Experimental and constructive algebra

Computing Parabolic Induction Maps of Iwahori-Hecke Algebras

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Coxeter Groups and Parabolic Subgroups

A finite **Coxeter group** is a finite group W presented by a generating set $S \subseteq W$ and **braid relations** and **quadratic relations**

$$\underbrace{stst\ldots}_{m_{st}} = \underbrace{tsts\ldots}_{m_{st}}, \text{ (b.rel.)} \quad \text{and} \quad s^2 = 1, \text{ (q.rel.)}$$

for some integers $m_{st} \geq 2$. We call (W, S) a Coxeter system.
If $J \subseteq S$, then $W_J := \langle J \rangle \leq W$, too, is a Coxeter group called a **parabolic subgroup of** W and (W_J, J) is a Coxeter system.

Iwahori-Hecke Algebras and Parabolic Subalgebras

Let (W, S) be a Coxeter system, F a field and $q \in F^*$. Then the **Iwahori-Hecke algebra** $H_F(W, q)$ is the associative unital algebra presented by a generating set $\{T_s \mid s \in S\}$ and **braid relations** and **quadratic relations**

$$\underbrace{T_sT_tT_sT_t\ldots}_{m_{st}} = \underbrace{T_tT_sT_tT_s\ldots}_{m_{st}} \text{(b.rel.)} \quad \text{and} \quad T_s^2 = q \cdot 1 + (q-1)T_s, \text{ (q.rel.)}.$$

$\rightsquigarrow$ Deformation of the group algebra
If $J \subseteq S$ then the **parabolic subalgebra** $H_F(W_J, q)$ embeds naturally into $H_F(W, S, q)$.

Parabolic Induction

Let $FH := H_F(W, q)$ and $FH_J := H_F(W_J, q) \lesssim FH$. Then there is a **parabolic induction** functor

$$F\text{-}\operatorname{Ind} : FH_J\text{-mod} \to FH\text{-mod}; \; M \mapsto M \otimes_{FH_J} FH$$

between the categories of finitely generated (right) modules.
It is exact and hence defines a homomorphism

$$F\text{-}\widehat{\operatorname{Ind}} : K_0(FH_J) \to K_0(FH)$$

between the corresponding Grothendieck groups.

Objective

Describe the structure of $F\text{-}\operatorname{Ind}(M)$ for all FH-modules M if F is a splitting field for both FH and FH_J.

Sub-objectives

Compute $F\text{-}\widehat{\operatorname{Ind}}(M)$.

Theorem [S.]: Simplicity of Induced Modules

If $M \neq 0$ and $J \neq S$, then $F\text{-}\operatorname{Ind}(M)$ is *not* simple. In particular, it has at least two simple constituents, counting multiplicities.

A Commuting Diagram from Specialisation

There exists a field k of characteristic zero and some $x \in k^*$ s.t. $kH := H_k(W, x)$ and $kH_J := H_k(W_J, x)$ are both split semisimple. Then we have well-defined **decomposition maps**

$$d^S : K_0(kH) \to K_0(FH) \quad \text{and} \quad d^J : K_0(kH_J) \to K_0(FH_J).$$

If we denote by $k\text{-}\widehat{\operatorname{Ind}}$ and $F\text{-}\widehat{\operatorname{Ind}}$ the induction maps for kH_J and FH_J respectively, the following is a commuting diagram of homomorphisms:

$$\begin{array}{ccc} K_0(kH_J) & \xrightarrow{d^J} & K_0(FH_J) \\ \downarrow{\scriptstyle k\text{-}\widehat{\operatorname{Ind}}} & & \downarrow{\scriptstyle F\text{-}\widehat{\operatorname{Ind}}} \\ K_0(kH) & \xrightarrow{d^S} & K_0(FH) \end{array}$$

Computing $F\text{-}\widehat{\operatorname{Ind}}$

If d^J is surjective, let c^J be a right inverse. Then

$$F\text{-}\widehat{\operatorname{Ind}} = d^S \circ k\text{-}\widehat{\operatorname{Ind}} \circ c^J.$$

How and when can this be applied?

- **Surjectivity:** d^J is known to be surjective in most cases.
- **Computing** $k\text{-}\widehat{\operatorname{Ind}}$**:** This is easy using ordinary representation theory.
- **Computing** d^S **and** d^J**:** This is hard, but it has been solved for many parameter choices. For W of exceptional type d^S is known unless $W \cong E_8$ and $\operatorname{char}(F) \leq 5$. However, there are large families of cases where d^S or d^J are not yet known (e.g. W a symmetric group and $\operatorname{char}(F) > 0$).

