IMPRIMITIVE IRREDUCIBLE MODULES FOR FINITE QUASISIMPLE GROUPS, II

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Abstract. This work completes the classification of the imprimitive irreducible modules, over algebraically closed fields of characteristic 0, of the finite quasisimple groups.

1. Introduction

This is a continuation of our work in [15], where we began the classification of the irreducible imprimitive modules of the finite quasisimple groups over algebraically closed fields. We completed this program for the quasisimple covering groups of the sporadic simple groups and for the Tits simple group, as well as for the covering groups of the finite groups of Lie type with an exceptional Schur multiplier or with more than one defining characteristic. For the covering groups of the alternating groups, the desired classification is complete over fields of characteristic 0 by the work of Djoković and Malzan [8], respectively Nett and Noeske [23]. In this paper we finish the classification for all quasisimple groups G and fields of characteristic 0.

Let G be a finite group. Write $\mathrm{Irr}(G)$ for the set of irreducible characters of G over some algebraically closed field of characteristic 0. An element $\chi \in \mathrm{Irr}(G)$ is *imprimitive*, if it is induced from a proper subgroup of G. Now let G be a finite group with a split BN-pair, for example a finite reductive group. We say that $\chi \in \mathrm{Irr}(G)$ is *Harish-Chandra imprimitive*, if there is a proper Levi subgroup $L \leq G$ (in the sense of groups with a split BN-pair) and $\vartheta \in \mathrm{Irr}(L)$ such that $\chi = R_L^G(\vartheta)$. Here, R_L^G denotes Harish-Chandra induction from L to G. Next, let G be a quasisimple finite reductive group. Suppose that $G/Z(G)$ does not have an exceptional Schur multiplier and that G has a unique defining characteristic. Then [15, Theorem 6.1] states that $\chi \in \mathrm{Irr}(G)$ is imprimitive if and only if it is Harish-Chandra imprimitive. This leads

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to the program to classify the Harish-Chandra imprimitive elements of $\mathrm{Irr}(G)$ for finite reductive groups G. In [15, Theorem 7.3, Theorem 8.4], such a classification has been achieved for finite reductive groups G arising from algebraic groups with a connected center (but G not necessarily quasisimple). Also, by [15, Propositions 10.2–10.4], the classification is complete for the Suzuki and Ree groups, so that we will not consider these groups in the following.

To formulate our main result, let us introduce some more notation. Let $\mathbf{G}$ denote a connected reductive algebraic group defined over the algebraic closure of a finite field, and let F denote a Frobenius morphism of $\mathbf{G}$. Put $G := \mathbf{G}^F$, the finite group of F-fixed points on $\mathbf{G}$. We also let $(\mathbf{G}^*, F)$ denote a pair of a reductive group and a Frobenius morphism in duality to $(\mathbf{G}, F)$. Let $s \in G^*$ be semisimple and write $\mathcal{E}(G, [s]) \subseteq \mathrm{Irr}(G)$ for the Lusztig series defined by the G^*-conjugacy class $[s]$ of s. By Lusztig's generalized Jordan decomposition of characters, there is an equivalence relation on $\mathcal{E}(G, [s])$ and a bijection between the equivalence classes on $\mathcal{E}(G, [s])$ and the $C_{\mathbf{G}^*}(s)^F$-orbits on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$. (This bijection is, in general, not unique, but our results are true for any such bijection.) We write $[\chi]$ for the equivalence class of $\chi \in \mathcal{E}(G, [s])$, and $[\lambda]$ for the $C_{\mathbf{G}^*}(s)^F$-orbit of $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$. For such a λ we let $C_{\mathbf{G}^*}(s)^F_\lambda$ denote its stabilizer in $C_{\mathbf{G}^*}(s)^F$. We can now formulate one of the main results of our paper.

Theorem 1.1. *Let $\chi \in \mathrm{Irr}(G)$. Suppose that $\chi \in \mathcal{E}(G, [s])$ for a semisimple element $s \in G^*$, and let $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ such that $[\chi]$ corresponds to $[\lambda]$ under Lusztig's generalized Jordan decomposition of characters. Then the following assertions hold.*

(a) *If*

$$C_{\mathbf{G}^*}(s)^F_\lambda \; C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*, \tag{1}$$

for some split F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$, then χ is Harish-Chandra induced from $L = \mathbf{L}^F$, where $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$ (hence split).

(b) *Suppose that $\mathbf{G}$ is simple and simply connected. If χ is Harish-Chandra imprimitive, there is a proper split F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ such that Condition* (1) *is satisfied.*

If, in the notation above, $C_{\mathbf{G}^*}(s)$ is connected, then $C_{\mathbf{G}^*}(s)^F_\lambda \; C^\circ_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(s)$. Thus Theorem 1.1 generalizes parts of [15, Theorem 7.3, Theorem 8.4].

Let us now comment on the further results and the content of the individual sections of our paper. Section 2 contains preliminaries and introduces some notation. In Section 3 we prove Theorem 1.1(a) completely, as well as Theorem 1.1(b) under some restrictions. To remove these restricitions we have to resort to a case by case analysis, investigating all the possible groups $\mathbf{G}$. This is achieved in Section 5, where we explicitly decide the Harish-Chandra imprimitivity of an element $\chi \in \mathcal{E}(G, [s])$ in terms of the label of the unipotent character $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$, where $[\chi]$ and $[\lambda]$ correspond under Lusztig's generalized Jordan decomposition of characters. For the classical groups $\mathbf{G}$ this requires detailed knowledge on the semisimple elements $s \in G^*$ with $C_{\mathbf{G}^*}(s)$ non-connected, as well as on the action of $C_{\mathbf{G}^*}(s)^F$ on the unipotent characters of $C^\circ_{\mathbf{G}^*}(s)^F$. This information is collected in Section 4 of our paper, and is given in terms of the natural representation of a classical group $\tilde{\mathbf{G}}^*$, respectively $\check{\mathbf{G}}^*$, having $\mathbf{G}^*$ as epimorphic image. More precisely, we choose an element $\tilde{s} \in \tilde{G}^*$ mapping onto s, and give the conditions for the Harish-Chandra imprimitivity of χ in terms of the minimal polynomial of $\tilde{s}$ in the natural representation of $\tilde{G}^*$ (and similarly for $\check{G}^*$). One could as well work entirely within a maximally split torus of $\mathbf{G}^*$ and the action of the Weyl group on this torus, but this again is best achieved with the help of a natural representation of a group surjecting onto $\mathbf{G}^*$. The corresponding information for the exceptional groups is taken from the tables [18] computed by Frank Lübeck. Although not visible any more in the final version of our paper, we owe very much to explicit computations with substantial examples using GAP [12].

2. Notation and preliminaries

2.1. **Some general notation.** The $n \times n$ identity matrix is denoted by I_n, and by J_n we denote the $n \times n$ anti-diagonal matrix. If $n = 2m$, we put

$$\tilde{J}_n := \begin{bmatrix} 0 & J_m \\ -J_m & 0 \end{bmatrix}.$$

In each case we omit the index n if this is clear from the context. The transposed of a matrix a is denoted by a^T.

By a character of a finite group H we mean a complex character and $\mathrm{Irr}(H)$ denotes the set of irreducible characters of H.

2.2. **Finite reductive groups.** Throughout this paper, we let $\mathbb{F}$ denote an algebraic closure of the finite field with p elements, and we let q be a power of p. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{F}$, and let F be a Frobenius morphism of $\mathbf{G}$ arising from an $\mathbb{F}_q$-structure on $\mathbf{G}$. For any F-stable subgroup $\mathbf{H}$ of $\mathbf{G}$ we put $\mathbf{H}^F := \{h \in \mathbf{H} \mid F(h) = h\}$ for the finite group of F-fixed points in $\mathbf{H}$. We call $\mathbf{G}^F$ a *finite reductive group*. We adopt the typographical convention to write $H := \mathbf{H}^F$ for F-stable subgroups $\mathbf{H} \leq \mathbf{G}$ denoted by a single boldface letter. For reasons of clarity, we prefer the notation with exponent F for subgroups such as centralizers or normalizers. For example, we usually write $C_{\mathbf{G}}(s)^F$ rather than $C_G(s)$, if s is an F-stable element of $\mathbf{G}$. The connected component of a closed subgroup $\mathbf{H}$ of $\mathbf{G}$ is denoted by $\mathbf{H}^\circ$. In order to avoid double exponents, we put $Z^\circ(\mathbf{G}) := Z(\mathbf{G})^\circ$ and $C^\circ_{\mathbf{G}}(s) := C_{\mathbf{G}}(s)^\circ$, for $s \in \mathbf{G}$.

An F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is called *split*, if $\mathbf{L}$ is the Levi complement of an F-stable parabolic subgroup of $\mathbf{G}$. This is the case, if and only if $\mathbf{L}$ is the centralizer of a split torus of $\mathbf{G}$. The finite reductive group G is, in particular, a finite group with a split BN-pair of characteristic p. The Levi subgroups L of G, in the sense of a group with a split BN-pair, are of the form $L = \mathbf{L}^F$ for F-stable split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$.

Let L be a Levi subgroup of G and $\vartheta \in \mathrm{Irr}(L)$. Then $R_L^G(\vartheta)$ is the character of G obtained by Harish-Chandra inducing ϑ from L to G.

2.3. **The Weyl group.** Fix an F-stable maximally split torus $\mathbf{T}$ of $\mathbf{G}$ and let $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ denote the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$. For every $w \in W$ we choose an inverse image $\dot{w}$ of w under the canonical epimorphism $N_{\mathbf{G}}(\mathbf{T}) \to W$. For $t \in \mathbf{T}$ and $w \in W$ we write $t^w := \dot{w}^{-1}t\dot{w}$. This is independent of the chosen representative $\dot{w}$, and defines what we call the *conjugation action* of W on $\mathbf{T}$.

2.4. **Semisimple elements and centralizers.** An element $t \in \mathbf{T}$ is conjugate in $\mathbf{G}$ to an F-stable element, if and only if t is conjugate to $F(t)$ by an element of W. Indeed, let $s \in \mathbf{G}^F$ be semisimple. Then s lies in an F-stable maximal torus of $\mathbf{G}$. In particular, there is $w \in W$ such that s is conjugate to an element t of

$$\mathbf{T}^{Fw} := \{t \in \mathbf{T} \mid F(t)^w = t\} = \{t \in \mathbf{T} \mid \dot{w}^{-1}F(t)\dot{w} = t\}.$$

In turn, $C_{\mathbf{G}}(s)^F$ is conjugate in $\mathbf{G}$ to

$$C_{\mathbf{G}}(t)^{F\dot{w}} := \{x \in C_{\mathbf{G}}(t) \mid \dot{w}^{-1}F(x)\dot{w} = x\}.$$

Conversely, an element of $\mathbf{T}^{Fw}$ for some $w \in W$ is conjugate in $\mathbf{G}$ to an F-stable element.

2.5. **The component group.** For $s \in \mathbf{G}$ semisimple, we put

$$A_{\mathbf{G}}(s) := C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s).$$

Clearly, $A_{\mathbf{G}}(s)$ is F-stable, if s is F-stable. In this case we have $A_{\mathbf{G}}(s)^F \cong C_{\mathbf{G}}(s)^F/C^\circ_{\mathbf{G}}(s)^F$ as $C^\circ_{\mathbf{G}}(s)$ is connected. By the Lang-Steinberg theorem, there is a bijection between the F-conjugacy classes in $A_{\mathbf{G}}(s)^F$ and the G-conjugacy classes in the set of F-stable elements in the $\mathbf{G}$-conjugacy class of s (see [7, (3.25)]). If $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$ containing s, then $C^\circ_{\mathbf{G}}(s) \cap \mathbf{L} = C^\circ_{\mathbf{L}}(s)$. In particular, the inclusion $\mathbf{L} \to \mathbf{G}$ induces an injective group homomorphism $A_{\mathbf{L}}(s) \to A_{\mathbf{G}}(s)$ (see [1, 8.B]).

2.6. **Characters and Lusztig series.** Let $\mathbf{G}^*$ be a reductive group dual to $\mathbf{G}$, and denote by F the Frobenius morphism of $\mathbf{G}^*$ in duality with the Frobenius morphism on $\mathbf{G}$. For semisimple $s \in G^*$ we denote by $[s]$ the G^*-conjugacy of s and by $\mathcal{E}(G, [s]) \subseteq \mathrm{Irr}(G)$ the rational Lusztig series of characters defined by $[s]$ (see [4, Definition 8.23]).

2.7. **Direct products.** For later use, we record an easy result for direct products of groups with a BN-pair.

Lemma 2.1. *Let G_0 be a finite group with a split BN-pair. Put $G_i := G_0$, $1 \leq i \leq n$, and*

$$G := G_1 \times G_2 \times \cdots \times G_n.$$

Then the symmetric group S_n on n letters acts on G (by permuting the factors G_i), hence S_n acts on $\mathrm{Irr}(G)$.

