 + 48q^4 - 22q^2 - 2$
9	$\theta(336_a)$	336	$4q^8 - 22q^6 + 48q^4 - 36q^2 + 6$
	$\theta(336'_a)$	336	$6q^8 - 36q^6 + 48q^4 - 22q^2 + 4$
10	$\theta(56_a)$	56	$-18q^6 + 44q^4 - 32q^2 + 2$
	$\theta(56'_a)$	56	$-30q^6 + 42q^4 - 16q^2 - 2$
	$\theta(336_a)$	336	$5q^4 - 11q^2 + 10$
	$\theta(336'_a)$	336	$-q^6 - 5q^4 + 10q^2 - 10$
12	$\theta(280_a)$	280	$-14q^6 + 38q^4 - 36q^2 + 12$
	$\theta(280'_a)$	280	$-36q^6 + 50q^4 - 14q^2 - 12$
14	$\theta(56_a)$	56	$4q^8 - 22q^6 + 48q^4 - 36q^2 + 6$
	$\theta(56'_a)$	56	$6q^8 - 36q^6 + 48q^4 - 22q^2 + 4$
	$\theta(336_a)$	336	$5q^4 - 11q^2 + 10$
	$\theta(336'_a)$	336	$10q^8 - 11q^6 + 5q^4$
18	$\theta(56_a)$	56	$4q^8 - 22q^6 + 48q^4 - 36q^2 + 6$
	$\theta(56'_a)$	56	$6q^8 - 36q^6 + 48q^4 - 22q^2 + 4$
	$\theta(336_a)$	336	$5q^4 - 11q^2 + 10$
	$\theta(336'_a)$	336	$10q^8 - 11q^6 + 5q^4$

Table B.3.: Irreducible representations M_J of $LH_(E_7)$ whose degree is that of an irreducible $LH(E_8)$-representation divided by $[E_8 : E_7] = 240$

e	$[M_J] = \sum_N b_N \ d_{\theta,J}([N])$	$\dim(V)$	$\theta\left(\sum_N b_N \ \dot{\chi}_{\mathrm{Ind}_{J(N)}^S}(h_8)\right)$
4	$\theta(7_a)$	7	$604q^2 + 736$
4	$\theta(7'_a)$	7	$-604q^2 + 736$
5	$\theta(7_a)$	7	$-1072q^6 - 477q^4 - 555q^2 - 305$
5	$\theta(7'_a)$	7	$1072q^6 + 767q^4 + 517q^2 + 595$
7	$\theta(7_a)$	7	$78q^{10} + 555q^8 - 526q^6 + 78q^4 + 87q^2 + 250$
7	$\theta(7'_a)$	7	$250q^{10} + 87q^8 + 78q^6 - 526q^4 + 555q^2 + 78$
8	$\theta(7_a)$	7	$-604q^6 + 78q^4 - 296$
8	$\theta(7'_a)$	7	$78q^4 - 604q^2 + 296$
9	$\theta(7_a)$	7	$-9q^{10} + 477q^8 - 526q^6 - 9q^4 + 250$
9	$\theta(7'_a)$	7	$-477q^{10} + 9q^8 + 250q^6 - 477q^4 - 526$
10	$\theta(7_a)$	7	$-136q^6 - 477q^4 + 555q^2 - 305$
10	$\theta(7'_a)$	7	$136q^6 - 441q^4 + 691q^2 - 613$
12	$\theta(7_a)$	7	$-604q^6 + 468q^4 - 227$
12	$\theta(7'_a)$	7	$604q^6 - 468q^4 + 241$
14	$\theta(7_a)$	7	$-78q^{10} + 555q^8 - 682q^6 + 78q^4 - 87q^2 + 250$
14	$\theta(7'_a)$	7	$-250q^{10} + 87q^8 - 78q^6 + 682q^4 - 555q^2 + 78$
15	$\theta(7_a)$	7	$9q^{14} - 78q^{12} - 9q^{10} + 486q^8 - 613q^6 + 9q^2 + 163$
15	$\theta(7'_a)$	7	$-299q^{14} + 526q^{12} - 477q^{10} + 486q^8 + 127q^6 - 776q^4 + 477q^2 + 127$
24	$\theta(7_a)$	7	$-78q^{12} + 486q^8 - 604q^6 + 163$
24	$\theta(7'_a)$	7	$-78q^{12} - 604q^{10} + 486q^8 + 604q^2 - 649$
30	$\theta(7_a)$	7	$-9q^{14} - 78q^{12} + 9q^{10} + 486q^8 - 595q^6 - 9q^2 + 163$
30	$\theta(7'_a)$	7	$-909q^{14} - 682q^{12} + 477q^{10} + 486q^8 + 1081q^6 + 432q^4 - 477q^2 - 1081$

Table B.4.: Irreducible representations M of $LH(E_8)$ whose degree is divisible by $[E_8 : E_7] = 240$ for which there exists an irreducible representation of $LH_J(E_7)$ whose degree is exactly that quotient

e	$[M] = \sum_V a_V\ d_\theta([V])$	$\dim(V)$	$\theta\left(\sum_N b_N\ \dot{\chi}_{\mathrm{Ind}_J^S(N)}(h_8)\right)$
4	$\theta(1680_y)$	1680	432
5	$\theta(1680_y)$	1680	$4q^4 - 22q^2 + 4$
7	$\theta(1680_y)$	1680	$26q^{10} + 214q^8 - 176q^6 - 176q^4 + 214q^2 + 26$
8	$\theta(1680_y)$	1680	$-192q^6 + 16q^4 - 192q^2$
9	$\theta(1680_y)$	1680	$-198q^{10} + 198q^8 - 166q^6 - 198q^4 - 166$
10	$\theta(1680_y)$	1680	$-380q^4 + 406q^2 - 380$
12	$\theta(1680_y)$	1680	-162
14	$\theta(1680_y)$	1680	$-26q^{10} + 214q^8 - 208q^6 + 208q^4 - 214q^2 + 26$
15	$\theta(1680_y)$	1680	$-4q^{14} + 176q^{12} - 198q^{10} + 396q^8 - 198q^6 - 202q^4 + 198q^2 + 4$
24	$\theta(1680_y)$	1680	$-16q^{12} - 192q^{10} + 396q^8 - 192q^6 + 192q^2 - 198$
30	$\theta(1680_y)$	1680	$-380q^{14} - 208q^{12} + 198q^{10} + 396q^8 + 198q^6 + 182q^4 - 198q^2 - 380$

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Eigenständigkeitserklärung

Hiermit versichere ich, die vorliegende Arbeit selbständig verfasst zu haben, und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.