Example & Remarks

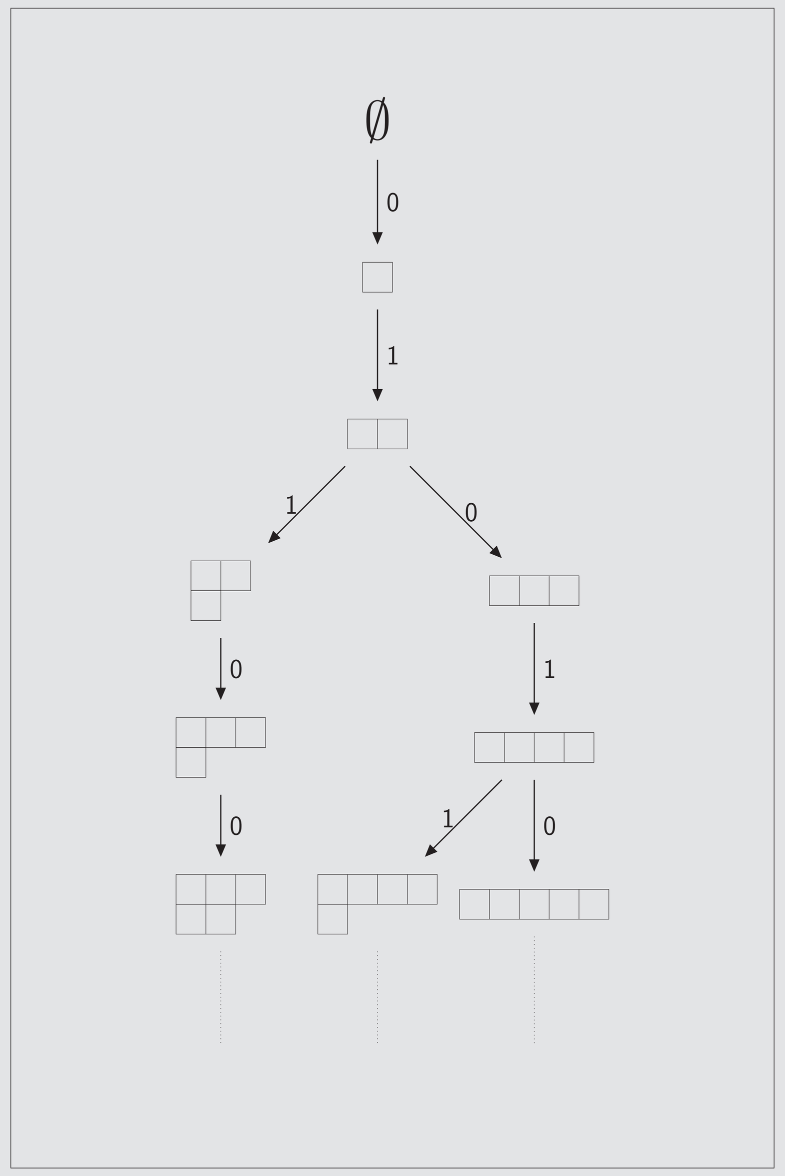
- The dependence on d^J and d^S means that this approach is most useful for the computation of concrete examples but not as helpful in obtaining general results.
- We have used this to compute $F\text{-}\widehat{\operatorname{Ind}}$ for all exceptional Weyl groups W, for arbitrary F and q, the only exception being $W \cong E_8$ for $\operatorname{char}(F) \leq 5$.

An approach for type A_{n-1}

The symmetric group $\mathfrak{S}_n$ is a Coxeter group and $\mathfrak{S}_{n-1}$ is a parabolic subgroup, so let $W \cong \mathfrak{S}_n$ and $W_J \cong \mathfrak{S}_{n-1}$. If $\operatorname{char}(F) = 0$, then $F\text{-}\widehat{\operatorname{Ind}} = k\text{-}\widehat{\operatorname{Ind}}$ unless q is an eth root of unity for some integer $e \geq 2$.

Simple Modules: The simple modules of both FH and FH_J are indexed by certain partitions of n and $n-1$ respectively.
Crystal Graph: The crystal graph is a graph with directed edges labeled by the elements of $\mathbb{Z}/e\mathbb{Z}$ whose vertices are certain partitions. It is defined completely combinatorially.
Ariki: Let D^λ be a simple FH_J-module indexed by the partition λ. Then the head and socle of $F\text{-}\widehat{\operatorname{Ind}}(D^\lambda)$ are multiplicity free and can be read off the crystal graph.
Grojnowski: A lower bound for the multiplicity of D^μ in $F\text{-}\widehat{\operatorname{Ind}}(D^\lambda)$ is given by the length of a certain i-path in the crystal graph.

Example: Part of the crystal graph for $e = 2$



The General Case: Ariki-Koike Algebras

The above theory generalises to Ariki-Koike algebras for complex reflection groups of type $G(\ell, 1, n)$ for $\ell \geq 1$. In particular, it applies to Iwahori-Hecke algebras of type B_n as this is type $G(2, 1, n)$. For an Ariki-Koike algebra the crystal graph is defined on ℓ-multipartitions.
Theorem [S.]: If $\mathcal{H}_n$ is a cyclotomic Ariki-Koike algebra over $\mathbb{C}$ for the complex reflection group $G(\ell, 1, n)$, then the induction of a non-zero $\mathcal{H}_{n-1}$-module has at least $\ell + 1$ simple constituents.

Ongoing & Future Work

- For type A_n and B_n consider subgroups that are not A_{n-1} or B_{n-1}.
- Apply the theory of e-weights, abaci etc. to further study the case A_n.
- Use Clifford theory to carry the results on type B_n over to type D_n.
- Use the KZ functor to exploit results on Cherednik algebras.