Let $\alpha_i \in \mathrm{Irr}(G_i)$, $1 \leq i \leq n$, and put

$$\alpha := \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n.$$

For $1 \leq i \leq n$, let L_i denote a Levi subgoup of G_i, and put

$$L := L_1 \times L_2 \times \cdots \times L_n.$$

Let $\beta_i \in \mathrm{Irr}(L_i)$, $1 \leq i \leq n$, and put

$$\beta := \beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_n.$$

Suppose that $R_L^G(\beta)$ is multiplicity free and that α is an irreducible constituent of $R_L^G(\beta)$. Finally, suppose that there is some $1 \leq j \leq n$ such that $R_{L_j}^{G_j}(\beta_j)$ is not irreducible.

Then there is an irreducible constituent of $R_L^G(\beta)$ which does not lie in the S_n-orbit of α.

Proof. We have

$$R_L^G(\beta) = R_{L_1}^{G_1}(\beta_1) \otimes R_{L_2}^{G_2}(\beta_2) \otimes \cdots \otimes R_{L_n}^{G_n}(\beta_n). \tag{2}$$

As α is an irreducible constituent of $R_L^G(\beta)$, each α_i is an irreducible constituent of $R_{L_i}^{G_i}(\beta_i)$ for $1 \leq i \leq n$. As $R_L^G(\beta)$ is multiplicity free, the same is true for each $R_{L_i}^{G_i}(\beta_i)$, $1 \leq i \leq n$. By renumbering, we may assume that $j = 1$. Let $\varphi \neq \psi$ be two irreducible constituents of $R_{L_1}^{G_1}(\beta_1)$. By Equation (2) and the subsequent remark, $\varphi \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$

and $\psi \otimes \alpha_2 \otimes \cdots \otimes \alpha_n$ are distinct irreducible constituents of $R_L^G(\beta)$. At most one of these lies in the S_n-orbit of α. $\diamond$

3. Decent to commutator subgroups

In this section we generalize the results of [15, Chapters 8, 9] to quasisimple groups. (Recall that in [15, Chapters 8, 9] we have assumed that our groups arise from algebraic groups with connected center.) As in the proof of [15, Theorem 8.4], our approach uses regular embeddings, a standard technique introduced by Deligne and Lusztig in [6, Corollary 5.18].

3.1. **Regular embeddings.** Let $\mathbf{G}$ and F be as in Subsection 2.2. Then there is a connected reductive group $\tilde{\mathbf{G}}$, and a Frobenius morphism of $\tilde{\mathbf{G}}$, also denoted by F, such that the following conditions are satisfied: The center of $\tilde{\mathbf{G}}$ is connected, $\mathbf{G}$ is an F-stable closed subgroup of $\tilde{\mathbf{G}}$ containing the derived subgroup $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}]$ of $\tilde{\mathbf{G}}$, and the restriction of F from $\tilde{\mathbf{G}}$ to $\mathbf{G}$ is the original Frobenius morphism F on $\mathbf{G}$. Let us denote by $i : \mathbf{G} \to \tilde{\mathbf{G}}$ the embedding of $\mathbf{G}$ into $\tilde{\mathbf{G}}$. This induces a surjective morphism $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ of dual groups, compatible with F, with kernel contained in the center of $\tilde{\mathbf{G}}^*$.

Alternatively, one can start with a connected reductive algebraic group $\tilde{\mathbf{G}}$ with connected center, defined over $\mathbb{F}$, and equipped with a Frobenius morphism F with respect to some $\mathbb{F}_q$-structure on $\tilde{\mathbf{G}}$. Then if we set $\mathbf{G} := [\tilde{\mathbf{G}}, \tilde{\mathbf{G}}]$, the pair $\mathbf{G}$, $\tilde{\mathbf{G}}$ satisfies the conditions above.

There is a bijection $\mathbf{L} \mapsto \tilde{\mathbf{L}}$ between the F-stable Levi subgroups of $\mathbf{G}$ and those of $\tilde{\mathbf{G}}$, such that $\mathbf{L} = i^{-1}(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}} \cap \mathbf{G}$ for an F-stable Levi subgroup $\tilde{\mathbf{L}}$ of $\tilde{\mathbf{G}}$. Moreover, the restriction of i to $\mathbf{L}$ yields a regular embedding $\mathbf{L} \to \tilde{\mathbf{L}}$, and $\mathbf{L}$ is split if and only if $\tilde{\mathbf{L}}$ is split. Dually, $(i^*)^{-1}$ induces a bijection $\mathbf{L}^* \mapsto \tilde{\mathbf{L}}^* := (i^*)^{-1}(\mathbf{L}^*)$ between the F-stable Levi subgroups of $\mathbf{G}^*$ and those of $\tilde{\mathbf{G}}^*$. The F-stable Levi subgroups $\mathbf{L}$ and $\mathbf{L}^*$ of $\mathbf{G}$ and $\mathbf{G}^*$ are in duality if and only if $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{L}}^*$ are in duality.

Let $\tilde{\chi} \in \mathrm{Irr}(\tilde{G})$. By a result of Lusztig, the restriction $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is multiplicity free (see [21, Section 10] and [4, Proposition 15.11]). The following easy observation is crucial.

Lemma 3.1. *Let $s \in G^*$ be semisimple and let $\tilde{s} \in \tilde{G}^*$ with $i^*(\tilde{s}) = s$. Then $(i^*)^{-1}(C^\circ_{\mathbf{G}^*}(s)) = C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. In particular $C^\circ_{\mathbf{G}^*}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$ if and only if $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ is contained in a proper split F-stable Levi subgroup of $\tilde{\mathbf{G}}^*$.*

Proof. We have $i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C^\circ_{\mathbf{G}^*}(s)$ (see [1, p. 36]). As the kernel of i^* is contained in $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$, the first result follows.

As i^* induces a bijection between the proper split F-stable Levi subgroups of $\tilde{\mathbf{G}}^*$ and of $\mathbf{G}^*$, the second statement follows from the first. $\diamond$

3.2. **Restriction of characters.** An irreducible character $\chi \in \mathrm{Irr}(G)$ is called *Harish-Chandra imprimitive*, if there is a proper split F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, and $\vartheta \in \mathrm{Irr}(L)$ such that $\chi = R_L^G(\vartheta)$. Otherwise, χ is called *Harish-Chandra primitive*.

Lemma 3.2. *Let $\tilde{\chi}$ be an irreducible character of $\tilde{G}$. If one irreducible constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is imprimitive or Harish-Chandra imprimitive, all of them are.*

Proof. The irreducible constituents of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ are conjugate in $\tilde{G}$. As conjugation of characters commutes with induction and Harish-Chandra induction, respectively, our claim follows. $\diamond$

The following well-known result will also be very useful later on.

Lemma 3.3. *Let $\tilde{\mathbf{L}}$ be a split F-stable Levi subgroup of $\tilde{\mathbf{G}}$ and $\mathbf{L} := \tilde{\mathbf{L}} \cap \mathbf{G}$. Then for every character $\tilde{\vartheta}$ of $\tilde{L}$ and every character ϑ of L we have*

$$\mathrm{Res}_G^{\tilde{G}}(R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})) = R_L^G(\mathrm{Res}_L^{\tilde{L}}(\tilde{\vartheta}))$$

and

$$\mathrm{Ind}_G^{\tilde{G}}(R_L^G(\vartheta)) = R_{\tilde{L}}^{\tilde{G}}(\mathrm{Ind}_L^{\tilde{L}}(\vartheta)).$$

Proof. See [1, Proposition 10.10]. $\diamond$

3.3. **Jordan decomposition of characters.** Let $s \in G^*$ be semisimple. Choose a semisimple element $\tilde{s} \in \tilde{G}^*$ with $i^*(\tilde{s}) = s$. Let $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$. Then every irreducible constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is contained in $\mathcal{E}(G, [s])$ (see [1, Proposition 11.7(a)]). Also, if $\chi \in \mathcal{E}(G, [s])$, there is $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$ such that χ is a constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ (see [1, Proposition 11.7(b)]). Moreover, $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is multiplicity free (see [21, Section 10] and [4, Proposition 15.11]). The conjugation action of $\tilde{G}$ on G permutes these irreducible constituents, and G fixes all of these. Thus $\tilde{G}$ acts on $\mathcal{E}(G, [s])$. We write $[\chi]$ for the $\tilde{G}$-orbit of $\chi \in \mathcal{E}(G, [s])$ and $c(\chi)$ for the number of elements in $[\chi]$. Thus $c(\chi)$ equals the number of $\tilde{G}$-conjugates of χ. The orbit of χ does not depend on the chosen regular embedding of $\mathbf{G}$ (see [4, Corollary 15.14(i)]). On the other hand, $A_{\mathbf{G}^*}(s)^F$ acts on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ and we write $[\lambda]$ for the $A_{\mathbf{G}^*}(s)^F$-orbit of $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ and $A_{\mathbf{G}^*}(s)^F_\lambda$ for its stabilizer in $A_{\mathbf{G}^*}(s)^F$. Thus $A_{\mathbf{G}^*}(s)^F_\lambda = C_{\mathbf{G}^*}(s)^F_\lambda / C^\circ_{\mathbf{G}^*}(s)^F$, where $C_{\mathbf{G}^*}(s)^F_\lambda$ is the stabilizer of λ in $C_{\mathbf{G}^*}(s)^F$.

The following notation will also be useful in the sequel. We put $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}) := (i^*)^{-1}(C_{\mathbf{G}^*}(s))$ and $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s}) := \tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})/C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Then i^* induces an isomorphism $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F \to A_{\mathbf{G}^*}(s)^F$ which commutes with the actions of $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ on $\mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1])$ and of $A_{\mathbf{G}^*}(s)^F$ on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$, respectively. Let $\tilde{\lambda} \in \mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1])$. Identify $\tilde{\lambda}$ with an element λ of $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ through the surjective homomorphism $i^* : C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F \to C^\circ_{\mathbf{G}^*}(s)^F$ (see [7, Proposition 13.20]). Write $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_{\tilde{\lambda}}$ for the stabilizer of $\tilde{\lambda}$ in $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ and $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_{\tilde{\lambda}} := \tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_{\tilde{\lambda}}/C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$. Thus i^* induces an isomorphism between $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_{\tilde{\lambda}}$ and $A_{\mathbf{G}^*}(s)^F_\lambda$.

Lusztig has shown in [21], that a Jordan decomposition of characters between $\mathcal{E}(\tilde{G}, [\tilde{s}])$ and $\mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1])$ induces a bijection between the $\tilde{G}$-orbits on $\mathcal{E}(G, [s])$ and the $A_{\mathbf{G}^*}(s)^F$-orbits on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$. Suppose that $[\chi] \subseteq \mathcal{E}(G, [s])$ and $[\lambda] \subseteq \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ are corresponding orbits under this bijection. We then write $[\chi] \leftrightarrow [\lambda]$. The bijection is obtained as follows. First, choose $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$ such that $\mathrm{Res}^{\tilde{G}}_G(\tilde{\chi})$ contains χ as a constituent. Next, let $\tilde{\lambda} \in \mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1])$ correspond to $\tilde{\chi}$ in the given Jordan decomposition of characters. As above, identify $\tilde{\lambda}$ with an element λ of $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$. Then $[\chi] \leftrightarrow [\lambda]$. An important property of any such bijection is the fact that

$$c(\chi) = |A_{\mathbf{G}^*}(s)^F_\lambda|, \tag{3}$$

if $[\chi] \leftrightarrow [\lambda]$ (see [21, Proposition 5.1]). In the following, we will call any such bijection *Lusztig's generalized Jordan decomposition of characters.*

3.4. **The proof of Theorem 1.1(a).** We begin by generalizing [15, Theorem 7.3].

Theorem 3.4. *Let $s \in G^*$ be semisimple such that $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$, where $\mathbf{L}^*$ is a split F-stable Levi subgroup of $\mathbf{G}^*$. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$.*

Let $\chi \in \mathcal{E}(G, [s])$. Then there is $\vartheta \in \mathcal{E}(L, [s])$ such that χ is an irreducible constituent of $R^G_L(\vartheta)$. Any two such elements of $\mathcal{E}(L, [s])$ are conjugate by an element of $\tilde{L}$.

The number of $\tilde{L}$-conjugates of ϑ is less than or equal to the number of $\tilde{G}$-conjugates of χ. If equality holds, then $\chi = R^G_L(\vartheta)$.

Proof. Let $\tilde{s} \in \tilde{G}^*$ be semisimple with $i^*(\tilde{s}) = s$. Choose $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$, such that χ is a constituent of $\mathrm{Res}^{\tilde{G}}_G(\tilde{\chi})$. By Lemma 3.1, we have $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) = (i^*)^{-1}(C^\circ_{\mathbf{G}^*}(s)) \leq \tilde{\mathbf{L}}^*$, and hence $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) = C_{\tilde{\mathbf{L}}^*}(\tilde{s})$. By a result of Lusztig (see [19, (7.9.1)]), Harish-Chandra induction yields a bijection $\mathcal{E}(\tilde{L}, [\tilde{s}]) \to \mathcal{E}(\tilde{G}, [\tilde{s}])$. Thus there is $\tilde{\vartheta} \in \mathcal{E}(\tilde{L}, [\tilde{s}])$ with $R^{\tilde{G}}_{\tilde{L}}(\tilde{\vartheta}) = \tilde{\chi}$.

By Lemma 3.3 we have

$$\mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi}) = \mathrm{Res}^{\tilde{G}}_{G}(R^{\tilde{G}}_{\tilde{L}}(\tilde{\vartheta})) = R^{G}_{L}(\mathrm{Res}^{\tilde{L}}_{L}(\tilde{\vartheta})). \tag{4}$$

As χ is an irreducible constituent of $\mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi})$, this gives our first claim.

Let $\rho \in \mathcal{E}(L, [s])$ such that χ is an irreducible constituent of $R^{G}_{L}(\rho)$. By Lemma 3.3 we have $\mathrm{Ind}^{\tilde{G}}_{G}(R^{G}_{L}(\rho)) = R^{\tilde{G}}_{\tilde{L}}(\mathrm{Ind}^{\tilde{L}}_{L}(\rho))$. As $\tilde{\chi}$ is a constituent of $\mathrm{Ind}^{\tilde{G}}_{G}(\chi)$, there is an irreducible constituent $\tilde{\rho}$ of $\mathrm{Ind}^{\tilde{L}}_{L}(\rho)$, such that $\tilde{\chi}$ occurs in $R^{\tilde{G}}_{\tilde{L}}(\tilde{\rho})$. But then $\tilde{\rho} \in \mathcal{E}(\tilde{L}, [\tilde{s}])$ (see [4, Proposition 15.7]), and thus $R^{\tilde{G}}_{\tilde{L}}(\tilde{\rho}) = \tilde{\chi}$. It follows that $\tilde{\rho} = \tilde{\vartheta}$ which gives our second claim.

Since the characters $\mathrm{Res}^{\tilde{G}}_{G}(\tilde{\chi})$ and $\mathrm{Res}^{\tilde{L}}_{L}(\tilde{\vartheta})$ are multiplicity free (see above), the last two assertions follow from Equation (4). $\diamond$

We also have a kind of converse to Theorem 3.4.

Theorem 3.5. *Let $s \in G^*$ be semisimple, let $\mathbf{L}^*$ be an F-stable split Levi subgroup of $\mathbf{G}^*$ containing s, and let $\mathbf{L}$ be an F-stable split Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$.*

Suppose that there is $\chi \in \mathcal{E}(G, [s])$ and $\vartheta \in \mathcal{E}(L, [s])$ such that $R^{G}_{L}(\vartheta) = \chi$. Let $\tilde{\vartheta}$ be an irreducible constituent of $\mathrm{Ind}^{\tilde{L}}_{L}(\vartheta)$. Then all irreducible constituents of $R^{\tilde{G}}_{\tilde{L}}(\tilde{\vartheta})$ have the same degree, and the number of these constituents equals $c(\vartheta)/c(\chi)$. In particular, $c(\vartheta) \geq c(\chi)$. If $c(\chi) = c(\vartheta)$, then $C^{\circ}_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$.

Proof. By Lemma 3.3 we have

$$\mathrm{Ind}^{\tilde{G}}_{G}(\chi) = \mathrm{Ind}^{\tilde{G}}_{G}(R^{G}_{L}(\vartheta)) = R^{\tilde{G}}_{\tilde{L}}(\mathrm{Ind}^{\tilde{L}}_{L}(\vartheta)). \tag{5}$$

The restriction of every irreducible character of $\tilde{G}$ to G (and of $\tilde{L}$ to L) is multiplicity free (see [21, Section 10] and [4, Proposition 15.11]). Clifford theory then implies that the irreducible constituents of $\mathrm{Ind}^{\tilde{G}}_{G}(\chi)$ and of $\mathrm{Ind}^{\tilde{L}}_{L}(\vartheta)$ have degrees $c(\chi)\chi(1)$ and $c(\vartheta)\vartheta(1)$, respectively. In particular, all constituents of $R^{\tilde{G}}_{\tilde{L}}(\tilde{\vartheta})$ have the same degree. Using the fact that $\tilde{G}/G \cong \tilde{L}/L$ (see [1, Corollaire 2.6, Démonstration]) and comparing the degree of $\vartheta(1)$ with the degree of $R^{\tilde{G}}_{\tilde{L}}(\tilde{\vartheta})$, we obtain the asserted number of constituents of the latter character.

If $c(\chi) = c(\vartheta)$, then $R^{\tilde{G}}_{\tilde{L}}(\tilde{\vartheta})$ is irreducible. It follows from the first part of the proof of [15, Theorem 8.4] that $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) \leq \tilde{\mathbf{L}}^*$. This gives the third claim by Lemma 3.1. $\diamond$

Corollary 3.6. *Let $s \in G^*$ be semisimple such that $C^{\circ}_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$, where $\mathbf{L}^*$ is a split F-stable Levi subgroup of $\mathbf{G}^*$. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$.*

Let $\chi \in \mathcal{E}(G,[s])$ and let λ be a unipotent character of $C^\circ_{\mathbf{G}^*}(s)^F$ such that $[\chi]$ corresponds to $[\lambda]$ in Lusztig's generalized Jordan decomposition of characters. Then the following two assertions are equivalent.

(a) *We have $C_{\mathbf{G}^*}(s)^F_\lambda \leq \mathbf{L}^*$.*

(b) *There is $\vartheta \in \mathcal{E}(L,[s])$ with $R_L^G(\vartheta) = \chi$.*

Proof. Let $\tilde{s} \in \tilde{G}^*$ be semisimple with $s = i^*(\tilde{s})$. Let $\tilde{\chi} \in \mathcal{E}(\tilde{G},[\tilde{s}])$ such that χ is an irreducible constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$. As in the proof of Theorem 3.4, we let $\tilde{\vartheta} \in \mathcal{E}(\tilde{L},[\tilde{s}])$ with $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta}) = \tilde{\chi}$. Moreover, we choose an irreducible constituent ϑ of $\mathrm{Res}_L^{\tilde{L}}(\tilde{\vartheta})$ such that χ occurs in $R_L^G(\vartheta)$.

Composing the Jordan decomposition of characters between $\mathcal{E}(\tilde{L},[\tilde{s}])$ and $\mathcal{E}(C_{\tilde{\mathbf{L}}^*}(\tilde{s})^F,[1])$ with Harish-Chandra induction, we obtain the Jordan decomposition between $\mathcal{E}(\tilde{G},[\tilde{s}])$ and $\mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F,[1])$. Thus $\tilde{\vartheta} \in \mathcal{E}(\tilde{L},[\tilde{s}])$ and $R_{\tilde{L}}^{\tilde{G}}(\vartheta) = \tilde{\chi} \in \mathcal{E}(\tilde{G},[\tilde{s}])$ correspond to the same unipotent character of $C_{\tilde{\mathbf{L}}^*}(\tilde{s})^F = C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$. Viewed as a character of $C^\circ_{\mathbf{G}^*}(s)^F$, this lies in the $A_{\mathbf{G}^*}(s)^F$-orbit $[\lambda]$. Moreover, $[\vartheta] \leftrightarrow [\lambda]$.

Suppose that (a) holds. Then $|A_{\mathbf{G}^*}(s)^F_\lambda| = |A_{\mathbf{L}^*}(s)^F_\lambda|$, and (b) follows from (3) and the last statement of Theorem 3.4.

Now suppose that (b) is satisfied. It follows from Theorem 3.5 and (3) that $|A_{\mathbf{G}^*}(s)^F_\lambda| \geq |A_{\mathbf{L}^*}(s)^F_\lambda|$. We thus have equality, implying (a). $\diamond$

Notice that assertion (a) of Theorem 1.1 follows from Corollary 3.6. Assertion (b) of that theorem will be proved case by case. We start by investigating a special situation.

Corollary 3.7. *Let $s \in G^*$ be semisimple, let $\chi \in \mathcal{E}(G,[s])$ and $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F,[1])$ such that $[\chi] \leftrightarrow [\lambda]$. Assume that $A_{\mathbf{G}^*}(s)^F$ fixes λ, i.e. $A_{\mathbf{G}^*}(s)^F_\lambda = A_{\mathbf{G}^*}(s)^F$.*

(a) *Suppose that there is a proper split F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ with $s \in \mathbf{L}^*$ and $\vartheta \in \mathcal{E}(L,[s])$ such that $\chi = R_L^G(\vartheta)$ (where $\mathbf{L}$ is an F-stable split Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$). Then $C_{\mathbf{G}^*}(s)^F C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$.*

(b) *If $C_{\mathbf{G}^*}(s)^F$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$, then χ is Harish-Chandra primitive.*

Proof. It suffices to prove (a). By (3) and our assumption, we have $c(\vartheta) \leq |A_{\mathbf{L}^*}(s)^F|$ and $c(\chi) = |A_{\mathbf{G}^*}(s)^F_\lambda| = |A_{\mathbf{G}^*}(s)^F|$. Theorem 3.5 implies $c(\chi) \leq c(\vartheta)$, and thus $|A_{\mathbf{G}^*}(s)^F| \leq |A_{\mathbf{L}^*}(s)^F|$. As the reverse inequality also holds (see 2.5), we have $c(\chi) = c(\vartheta)$, and then $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$, again by Theorem 3.5. Finally, $C_{\mathbf{G}^*}(s)^F = C_{\mathbf{G}^*}(s)^F_\lambda$ by assumption, and $C_{\mathbf{G}^*}(s)^F_\lambda \leq \mathbf{L}^*$ by Corollary 3.6. $\diamond$

3.5. **Some results on Weyl groups.** Before we proceed, we need to strengthen [15, Lemma 8.2]. In the lemma below we adopt the ususal exponential notation for partitions.

Lemma 3.8. *Let W be a finite irreducible Coxeter group and let W_0 be a parabolic subgroup of W. Furthermore, let $\rho \in \mathrm{Irr}(W_0)$.*

(a) *Suppose that $\mathrm{Ind}_{W_0}^W(\rho) = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in \mathrm{Irr}(W)$. Then one of the following holds.*

(i) *We have $W = S_n$ and $W_0 = S_{n-1}$ for some $n \geq 1$, and ρ is labelled by one of the partitions (a^b) of $n-1$ (where $(a,b) = (0,1)$ if $n = 1$). The characters ψ_1 and ψ_2 are then labelled by $(a+1, a^{b-1})$ and $(a^b, 1)$, respectively.*

(ii) *We have $W = S_n$ and $W_0 = S_{n-k} \times S_k$ for some $n \geq 4$ and some $1 < k < n-1$, and ρ is labelled by $(1^{n-k}) \times (k)$. In this case, ψ_1 and ψ_2 are labelled by $(k+1, 1^{n-k-1})$ and $(k, 1^{n-k})$, respectively.*

(iii) *We have W of type D_m and W_0 of type D_{m-1} for some odd $m \geq 5$, and ρ is one of the two characters labelled by an unordered pair of partitions of the form $\{(a^b), (a^b)\}$, where $ab = (m-1)/2$. Moreover, the unordered pairs of partitions $\{(a^b), (a+1, a^{b-1})\}$ and $\{(a^b), (a^b, 1)\}$ label ψ_1 and ψ_2, respectively.*

(b) *Assume that W is of type A_1, A_2, B_2, G_2, B_3 or D_4. Suppose further that $\mathrm{Ind}_{W_0}^W(\rho) = \psi_1 + \psi_2 + \psi_3$ with (not necessarily distinct) $\psi_i \in \mathrm{Irr}(W)$, $i = 1, 2, 3$. Then W is of type B_2, B_3 or D_4 and the labels of ψ_1, ψ_2, ψ_3 are as given in the following lists, where each line corresponds to one pair (W_0, ρ).*

B_2 :

ψ_1	ψ_2	ψ_3
$(1^2, -)$	$(1,1)$	$(-, 1^2)$
$(1,1)$	$(2,-)$	$(-,2)$
$(1,1)$	$(-,1^2)$	$(-,2)$
$(1^2,-)$	$(1,1)$	$(2,-)$

B_3 :

ψ_1	ψ_2	ψ_3
$(1^3, -)$	$(1^2, 1)$	$(21, -)$
$(1, 1^2)$	$(-, 1^3)$	$(-, 21)$
$(21, -)$	$(2, 1)$	$(3, -)$
$(1, 2)$	$(-, 21)$	$(-, 3)$

D_4 :

ψ_1	ψ_2	ψ_3
$(1^2, 1^2)^-$	$(1, 1^3)$	$(-, 1^4)$
$(2, 2)^-$	$(1, 3)$	$(-, 4)$
$(1^2, 1^2)^+$	$(1, 1^3)$	$(-, 1^4)$
$(2, 2)^+$	$(1, 3)$	$(-, 4)$
$(1, 1^3)$	$(-, 1^4)$	$(-, 21^2)$
$(1, 3)$	$(-, 31)$	$(-, 4)$

(c) *Suppose that* (W, ψ_1, ψ_2) *is as in* (a). *For* $1 \leq i \leq 2$ *let* a_i *denote the value of Lusztig's* $\mathbf{a}$*-function associated to the irreducible representation* ψ_i *of* W *(see* [20, (4.1.1)]*). Then* $a_1 \neq a_2$.

(d) *Suppose that* $(W, \psi_1, \psi_2, \psi_3)$ *is as in* (b). *For* $1 \leq i \leq 3$ *let* d_i *denote the generic degree associated to the irreducible representation* ψ_i *(see* [20, p. 61] *or* [14, 8.1.8]*). Then* d_i *is a polynomial in one variable if* W *is of type* D_4*, and in two variables, otherwise. Let* k *and* ℓ *be positive integers. Then, in the first case,* $|\{d_i(q^k) \mid i = 1, 2, 3\}| > 1$*, and in the second case,* $|\{d_i(q^k, q^\ell) \mid i = 1, 2, 3\}| > 1$.

Proof. (a) Let (W, W_0, ρ) satisfy the hypothesis of (a). If W is a dihedral group of order $2d$, then $d \leq 3$, as W_0 has index at least d, and the character degrees of W are 1 and 2. Thus W is of type A_2 and cannot be of type G_2. Using CHEVIE (see [13]), it is easy to check that W is not one of the Coxeter groups H_3 or H_4. Now [15, Lemma 8.2] implies that W_0 is a maximal parabolic subgroup of W.

A further application of CHEVIE shows that W is not one of the exceptional Weyl groups F_4, or E_i, $i = 6, 7, 8$. Next, [15, Lemma 8.1] rules out the case that W is of type B_m for some $m \geq 2$. It thus remains to consider the cases that W is of type A_{n-1}, i.e. $W = S_n$ for $n > 1$, or of type D_m for $m \geq 4$. We begin by investigating the case $W = S_n$, $n \geq 1$, and $W_0 = S_{n-1}$. In this case, the branching rule (see e.g. [17, 2.4.3]) easily gives our claim. Now suppose that $W = S_n$, $n \geq 4$, and $W_0 = S_{n-k} \times S_k$ for some $1 < k < n-1$. Here, we use the Littlewood-Richardson rule (see the version of [17, Corollary 2.8.14]) to show that exactly the cases listed in (ii) satisfy the hypothesis. Finally, suppose that W is of type D_m for some $m \geq 4$. We proceed as in the last two paragraphs of the proof of [15, Lemma 8.1]. Namely, we embed W into a Weyl group $\hat{W}$ of type B_m. Then W_0 is of the form $W_0 = W' \times S_k$, for some $1 \leq k \leq m-1$, where W' denotes a Weyl group of type D_{m-k}, naturally embedded into W. Further, $\hat{W}_0 := \hat{W}' \times S_k$ is a maximal parabolic subgroup of $\hat{W}$, where W' is embedded into $\hat{W}'$ in the same way as W is embedded into $\hat{W}$. Now

(6) $\mathrm{Ind}_{\hat{W}_0}^{\hat{W}}(\mathrm{Ind}_{W_0}^{\hat{W}_0}(\rho)) = \mathrm{Ind}_{W_0}^{\hat{W}}(\rho) = \mathrm{Ind}_{W}^{\hat{W}}(\mathrm{Ind}_{W_0}^{W}(\rho)) = \mathrm{Ind}_{W}^{\hat{W}}(\psi_1 + \psi_2)$.

Hence $\mathrm{Ind}_{\hat{W}_0}^{\hat{W}}(\mathrm{Ind}_{W_0}^{\hat{W}_0}(\rho))$ has at most four irreducible constituents, as W has index 2 in $\hat{W}$. It follows from [15, Lemma 8.1] that $k = 1$ and also that $\hat{\rho} := \mathrm{Ind}_{W_0}^{\hat{W}_0}(\rho)$ is irreducible. Thus $\hat{\rho}$ is labelled by an unordered pair of partitions of the form $\{\pi, \pi\}$. In particular, $m-1$ is even. If π is not of the form $\pi = (a^b)$ with $ab = (m-1)/2$, then $\mathrm{Ind}_{\hat{W}_0}^{\hat{W}}(\hat{\rho})$ has more

Table 1. Some generic degrees in B_3

Bipartition	Generic degree
$(3,-)$	1
$(21,-)$	$\frac{X^2(X^2Y+1)(X+1)}{X+Y}$
$(1^3,-)$	$\frac{X^6(XY+1)(X^2Y+1)}{(X+Y)(X^2+Y)}$
$(2,1)$	$\frac{XY(XY+1)(X^2+X+1)}{X+Y}$
$(1^2,1)$	$\frac{X^3Y(X^2Y+1)(X^2+X+1)}{X^2+Y}$
$(1,2)$	$\frac{XY^2(X^2Y+1)(X^2+X+1)}{X^2+Y}$
$(1,1^2)$	$\frac{X^3Y(XY+1)(X^2+X+1)}{X+Y}$
$(-,3)$	$\frac{Y^3(XY+1)(X^2Y+1)}{(X+Y)(X^2+Y)}$
$(-,21)$	$\frac{X^2Y^3(X^2Y+1)(X+1)}{X+Y}$
$(-,1^3)$	X^6Y^3

than four irreducible constituents by the branching rule for irreducible characters of $\hat{W}$. This contradicts (6), proving our claim.

(b) This is easily checked with CHEVIE.

(c) The a_i can be computed from the labels of the ψ_i (see [20, (4.4)] for W of type A, and [20, (4.6)] for W of type D).

(d) The first part for the possibilities for $\{\psi_1, \psi_2, \psi_3\}$ is again checked with CHEVIE. The generic degrees for W can be found, for example, in [5, 13.5]. If W is of type B_2, the generic degrees for the characters labelled by $(2,-)$, $(1,1)$ and $(-,1^2)$ are 1, $XY(X+1)(Y+1)/(X+Y)$ and X^2Y^2, respectively. Substituting (X,Y) by (q^k, q^ℓ) in the second of these polynomials will neither evaluate to 1 nor to $q^{2(k+\ell)}$. This yields our claim for W of type B_2. Now assume that W is of type B_3. The generic degrees of W can be computed with CHEVIE and are given in Table 1. The proof is now completed as in the precious case. We omit the details. Finally, let W be of type D_4. If $d_i \in \mathbb{Z}[X]$, write

$$d_i(X) = c_i X^{a_i} + \text{ sum of higher powers of } X$$

with a_i a non-negative integer and $0 \neq c_i \in \mathbb{Z}$. Computing the generic degrees for the characters of W, we observe: For each possible set $\{\psi_1, \psi_2, \psi_3\}$ there are $i \neq j \in \{1,2,3\}$ such that $d_i, d_j \in \mathbb{Z}[X]$ with $a_i \neq a_j$ and $c_i = c_j = 1$. Hence $d_i(q^k)$ and $d_j(q^k)$ are divisible by different powers of q, and thus cannot be equal. $\diamond$

Corollary 3.9. *Resume the notation of* 3.1. *Let* $\tilde{\mathbf{L}}$ *denote a split F-stable Levi subgroup of* $\tilde{\mathbf{G}}$*, and let* $\tilde{\vartheta} \in \mathrm{Irr}(\tilde{L})$.

(a) *If q is odd, then $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$ is not a sum of two irreducible characters of the same degree.*

(b) *Suppose that $\tilde{\mathbf{G}}$ is of type E_6. Let $\tilde{s} \in \tilde{L}^*$ be semisimple such that $\tilde{\vartheta} \in \mathcal{E}(\tilde{L}, [\tilde{s}])$, and put $s := i^*(\tilde{s}) \in L^*$. If $A_{\mathbf{G}^*}(s)^F \neq 1$, then $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$ is not a sum of three irreducible characters of equal degrees.*

Proof. Let $(\tilde{L}_0, \tilde{\vartheta}_0)$ be a cuspidal pair below $\tilde{\vartheta}$. Then $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is a Coxeter group containing $W_{\tilde{L}}(\tilde{L}_0, \tilde{\vartheta}_0)$ as a parabolic subgroup (see [20, Theorem 8.6]). Let ρ be an irreducible character of $W_{\tilde{L}}(\tilde{L}_0, \tilde{\vartheta}_0)$ corresponding to $\tilde{\vartheta}$ via Harish-Chandra theory. By the Howlett-Lehrer comparison theorem [16, Theorem 5.9], there is a multiplicity preserving bijection between the irreducible constituents of $\mathrm{Ind}_{W_{\tilde{L}}(\tilde{L}_0, \tilde{\vartheta}_0)}^{W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)}(\rho)$ and those of $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$. The formula of [20, Corollary 8.7] gives the degrees of the latter characters in terms of the generic degrees associated to the irreducible constituents of $\mathrm{Ind}_{W_{\tilde{L}}(\tilde{L}_0, \tilde{\vartheta}_0)}^{W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)}(\rho)$. More precisely, the generic degrees are evaluated at the parameters of the Iwahori-Hecke algebra $\mathrm{End}_{\mathbb{C}\tilde{G}}(R_{\tilde{L}_0}^{\tilde{G}}(\tilde{\vartheta}_0))$. By [20, Theorem 8.6], these parameters are positive powers of q. If two generic degrees of $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ evaluate to different integers, then the corresponding characters of $R_{\tilde{L}_0}^{\tilde{G}}(\tilde{\vartheta}_0)$ have different degrees.

(a) Suppose that $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$ has exactly two constituents. Under this hypothesis, one of the irreducible Weyl groups occuring in the decomposition of $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ satisfies the hypothesis of Lemma 3.8(a). As the generic degrees behave well with respect to the factorisation of a Weyl group into irreducibles, we may assume, for the sake of the argument, that $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is irreducible. Then $(W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0), W_{\tilde{L}}(\tilde{L}_0, \tilde{\vartheta}_0, \rho))$ is one of the pairs listed in Lemma 3.8(a)(i)–(iii). In this case, the generic degree associated to a character ψ of $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is of the form

$$X^{a_\psi}/f_\psi + \text{ sum of higher powers of } X,$$

where $f_\psi = 1$ if $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is of type A, and f_ψ is a power of 2, otherwise (see [14, Corollary 9.3.6] and [20, (4.1.1),(4.14.2)]). Here, a_ψ is the value of Lusztig's $\mathbf{a}$-function associated to ψ. As q is odd, two generic degrees with different $\mathbf{a}$-values evaluate to different integers under substituting the indeterminate X by a positive power of q. An application of Lemma 3.8(c) proves our claim.

(b) Suppose that $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$ has exactly three constituents. By [20, (8.5.4),(8.5.7),(8.5.8)], the group $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is isomorphic to the F-fixed points of a relative Weyl group in $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Let us write $W(\tilde{s})$ for the Weyl group of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Using the assumption that $A_{\mathbf{G}^*}(s)^F \neq 1$, the tables of Lübeck [18] show that the irreducible components of $W(\tilde{s})^F$ have rank at most 2 or else have type B_3 or D_4. It is easy to check with CHEVIE that the same statement then holds for the relative Weyl groups occuring, i.e. for $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$. As in the proof of (a) we may assume that $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is irreducible and that $W_{\tilde{L}}(\tilde{L}_0, \tilde{\vartheta}_0)$ is a maximal parabolic subgroup of $W_{\tilde{G}}(\tilde{L}_0, \tilde{\vartheta}_0)$. Our claim now follows from Lemma 3.8(b),(d). $\diamond$

3.6. **Special cases of Theorem 1.1(b).** Let $\mathbf{G}$ and F be such that G is one of the following groups:

(a) a symplectic group $\mathrm{Sp}_n(q)$ with q odd and $n \geq 4$ even;
(b) a spin group $\mathrm{Spin}_{n+1}(q)$ with q odd and $n \geq 6$ even;
(c) the universal covering group $E_6(q)_{\mathrm{sc}}$ of the simple Chevalley group $E_6(q)$;
(d) the covering group ${}^2E_6(q)_{\mathrm{sc}}$ (which is the universal covering group if $q > 2$) of the simple twisted Steinberg group ${}^2E_6(q)$;
(e) the universal covering group $E_7(q)_{\mathrm{sc}}$ of the simple Chevalley group $E_7(q)$.

For the remainder of this subsection we will assume that $(\mathbf{G}, F)$ is one of the pairs introduced above. Choose $(\tilde{\mathbf{G}}, F)$ and the notation as in 3.1.

Theorem 3.10. *Let $\tilde{\mathbf{G}}$ be one of the groups introduced above. If $\tilde{\chi}$ is an irreducible Harish-Chandra primitive character of $\tilde{G}$, then every irreducible constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is Harish-Chandra primitive.*

Proof. Suppose that an irreducible constituent χ of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$ is Harish-Chandra imprimitive. There is a semisimple element $\tilde{s} \in \tilde{G}^*$ such that $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$ and $\chi \in \mathcal{E}(G, [s])$, where $s := i^*(\tilde{s}) \in G^*$. Let $\mathbf{L}$ be a proper split F-stable Levi subgroup of $\mathbf{G}$ such that L contains an irreducible character ϑ with $R_L^G(\vartheta) = \chi$. We may assume that $s \in L^*$ for some F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ dual to $\mathbf{L}$, and that $\vartheta \in \mathcal{E}(L, [s])$ (see [4, Proposition 15.7]). Using the notation of Theorem 3.5, we have $c(\chi) < c(\vartheta)$. (Otherwise, $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$, implying that $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) = (i^*)^{-1}(C^\circ_{\mathbf{G}^*}(s)) \leq \tilde{\mathbf{L}}^*$, and then $\mathcal{E}(\tilde{G}, [\tilde{s}])$ would contain only Harish-Chandra imprimitive characters by [15, Theorem 7.3].) Now $c(\vartheta)$ divides $|A_{\mathbf{L}^*}(s)|$ by (3), and $A_{\mathbf{L}^*}(s)$ is isomorphic to a subgroup of $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ (see 2.5 and [1, Lemme 8.3]). This is a group of order 2 if $\mathbf{G}$ is as in (a) or (b), a group of order 1 or 2 if $\mathbf{G}$ is as in (e) and q is even or odd, respectively, and a group of order 1 or 3 if $\mathbf{G}$ is of type E_6. It follows that $1 = c(\chi) < c(\vartheta) = |Z(\mathbf{G})/Z^\circ(\mathbf{G})|$. In particular, q is odd if $\mathbf{G}$ is of type E_7. The last assertion of Theorem 3.5 states that $c(\vartheta)$ equals the number of irreducible constituents of $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$, and that all of these constituents have the same degree. This contradicts Corollary 3.9. $\diamond$

Corollary 3.11. *Let $s \in G^*$ be semisimple. If $\mathcal{E}(G, [s])$ contains a Harish-Chandra imprimitive element, then $C^\circ_{\mathbf{G}^*}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$.*

Proof. Let $\chi \in \mathcal{E}(G, [s])$ be Harish-Chandra imprimitive. Choose $\tilde{s} \in \tilde{G}^*$ with $i^*(\tilde{s}) = s$. By [1, Proposition 11.7(b)], there is an irreducible character $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$ such that χ is a constituent of the restriction of $\tilde{\chi}$ to G. It follows from Theorem 3.10 that $\tilde{\chi}$ is Harish-Chandra imprimitive.

By [15, Theorem 8.4], we have $C_{\tilde{\mathbf{G}}^*}(\tilde{s}) \leq \tilde{\mathbf{L}}^*$ for a proper split F-stable Levi subgroup $\tilde{\mathbf{L}}^*$ of $\tilde{\mathbf{G}}^*$. Now $i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C^\circ_{\mathbf{G}^*}(s)$ by Lemma 3.1, and thus $C^\circ_{\mathbf{G}^*}(s) \leq i^*(\tilde{\mathbf{L}}^*)$. The latter is a proper split F-stable Levi subgroup of $\mathbf{G}^*$. $\diamond$

We can now prove Theorem 1.1(b) for the groups considered in this subsection.

Corollary 3.12. *Theorem* 1.1(b) *holds for G.*

Proof. Let $s \in G^*$ and let $\chi \in \mathcal{E}(G, [s])$ be Harish-Chandra imprimitive. Let $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ with $[\chi] \leftrightarrow [\lambda]$. By Corollary 3.11, there is a proper split F-stable Levi subgroup $\mathbf{L}^* \leq \mathbf{G}^*$ with $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$. Suppose first that $A_{\mathbf{G}^*}(s)^F_\lambda = \{1\}$. Then $C_{\mathbf{G}^*}(s)^F_\lambda = C^\circ_{\mathbf{G}^*}(s)^F \leq \mathbf{L}^*$ and our claim follows.

Now assume that $A_{\mathbf{G}^*}(s)^F_\lambda \neq \{1\}$. As $A_{\mathbf{G}^*}(s)$ is isomorphic to a subgroup of $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ (see [1, Lemme 8.3]) and this is a group of prime order, we conclude that $A_{\mathbf{G}^*}(s)^F_\lambda = A_{\mathbf{G}^*}(s)^F$. By Corollary 3.7(a), Theorem 1.1(b) holds in this case as well. $\diamond$

Although we know that Theorem 1.1 holds for the groups considered here, it remains to determine the pairs (s, λ) for which Condition (1) is satisfied. It also remains to prove Theorem 1.1 for the other quasisimple groups not treated here, namely $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$ and $\mathrm{Spin}^\pm_{2m}(q)$. Both tasks will be achieved in Section 5.

4. Preliminary results on some classical groups

Recall that p denotes a prime number, q a power of p, and $\mathbb{F}$ an algebraic closure of the field with p elements. The finite fields of characteristic p are viewed as subfields of $\mathbb{F}$.

4.1. **Polynomials and linear transformations.** We collect some notations and results on polynomials, linear transformations and centralizers of classical groups needed later on.

4.1.1. *Polynomials.* Let $\mathbb{F}[X]^0$ denote the set of monic, separable polynomials over $\mathbb{F}$ in the indeterminate X, with non-zero constant coefficient. An element of $\mathbb{F}[X]^0$ is uniquely determined by its set of roots. For $\mu \in \mathbb{F}[X]^0$ of degree d and $\alpha \in \mathbb{F}^*$ we put

$$\mu^\alpha := \alpha^d \mu(X/\alpha).$$

This defines an action of $\mathbb{F}^*$ on $\mathbb{F}[X]^0$. Notice that $\zeta \in \mathbb{F}^*$ is a root of μ if and only if $\alpha\zeta$ is a root of μ^α. In case $\alpha = -1$, we put $\mu' := \mu^\alpha$. Thus the roots of μ' are the negatives of the roots of μ. We write μ^* for the element of $\mathbb{F}[X]^0$ whose roots are the inverses of the roots of μ. Finally, we put

$$\mu^{*\alpha} := (\mu^*)^\alpha.$$

Thus $\zeta \in \mathbb{F}^*$ is a root of μ if and only if $\alpha\zeta^{-1}$ is a root of $\mu^{*\alpha}$. Then $(\mu^{*\alpha})^{*\alpha} = \mu$. In addition, for $\mu \in \mathbb{F}[X]^0$, we write $\mu^\dagger$ for the element of $\mathbb{F}[X]^0$ whose roots are the $(-q)$ths powers of the roots of μ.

Lemma 4.1. *Let $\alpha \in \mathbb{F}_q^*$, and let $\mu \in \mathbb{F}_q[X] \cap \mathbb{F}[X]^0$ be of degree d.*

(a) *Suppose that $\mu = \mu^{*\alpha}$. If $\zeta^2 \neq \alpha$ for every root $\zeta \in \mathbb{F}$ of μ, then $d = 2e$ is even, and the product of the roots of μ equals α^e.*

If $\mu = \mu^{*\alpha}$ and μ is irreducible in $\mathbb{F}_q[X]$, one of the following cases occurs.

- (I) *There is $\zeta \in \mathbb{F}_q$ with $\alpha = \zeta^2$, $\mu = X - \zeta$ and $\mu' = X + \zeta$.*
- (II) *There is $\zeta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ with $\alpha = \zeta^2$ and $\mu = X^2 - \alpha = \mu'$.*
- (III) *There is $\zeta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ with $\alpha = -\zeta^2$ and $\mu = X^2 + \alpha = \mu'$.*
- (IV) *For all roots $\zeta \in \mathbb{F}$ of μ we have $\alpha \neq \pm\zeta^2$. In this case, $\zeta^{q^e} = \alpha\zeta^{-1}$. Moreover, $\mu \neq \mu'$ if q is odd.*

Cases (II) *and* (III) *only occur if q is odd.*

(b) *If μ is irreducible, d and q are odd, and $\mu' = \mu^{*\alpha}$, then $-\alpha$ is a square in $\mathbb{F}_q$.*

Proof. (a) Let ζ be a root of μ in $\mathbb{F}$. If $\zeta^2 \neq \alpha$, then $\alpha\zeta^{-1} \neq \zeta$. This implies the first statement.

Assume now that μ is irreducible. Then the roots of μ are ζ^{q^i}, $0 \leq i \leq d-1$. If $d = 1$, then $\zeta \in \mathbb{F}_q^*$ and $\zeta = \alpha\zeta^{-1}$, i.e. $\alpha = \zeta^2$ is a square in $\mathbb{F}_q$. Thus μ is as in Case (I). Now suppose that $d > 1$. If $\alpha = \zeta^2$, then $d = 2$, q is odd, $\mu = \mu'$, and μ is as in (II).

Suppose then that $\zeta^2 \neq \alpha$. Then, by the first statement, $d = 2e$ is even and $\zeta^{q^e} = \alpha\zeta^{-1}$ by an easy argument. Suppose that q is odd and $\mu = \mu'$. A similar argument as above shows that $\zeta^{q^e} = -\zeta$. We obtain $\zeta^2 = -\alpha \in \mathbb{F}_q$. Thus $d = 2$ and $\alpha\zeta^{-1} = \zeta^q = -\zeta$, i.e. μ is as in (III). If q is even or if q is odd and $\mu \neq \mu'$, then μ is as in (IV).

(b) This is an elementary exercise. $\diamond$

4.1.2. *Linear transformations.* Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{F}$, and let s be a semisimple element of $\mathrm{GL}(\mathbf{V})$. If $\mu \in \mathbb{F}[X]^0$ and $U \subseteq \mathbf{V}$, we put $U_\mu(s) := \ker(\mu(s)) \cap U$. Then $\mathbf{V}_\mu(s)$ is invariant under the centralizer of s in $\mathrm{GL}(\mathbf{V})$. More generally, suppose that $h \in \mathrm{GL}(\mathbf{V})$ such that $hsh^{-1} = \alpha s$ for some $\alpha \in \mathbb{F}$. Then $\mathbf{V}_\mu(s)h = \mathbf{V}_{\mu^\alpha}(s)$.

4.2. **The general unitary groups.** In this subsection we introduce the general unitary groups and investigate some of their semisimple elements.

4.2.1. *The groups.* Let $\mathbf{G} := \mathrm{GL}_n(\mathbb{F})$, acting on the natural vector space $\mathbf{V} := \mathbb{F}^n$ on the right. For a matrix $b = (\beta_{ij}) \in \mathbb{F}^{d\times e}$ we write $\bar{b} := (\beta_{ij}^q)$. We define the Frobenius morphism F on $\mathbf{G}$ by

$$F(a) := J\left(\bar{a}^T\right)^{-1} J^{-1}, \qquad a \in \mathbf{G}. \tag{7}$$

Then $G = \mathbf{G}^F = \mathrm{GU}_n(q) \leq \mathrm{GL}_n(q^2)$. Indeed, the sesqui-linear form

$$\mathbf{V} \times \mathbf{V} \to \mathbf{V}, (v, w) \mapsto vJ\bar{w}^T \text{ for } v, w \in \mathbf{V} \tag{8}$$

is non-degenerate, its restriction to $V := (\mathbb{F}_{q^2})^n$ is hermitian and G is the unitary group of this hermitian form, acting naturally on V.

For a semisimple element $s \in G$ we define $\mathcal{F}_s \subseteq \mathbb{F}_{q^2}[X] \cap \mathbb{F}[X]^0$ as the set of monic irreducible factors of the minimal polynomial of s (viewed as a linear transformation on $V = (\mathbb{F}_{q^2})^n$). Let $\mu, \nu \in \mathcal{F}_s$. Then $\mathbf{V}_\mu(s) \leq \mathbf{V}_\nu(s)^\perp$ if and only if $\nu \neq \mu^\dagger$. In particular, $\mathbf{V}_\mu(s)$ is totally isotropic if and only if $\mu \neq \mu^\dagger$. In the latter case, $\mathbf{V}_\mu(s) \oplus \mathbf{V}_{\mu^\dagger}(s)$ is non-degenerate. If $\mu = \mu^\dagger$, then $\mathbf{V}_\mu(s)$ is non-degenerate. Analogous statements hold for the finite spaces $V_\mu(s) = \mathbf{V}_\mu(s) \cap V$. Finally, as s is conjugate to $(\bar{s}^T)^{-1}$, we have $\dim \mathbf{V}_\mu(s) = \dim \mathbf{V}_{\mu^\dagger}(s)$ for all $\mu \in \mathcal{F}_s$. Let $\mathbf{L}$ denote an F-stable Levi subgroup of $\mathbf{G}$ containing s. We then put

$$\tilde{C}_{\mathbf{L}}(s) := \{g \in \mathbf{L} \mid gsg^{-1} = \gamma s \text{ for some } \gamma \in \mathbb{F}^*\}$$

and

$$\tilde{A}_{\mathbf{L}}(s) := \tilde{C}_{\mathbf{L}}(s)/C_{\mathbf{L}}(s).$$

Then $\tilde{C}_{\mathbf{L}}(s)$ is F-stable and thus there is a natural action of F as an endomorphism on $\tilde{A}_{\mathbf{L}}(s)$. Moreover, $\tilde{A}_{\mathbf{L}}(s)^F = \tilde{C}_{\mathbf{L}}(s)^F/C_{\mathbf{L}}(s)^F$, as $C_{\mathbf{L}}(s)$ is connected.

For $\zeta_1, \ldots, \zeta_n \in \mathbb{F}^*$, let $h(\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbf{G}$ denote the diagonal matrix with entry ζ_i at the (i,i)-position, and let

$$\mathbf{T} := \{h(\zeta_1, \ldots, \zeta_n) \mid \zeta_i \in \mathbb{F}^* \text{ for all } 1 \leq i \leq n\}.$$

Then $\mathbf{T}$ is a maximally split torus of $\mathbf{G}$ and the Weyl group $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ may and will be identified with the set of permutation matrices in $\mathbf{G}$.

4.2.2. *Semisimple elements.* We begin by constructing certain semisimple elements and determine their centralizers.

Lemma 4.2. *Let $\mu \in \mathbb{F}_{q^2}[X] \cap \mathbb{F}[X]^0$ be irreducible over $\mathbb{F}_{q^2}$ and of degree d. If $\mu = \mu^\dagger$, put $\tilde{\mu} := \mu$, otherwise, put $\tilde{\mu} := \mu\mu^\dagger$. Assume that $n = \deg(\tilde{\mu})k$ for some integer k.*

Then there exists a semisimple element $s \in G$ with characteristic polynomial $\tilde{\mu}^k$.

Let $s \in G$ be semisimple with characteristic polynomial $\tilde{\mu}^k$. Put $C := C_{\mathbf{G}}(s)^F$. Then one of the following cases occurs.

(u) *If $\mu = \mu^\dagger$, then d is odd and we have $C \cong \mathrm{GU}_k(q^d)$.*

(l) *If $\mu \neq \mu^\dagger$, we have $C \cong \mathrm{GL}_k(q^{2d})$.*

Proof. Let $t := h(\zeta_1, \ldots, \zeta_n) \in \mathbf{T}$, where $\{\zeta_1, \ldots, \zeta_n\}$ is the multiset of zeros of $\tilde{\mu}^k$. Then $F(t) = h(\zeta_n^{-q}, \ldots, \zeta_1^{-q})$ and t are conjugate by an element of W as $\tilde{\mu} = \tilde{\mu}^\dagger$. Thus there is an F-stable conjugate s of t (see 2.4).

Now let s be a semisimple, F-stable element with characteristic polynomial $\tilde{\mu}^k$. Then s is conjugate to t in $\mathbf{G}$. Let us sketch the arguments to derive the isomorphism type of C and the remaining claims in case $\mu = \mu^\dagger$. Let ζ be a root of μ. Then $\zeta^{-q} = \zeta^{q^{2i}}$ for some $0 \leq i < d$. As d is the smallest positive integer with $\zeta^{q^{2d}} = \zeta$, it follows that d is odd. Next, let F' denote the Frobenius endomorphism of $\mathbf{G}$ defined by $F'(a) = (\bar{a}^T)^{-1}$ for $a \in \mathbf{G}$, so that $F(a) = w_0 F'(a) w_0^{-1}$, where $w_0 \in W$ denotes the longest element of W. Then, if $F'(t)^w = t$ for some $w \in W$, we have $C_{\mathbf{G}}(s)^F \cong C_{\mathbf{G}}(t)^{Fw_0w} = C_{\mathbf{G}}(t)^{F'w}$ (see 2.4). Next, we may assume that

$$t = h(\zeta, \ldots, \zeta, \zeta^{q^2}, \ldots, \zeta^{q^2}, \ldots, \zeta^{q^{2d-2}}, \ldots, \zeta^{q^{2d-2}}),$$

where each eigenvalue of t occurs exactly k times. For $1 \leq i \leq d$, let $\mathbf{V}_i \leq \mathbf{V}$ denote the eigenspace of t for the eigenvalue $\zeta^{q^{2i-2}}$. Then

$$C_{\mathbf{G}}(t) = \mathrm{GL}(\mathbf{V}_1) \times \cdots \times \mathrm{GL}(\mathbf{V}_d),$$

where $\mathrm{GL}(\mathbf{V}_i)$ is viewed as a subgroup of $\mathbf{G}$ in the natural way, $i = 1, \ldots, d$. Let c denote the d-cycle on $\{1, \ldots, d\}$ such that $(\zeta^{q^{2i-2}})^{-q} = \zeta^{q^{2c(i)-2}}$ for $1 \leq i \leq d$. Let $w \in W$ be the corresponding block permutation so that $F'(t)^w = t$. Then

$$C_{\mathbf{G}}(t)^{F'w} = \{(x_1, \ldots, x_d) \mid x_i \in \mathrm{GL}(\mathbf{V}_i), F'(x_i) = x_{c(i)}, 1 \leq i \leq d\}.$$

As c is a d-cycle, and as d is odd, if follows that

$$C_{\mathbf{G}}(t)^{F'w} \cong \{x \in \mathrm{GL}(\mathbf{V}_1) \mid (F')^d(x) = x\} \cong \mathrm{GU}_k(q^d).$$

$\diamond$

We next consider the case of a general semisimple element in G.

Lemma 4.3. *Let $s \in G$ be semisimple. Then there is $\alpha \in \mathbb{F}^*$ with $\alpha^n = 1 = \alpha^{q+1}$, and an isomorphism*

$$\langle\alpha\rangle \to \tilde{A}_{\mathbf{G}}(s)^F, \tag{9}$$

such that the following conditions hold.

(a) *For all $\mu \in \mathcal{F}_s$ we have $\mu^\alpha \in \mathcal{F}_s$, $\mu^\dagger \in \mathcal{F}_s$ and $(\mu^\dagger)^\alpha = (\mu^\alpha)^\dagger$. In particular, $\tilde{A}_{\mathbf{G}}(s)^F$ acts on $\mathcal{F}_s$ through the isomorphism* (9)*, and this action is equivalent to the action of $\tilde{A}_{\mathbf{G}}(s)^F$ on $\{\mathbf{V}_\mu(s) \mid \mu \in \mathcal{F}_s\}$.*

(b) *Let $\mu \in \mathcal{F}_s$ be of degree d and let $\mathcal{O}$ denote the $\langle\alpha\rangle$-orbit of μ, and $\tilde{\mathcal{O}}$ the union of $\mathcal{O}$ with the $\langle\alpha\rangle$-orbit of $\mu^\dagger$. Let $\mathbf{V}_1 := \sum_{\nu\in\tilde{\mathcal{O}}} \mathbf{V}_\nu(s)$. Then $\mathbf{V}_1$ is non-degenerate with respect to the form defined in* (8)*. Write s_1 for the element of $\mathbf{G}$ that acts as s on $\mathbf{V}_1$, and as the identity on its orthogonal complement. There is a natural embedding of $\mathrm{GL}(\mathbf{V}_1)$ into $\mathbf{G}$ such that $\mathrm{GL}(\mathbf{V}_1)$ is F-stable, $s_1 \in \mathrm{GL}(\mathbf{V}_1)$, and $F(s_1) = s_1$. Put $C := C_{\mathrm{GL}(\mathbf{V}_1)}(s_1)^F$.*

If $\mu = \mu^\dagger$, we define $e := |\mathcal{O}|$. If $\mu \neq \mu^\dagger$, then $|\tilde{\mathcal{O}}|$ is even, and we put $e := |\tilde{\mathcal{O}}|/2$.

(u) *If $\mu = \mu^\dagger$, then d is odd and we have*

$$C \cong \mathrm{GU}_k(q^d) \times \cdots \times \mathrm{GU}_k(q^d) \qquad (e \text{ factors}), \tag{10}$$

and $\tilde{A}_{\mathbf{G}}(s)^F$ acts on C by transitively permuting the factors $\mathrm{GU}_k(q^d)$.

(l) *If $\mu \neq \mu^\dagger$, we have*

$$C \cong \mathrm{GL}_k(q^{2d}) \times \cdots \times \mathrm{GL}_k(q^{2d}) \qquad (e \text{ factors}), \tag{11}$$

and $\tilde{A}_{\mathbf{G}}(s)^F$ acts on C by transitively permuting the factors $\mathrm{GL}_k(q^{2d})$. Moreover, the following two subcases occur.

(ls) *The $\langle\alpha\rangle$-orbit of μ does not contain $\mu^\dagger$.*

(lt) *The $\langle\alpha\rangle$-orbit of μ contains $\mu^\dagger$. In this case, q is odd.*

Proof. The map $\tilde{C}_{\mathbf{G}}(s) \to \{\gamma I_n \mid \gamma \in \mathbb{F}^*\} \leq \mathbf{G}$, $h \mapsto [h,s]$ is a group homomorphism with kernel $C_{\mathbf{G}}(s)$. If γI_n lies in the image of this map, then s and γs have the same determinant, hence $\gamma^n = 1$. It follows that $\tilde{A}_{\mathbf{G}}(s)$ is cyclic of order dividing n. Choose $g \in \tilde{C}_{\mathbf{G}}(s)^F$ such that $\bar{g} := gC_{\mathbf{G}}(s)^F$ is a generator of $\tilde{A}_{\mathbf{G}}(s)^F$, and suppose that $[g,s] = \alpha I_n$. Then, as $F(g) = g$, we have $\alpha^{-q} = \alpha$, i.e. $\alpha^{q+1} = 1$. Moreover, $\langle \alpha \rangle \to \tilde{A}_{\mathbf{G}}(s)^F$, $\alpha^i \mapsto \bar{g}^i$, $i \in \mathbb{Z}$, is an isomorphism.

(a) As $F(s) = s$, we have $\mu^\dagger \in \mathcal{F}_s$ for all $\mu \in \mathcal{F}_s$. As $\alpha s = gsg^{-1}$, the minimal polynomial of αs is the same as that of s, and thus $\mu^\alpha \in \mathcal{F}_s$ for all $\mu \in \mathcal{F}_s$. From $\alpha^{-q} = \alpha$, it follows that $(\mu^\dagger)^\alpha = (\mu^\alpha)^\dagger$ for all $\mu \in \mathcal{F}_s$. Finally, $\mathbf{V}_\mu(s)g = \mathbf{V}_{\mu^\alpha}(s)$ for all $\mu \in \mathcal{F}_s$, and all parts of (a) are proved.

(b) Clearly, $\mathbf{V}_1$ is non-degenerate, and its orthogonal complement equals $\mathbf{V}_2 := \sum_{\nu \in (\mathcal{F}_s \setminus \tilde{\mathcal{O}})} \mathbf{V}_\nu(s)$ (see the second paragraph in 4.2.1). Let $\mathbf{H} \leq \mathbf{G}$ denote the stabilizer of $\mathbf{V}_1$ and $\mathbf{V}_2$. Then $\mathbf{H}$ factors as $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$, where $\mathbf{H}_i$ is naturally isomorphic to $\mathrm{GL}(\mathbf{V})_i$, $i = 1, 2$. Now $\mathbf{H}_i$ is F-stable, as the setwise stabilizer and the pointwise stabilizer in $\mathbf{G}$ of $\mathbf{V}_i$ are F-stable, $i = 1, 2$. Clearly, $s_1 \in \mathbf{H}_1$, and writing $s = s_1 s_2$ with $s_2 \in \mathbf{H}_2$, we obtain $F(s_i) = s_i$ from the fact that s and $\mathbf{H}_i$ are F-stable, $i = 1, 2$. By choosing an appropriate basis for $\mathbf{V}_1$, the action of F on $\mathrm{GL}(\mathbf{V}_1)$ is as in (7). We may thus assume that $\mathbf{V} = \mathbf{V}_1$.

If $\mu = \mu^\dagger$, the claims on d and C follow by applying Lemma 4.2 to each element of $\mathcal{O}$. Suppose that $\mu \neq \mu^\dagger$. Then $\nu \neq \nu^\dagger \in \tilde{\mathcal{O}}$ for each $\nu \in \tilde{\mathcal{O}}$ and hence $|\tilde{\mathcal{O}}|$ is even. If $\mu^\dagger \in \mathcal{O}$, then $\tilde{\mathcal{O}} = \mathcal{O}$, and as $|\mathcal{O}|$ divides the order of α, the latter is even as well. Thus q is odd, since $\alpha^{q+1} = 1$. The other statements follow from Lemma 4.2 applied to $\nu\nu^\dagger$ for $\nu \in \mathcal{O}$. $\diamond$

It is not hard to see that all three cases in part (b) of the above lemma occur for suitable q.

4.3. **Conformal groups.** Here, we introduce the conformal groups of symplectic and quadratic forms and investigate certain of their semisimple elements and their centralizers.

4.3.1. *Forms and groups.* Assume that $\dim(\mathbf{V}) = 2m$ with $m \geq 1$. Let $\mathbf{V}$ be equipped with a non-degenerate symplectic form β or a non-degenerate quadratic form Q. There is a basis

$$v_1, \ldots, v_m, v_m', \ldots, v_1' \tag{12}$$

of $\mathbf{V}$ such that if $(a_1, \ldots, a_m, a'_m, \ldots, a'_1), (b_1, \ldots, b_m, b'_m, \ldots, b'_1) \in \mathbb{F}^n$ are coordinates of $v, w \in \mathbf{V}$ with respect to (12), we have

$$\beta(v, w) = \sum_{i=1}^{m} a_i b'_i - a'_i b_i \tag{13}$$

and

$$Q(v) = \sum_{i=1}^{m} a_i a'_i. \tag{14}$$

If we identify $\mathbf{V}$ with $\mathbb{F}^n$ using the basis (12), we view β and Q as forms on $\mathbb{F}^n$ defined by the formulae (13) respectively (14). Let $\hat{\mathbf{G}}$ denote the conformal group of $\mathbf{V}$ with respect to the form β or Q, and put $\mathbf{G} := \hat{\mathbf{G}}^\circ$. If the form is symplectic, we have $\mathbf{G} = \hat{\mathbf{G}}$, whereas $\mathbf{G}$ has index 2 in $\hat{\mathbf{G}}$, if the form is quadratic. In this case we call $\mathbf{G}$ the special conformal group of Q. Usually we identify $\hat{\mathbf{G}}$ and $\mathbf{G}$ with their groups of matrices with respect to the basis (12), so that $\hat{\mathbf{G}} = \mathbf{G} = \mathrm{CSp}_{2m}(\mathbb{F})$ or $\hat{\mathbf{G}} = \mathrm{CO}_{2m}(\mathbb{F})$ and $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$, where $\mathrm{CSp}_n(\mathbb{F})$ denotes the group of all $g \in \mathrm{GL}_n(\mathbb{F})$ such that $\beta(vg, wg) = \alpha_g \beta(v, w)$ for some $\alpha_g \in \mathbb{F}^*$ and all $v, w \in \mathbb{F}^n$. Similarly, $\mathrm{CO}_n(\mathbb{F})$ is the group of all $g \in \mathrm{GL}_n(\mathbb{F})$ such that $Q(vg) = \alpha_g Q(v)$ for some $\alpha_g \in \mathbb{F}^*$ and all $v \in \mathbb{F}^n$. If q is odd, $\mathrm{CSO}_n(\mathbb{F}) = \{g \in \mathrm{CO}_n(\mathbb{F}) \mid \det(g) = \alpha_g^m\}$ (whereas $\mathrm{CSO}_n(\mathbb{F})$ simply denotes the connected component of $\mathrm{CO}_{2m}(\mathbb{F})$ if q is even). The element α_g in the above description is called the multiplier of g. The map $\mathbf{G} \to \mathbb{F}^*$, $g \mapsto \alpha_g$ is a surjective homomorphism of groups with kernel $[\mathbf{G}, \mathbf{G}]$. We have $[\mathbf{G}, \mathbf{G}] = \mathrm{Sp}_{2m}(\mathbb{F})$, respectively $[\mathbf{G}, \mathbf{G}] = \mathrm{SO}_{2m}(\mathbb{F})$ (where $\mathrm{SO}_{2m}(\mathbb{F})$ is defined to be the connected component of the full orthogonal group $\mathrm{GO}_{2m}(\mathbb{F})$ if q is even). If q is even, we have $\mathbf{G} = Z(\mathbf{G}) \times [\mathbf{G}, \mathbf{G}]$ as algebraic groups, and in view of our intended applications we could as well just work with $[\mathbf{G}, \mathbf{G}]$. We have chosen our approach for the sake of a uniform treatment.

Let F' denote the Frobenius map on $\mathrm{GL}(\mathbf{V})$ which raises every matrix entry (with respect to the basis (12)) to its qth power. Then $\mathbf{G}^{F'} = \mathrm{CSp}_{2m}(q)$ or $\mathrm{CSO}^+_{2m}(q)$, respectively. For $1 \leq i \leq m$ we let σ_i denote the automorphism of $\mathbf{V}$ which swaps the basis vectors v_i and v'_i (and fixes the other basis vectors). The Frobenius morphism F'' of $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ is defined by $F''(g) := \sigma_m^{-1} F'(g) \sigma_m$ for $g \in \mathbf{G}$. Then $\mathbf{G}^{F''} = \mathrm{CSO}^-_{2m}(q)$. The pairs $(\mathrm{CSO}_{2m}(\mathbb{F}), F')$ and $(\mathrm{CSO}_{2m}(\mathbb{F}), F'')$ or the corresponding finite groups are also called orthogonal groups of plus-type and minus-type, respectively. In the following we let F be one of F' or F'', where $F = F''$ implicitly assumes $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$. By V we denote the natural $\mathbb{F}_q$-vector space for $\mathbf{G}^F$ inside $\mathbf{V}$. This is $V = \langle v_1, \ldots, v_m, v_m', \ldots, v_1' \rangle_{\mathbb{F}_q}$ if $F = F'$, and $V = \{\sum_{i=1}^m a_i v_i + a_i' v_i' \mid a_i, a_i' \in \mathbb{F}_q, 1 \leq i \leq m-1, a_m \in \mathbb{F}_{q^2}, a_m' = a_m^q\}$ if $F = F''$.

4.3.2. *A maximally split torus and the Weyl group.* Let $\mathbf{T}$ denote the torus of diagonal matrices (with respect to the basis (12)) in $\mathbf{G}$. We write $h(\zeta_1, \ldots, \zeta_m; \alpha)$ for the element of $\mathrm{GL}(\mathbf{V})$ which acts by multiplication with ζ_i on v_i, and by multiplication with $\alpha\zeta_i^{-1}$ on v_i', $1 \leq i \leq m$. Then $h(\zeta_1, \ldots, \zeta_m; \alpha) \in \mathbf{T}$ has multiplier α. Moreover,

$$\mathbf{T} = \{h(\zeta_1, \ldots, \zeta_m; \alpha) \mid \zeta_i, \alpha \in \mathbb{F}^*, 1 \leq i \leq m\}.$$

Let $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ denote the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$. We will identify W with a group of permutations of the basis $\{v_i, v_i' \mid 1 \leq i \leq m\}$ as follows. First, the σ_i defined in 4.3.1 are viewed as permutations of this basis. Next, for $1 \leq i < m$ let τ_i denote the double transposition $(v_i, v_{i+1})(v_i', v_{i+1}')$. Define

$$\hat{W} := \langle \tau_1, \ldots, \tau_{m-1}, \sigma_m \rangle \leq \mathrm{Sym}(\{v_i, v_i' \mid 1 \leq i \leq m\}).$$

Notice that $\sigma_i \in \hat{W}$ for all $1 \leq i \leq m$. If $\mathbf{G}$ is symplectic, $\hat{W} = W$. If $\mathbf{G}$ is orthogonal, W is of index 2 in $\hat{W}$, namely

$$W = \langle \tau_1, \ldots, \tau_{m-1}, \sigma_m^{-1}\tau_{m-1}\sigma_m \rangle.$$

Notice that a product of σ_i is contained in W if and only if it consists of an even number of factors. In either case, $\hat{W}$ is a Coxeter group of type B_m which naturally acts on $\mathbf{T}$; in the orthogonal case, W is a Coxeter group of type D_m (which is trivial if $m = 1$). In the latter case, we also have $\hat{W} = N_{\hat{\mathbf{G}}}(\mathbf{T})/\mathbf{T}$ and $\hat{\mathbf{G}} = \langle \mathbf{G}, \dot{w} \rangle$ for any $w \in \hat{W} \setminus W$.

Suppose that $t \in \mathbf{T}$ such that $C_{\hat{W}}(t) \leq W$ and let $w \in \hat{W} \setminus W$ with $F(t)^w = t$. Then there is no element in W conjugating t to $F(t)$. Indeed, if $w' \in \hat{W}$ is such that $t^{w'} = F(t)$, then $w'w \in C_{\hat{W}}(t) \leq W$, and thus $w' \in \hat{W} \setminus W$. This argument will be used frequently in the sequel.

4.3.3. *Semisimple elements.* Let $s \in \mathbf{G}$ be semisimple with multiplier α. Then s is conjugate in $\mathbf{G}$ to an element $h(\zeta_1, \ldots, \zeta_m; \alpha) \in \mathbf{T}$. In particular, the multiplicities of $\zeta, \alpha\zeta^{-1} \in \mathbb{F}^*$ as eigenvalues of s are the same. If $\zeta^2 = \alpha$, then $\alpha\zeta^{-1} = \zeta$, and hence ζ occurs with even multiplicity as eigenvalue of s. More generally, let μ be a monic factor of the minimal polynomial of s. Then each root of μ occurs with the same multiplicity in the characteristic polynomial of s as the corresponding root of $\mu^{*\alpha}$. In particular, $\mu^{*\alpha}$ also divides the minimal polynomial of s. Moreover, if μ and $\mu^{*\alpha}$ are relatively prime, then $\mathbf{V}_\mu(s)$ is totally isotropic, and $\mathbf{V}_\mu(s) \oplus \mathbf{V}_{\mu^{*\alpha}}(s)$ is non-degenerate. On the other hand, if $\mu = \mu^{*\alpha}$, then $\mathbf{V}_\mu(s)$ is non-degenerate. Again, corresponding statements also hold for the finite vector spaces $V_\mu(s) = \mathbf{V}_\mu(s) \cap V$. (As in [15] we call a subset U of $\mathbf{V}$ *totally isotropic*, if the form vanishes on $U \times U$ respectively U. In case of a quadratic form, a set $U \subseteq \mathbf{V}$ with this property is sometimes called *totally singular*.)

Suppose that $s \in \mathbf{G}$. We then put

$$\tilde{C}_{\mathbf{G}}(s) := \{g \in \mathbf{G} \mid gsg^{-1} = \pm s\}$$

and

$$\tilde{A}_{\mathbf{G}}(s) := \tilde{C}_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s).$$

4.3.4. *Connectedness of centralizers.* Notice that centralizers of semisimple elements in $\mathbf{G}$ are connected if $\mathbf{G} = \mathrm{CSp}_{2m}(\mathbb{F})$ or if $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and q is even: In the first case, $[\mathbf{G}, \mathbf{G}] = \mathrm{Sp}_{2m}(\mathbb{F})$ is simply connected, and in the second case $Z(\mathbf{G}^*)$ is connected. We next recall the classification of the semisimple elements with a non-connected centralizer in case $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and q odd.

Lemma 4.4. *Let q be odd. Suppose that $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and let $s \in \mathbf{G}$ be semisimple with multiplier α. Let $\zeta \in \mathbb{F}^*$ with $\zeta^2 = \alpha$. If ζ is not an eigenvalue of s, then $C_{\mathbf{G}}(s)$ and $C_{[\mathbf{G},\mathbf{G}]}(s)$ are connected.*

Suppose that ζ and $-\zeta$ are eigenvalues of s. Then $|A_{\mathbf{G}}(s)| = 2$ and there is $h \in \mathbf{G}$ stabilizing $\mathbf{V}_{X-\zeta}(s) \oplus \mathbf{V}_{X+\zeta}(s)$ and acting trivially on its orthogonal complement, such that $C_{\mathbf{G}}(s) = \langle C^\circ_{\mathbf{G}}(s), h\rangle$.

Proof. We have $\mathbf{G} = Z(\mathbf{G})[\mathbf{G}, \mathbf{G}]$ and thus $C_{\mathbf{G}}(s) = Z(\mathbf{G})C_{[\mathbf{G},\mathbf{G}]}(s)$. It follows that $C_{\mathbf{G}}(s)$ is connected if $C_{[\mathbf{G},\mathbf{G}]}(s)$ is, as $Z(\mathbf{G})$ is a torus. We may assume that $s = h(\zeta_1, \ldots, \zeta_m; \alpha) \in \mathbf{T}$. We have an orthogonal decomposition

$$\mathbf{V} = \mathbf{V}_{X-\zeta}(s) \oplus \mathbf{V}_{X+\zeta}(s) \oplus \left(\bigoplus_{\xi} \tilde{\mathbf{V}}_\xi\right),$$

with $\tilde{\mathbf{V}}_\xi := \mathbf{V}_{X-\xi}(s) \oplus \mathbf{V}_{X-\alpha\xi^{-1}}(s)$, where ξ runs through a suitable subset of $\{\zeta_1, \ldots, \zeta_m\} \setminus \{\zeta, -\zeta\}$. Now $C_{\mathbf{G}}(s)$ fixes all these spaces, and $C_{[\mathbf{G},\mathbf{G}]}(s)$ induces the full linear group $\mathrm{GL}(\mathbf{V}_{X-\xi}(s))$ on each of $\tilde{\mathbf{V}}_\xi$.

We may thus assume that $\mathbf{V} = \mathbf{V}_{X-\zeta}(s) \oplus \mathbf{V}_{X+\zeta}(s)$. Then $C_{[\mathbf{G},\mathbf{G}]}(s)$ is connected if $\mathbf{V}_{X-\zeta}(s) = 0$. If none of the above two spaces vanishes, then

$$C_{\mathbf{G}}(s) = \langle \mathrm{CSO}(\mathbf{V}_{X-\zeta}(s)) \times \mathrm{CSO}(\mathbf{V}_{X+\zeta}(s)), h\rangle,$$

for an element $h \in \mathbf{G}$ of order 2 which simultaneously swaps the elements of a hyperbolic pair of $\mathbf{V}_{X-\zeta}(s)$ and of one of $\mathbf{V}_{X+\zeta}(s)$. Thus $C_{\mathbf{G}}(s)$ is not connected, and $A_{\mathbf{G}}(s)$ is of order 2, generated by the image of h. ◇

4.3.5. *Special semisimple elements.* Here we investigate certain critical semisimple elements and their centralizers. In the following, we will fix an irreducible, monic polynomial $\mu \in \mathbb{F}_q[X]$ of degree d with non-zero constant term, a root $\zeta \in \mathbb{F}^*$ of μ and an element $\alpha \in \mathbb{F}_q^*$. We will distinguish various types of μ. If $\mu = \mu^{*\alpha}$, we label the types of μ from (I) to (IV) as in Lemma 4.1(a). We say that μ has Type (V), if $\mu \neq \mu^{*\alpha}$. Recall that Types (II) and (III) only occur if q is odd.

Lemma 4.5. *Let $\mu \in \mathbb{F}_q[X]$ and α be as above. If $\mu = \mu^{*\alpha}$, put $\bar{\mu} := \mu$. Otherwise, $\bar{\mu} := \mu\mu^{*\alpha}$. Notice that Lemma* 4.1(a) *implies that* $\deg(\bar{\mu})$ *is even, unless μ has Type* (I). *Suppose that $2m = n = \deg(\bar{\mu})k$ for some integer k.*

Then there exists a semisimple element $s \in G$ with multiplier α and characteristic polynomial $\bar{\mu}^k$ if and only if $(\mathbf{G}, F)$ and k satisfy the conditions displayed in the following table. The centralizer $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$ of such an element s is as given in the table.

Type	m	$(\mathbf{G}, F)$	k	$C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$
I	$k/2$	$(\mathrm{CSp}_{2m}(\mathbb{F}), F')$	even	$\mathrm{Sp}_{2m}(q)$
		$(\mathrm{CSO}_{2m}(\mathbb{F}), F')$		$\mathrm{SO}^+_{2m}(q)$
		$(\mathrm{CSO}_{2m}(\mathbb{F}), F'')$		$\mathrm{SO}^-_{2m}(q)$
II	k	$(\mathrm{CSp}_{2m}(\mathbb{F}), F')$	even	$\mathrm{Sp}_m(q^2)$
		$(\mathrm{CSO}_{2m}(\mathbb{F}), F')$		$\mathrm{SO}^+_m(q^2)$
		$(\mathrm{CSO}_{2m}(\mathbb{F}), F'')$		$\mathrm{SO}^-_m(q^2)$
III, IV	ke	$(\mathrm{CSp}_{2m}(\mathbb{F}), F')$	any	$\mathrm{GU}_k(q^e)$
		$(\mathrm{CSO}_{2m}(\mathbb{F}), F')$	even	
		$(\mathrm{CSO}_{2m}(\mathbb{F}), F'')$	odd	
V	kd	$(\mathbf{G}, F')$	any	$\mathrm{GL}_k(q^d)$

In particular, there is no such element if $\mu \neq \mu^{*\alpha}$ and $(\mathbf{G}, F) = (\mathrm{CSO}_{2m}(\mathbb{F}), F'')$.

Proof. The proof is given separately for each type of μ. If μ has Type (I), the assertion is trivially satisfied.

Suppose that q is odd and that μ has Type (II). Assume that G contains an element with characteristic polynomial μ^k. As already observed in 4.3.3, the square roots ζ and $-\zeta$ of α occur with even multiplicity in μ^k and thus k is even. Conversely, if k is even, the element

$$t := h(\zeta, -\zeta, \zeta, -\zeta, \ldots, \zeta, -\zeta; \alpha) \in \mathbf{T}$$

is conjugate to an element of G with multiplier α and characteristic polynomial μ^k. Indeed, as $\zeta^q = -\zeta$, we have $F(t)^w = t$ for $w = \tau_1\tau_3\cdots\tau_{k-1} \in W$.

Suppose that μ has Type (IV) or that q is odd and μ has Type (III). By Lemma 4.1(a), the degree of μ is even, say $d = 2e$, and the roots of μ are $\zeta, \zeta^q, \ldots, \zeta^{q^{e-1}}, \alpha\zeta^{-q^{e-1}}, \ldots, \alpha\zeta^{-q}, \alpha\zeta^{-1}$ with $\zeta^{q^e} = \alpha\zeta^{-1}$. Hence

$$t := h(\zeta, \ldots, \zeta, \zeta^q, \ldots, \zeta^q, \ldots, \zeta^{q^{e-1}}, \ldots, \zeta^{q^{e-1}}; \alpha),$$

where each eigenvalue of t occurs exactly k times, has characteristic polynomial μ^k. Every element of $\mathbf{T}$ with characteristic polynomial μ^k is conjugate in W to t or to t^{σ_m}. In particular, there is $s \in G$ with characteristic polynomial μ^k, if and only if s is conjugate in $\mathbf{G}$ to t or to t^{σ_m}. As $\xi \neq \alpha\xi^{-1}$ for all roots ξ of μ, we have $C_{\hat{W}}(t) \leq W$. Clearly, $F(t)^w = t$ for an element $w \in \hat{W}$ of the form $w = \sigma_{(e-1)k+1}\cdots\sigma_{m'}\cdot\tau$ for some $\tau \in \langle\tau_1, \ldots, \tau_{m-1}\rangle$, where $m' = m$ if $F = F'$, and $m' = m - 1$ if $F = F''$. Thus $w \in W$ if and only if $(\mathbf{G}, F, k)$ are as in the assertion. The same argument works for t^{σ_m}.

Finally suppose that μ has Type (V). If such an element s exists, we have $\mathbf{V} = \mathbf{V}_\mu(s) \oplus \mathbf{V}_{\mu^{*\alpha}}(s)$, a direct sum of two totally isotropic subspaces. Thus $(\mathbf{G}, F) \neq (\mathrm{CSO}_{2m}(\mathbb{F}), F'')$. Hence assume that $F = F'$ in the following. The roots of μ are $\zeta, \zeta^q, \ldots, \zeta^{q^{d-1}}$, and the roots of $\mu^{*\alpha}$ are $\alpha\zeta^{-q^{d-1}}, \alpha\zeta^{-q^{d-2}}, \ldots, \alpha\zeta^{-1}$. Thus

$$t := h(\zeta, \ldots, \zeta, \zeta^q, \ldots, \zeta^q, \ldots, \zeta^{q^{d-1}}, \ldots, \zeta^{q^{d-1}}; \alpha), \tag{15}$$

where each eigenvalue of t occurs exactly k times, has characteristic polynomial $(\mu\mu^{*\alpha})^k$. Since $F = F'$, the element $F(t)$ is conjugate to t by an element $w \in \langle\tau_1, \ldots, \tau_{m-1}\rangle \leq W$, proving the existence of s as claimed.

To prove the claims on the structure of the centralizers, we employ the method introduced in 2.4. For each (t, w) as above with $F(t)^w = t$, choose $u \in \mathbf{G}$ with $F(u)u^{-1} = \dot{w}$; then $s := t^u$ is F-stable. To determine the structure of $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$, we have to compute $C^\circ_{[\mathbf{G},\mathbf{G}]}(t)^{F\dot{w}}$, which amounts to a routine calculation. We omit the details. ◇

4.3.6. *Critical semisimple elements.* In the investigations below we will use the following notation. Let $s \in G$ be semisimple and let ν be a monic factor of the minimal polynomial of s with $\nu = \nu^{*\alpha}$, so that $\mathbf{V}_\nu(s)$ is a non-degenerate subspace of $\mathbf{V}$. We then write $\hat{\mathbf{G}}_\nu(s)$ for the set of restrictions to $\mathbf{V}_\nu(s)$ of the elements of the setwise stabilizer of $\mathbf{V}_\nu(s)$ in $\hat{\mathbf{G}}$. Then $\hat{\mathbf{G}}_\nu(s)$ is the conformal group on $\mathbf{V}_\nu(s)$ with respect to the restricted symplectic, respectively quadratic form on $\mathbf{V}_\nu(s)$. If κ is another monic factor of the minimal polynomial of s with $\kappa = \kappa^{*\alpha}$

and ν and κ are relatively prime, then $\hat{\mathbf{G}}_{\nu\kappa}(s) \cong \hat{\mathbf{G}}_{\nu}(s) \times \hat{\mathbf{G}}_{\kappa}(s)$, and if $\nu\kappa$ equals the minimal polynomial of s, then $\hat{\mathbf{G}}_{\nu}(s) \times \hat{\mathbf{G}}_{\kappa}(s)$ can naturally be identified with the stabilizer of $\mathbf{V}_{\nu}(s)$ and $\mathbf{V}_{\kappa}(s)$ in $\hat{\mathbf{G}}$. The subgroup of $\hat{\mathbf{G}}_{\nu}(s)$ constituting the special conformal group on $\mathbf{V}_{\nu}(s)$ is denoted by $\mathbf{G}_{\mu}(s)$ (by convention, $\hat{\mathbf{G}}_{\nu}(s) = \mathbf{G}_{\nu}(s)$ if $\mathbf{G} = \mathrm{CSp}_{2m}(\mathbb{F})$). Finally, s_ν denotes the restriction of s to $\mathbf{V}_{\nu}(s)$, so that $s_\nu \in \hat{\mathbf{G}}_{\nu}(s)$. As the setwise stabilizer and the pointwise stabilizer of $\mathbf{V}_{\nu}(s)$ in $\hat{\mathbf{G}}$ are F-stable, F induces a well-defined Frobenius morphism on $\hat{\mathbf{G}}_{\nu}(s)$, which we also denote by F. Then $s_\nu \in \hat{\mathbf{G}}_{\nu}(s)^F$.

We now consider centralizers of critical elements $s \in G$ and the action of $\tilde{A}_{\mathbf{G}}(s)$ on these in case q is odd (recall the definition of $\tilde{A}_{\mathbf{G}}(s)$ in 4.3.3). The main results here are given in Tables 2 and 3, where we adopt the following notation. The cyclic group of order 2 is denoted by C_2. We write "f" and "g" for a field and a graph automorphism, respectively. Also, "sg" denotes an automorphism which simultaneously acts as a graph automorphism on each of the two factors of $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$. (If $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and $\mathbf{H} \leq \mathbf{G}$ is F-stable and invariant under conjugation by σ_m, the restriction of this conjugation to $\mathbf{H}^F$ is called a graph automorphism of $\mathbf{H}^F$.) We write "flip" for the automorphism which swaps the two factors of $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$ (more precisely, an element $xy \in C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$, where x and y lie in distinct direct factors of the factorization of $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$ given in Tables 2 and 3, is mapped to yx). The indices in I.1, I.2, II.1 and II.2 distinguish the two $\mathrm{CSO}_{2m}(\mathbb{F})^F$-conjugacy classes with characteristic polynomial $(X^2 - \alpha)^m$.

Lemma 4.6. *Assume that q is odd. Suppose that $\mu = \mu^{*\alpha}$ or $\mu' = \mu^{*\alpha}$. If $\mu = \mu'$, put $\tilde{\mu} := \mu$, otherwise put $\tilde{\mu} := \mu\mu'$. Assume that $2m = n = \deg(\tilde{\mu})k$ for some integer k.*

(a) *There exists a semisimple $s \in G$ with multiplier α and characteristic polynomial $\tilde{\mu}^k$, if and only if F and m satisfy the conditions displayed in Tables* 2 *and* 3.

(b) *Suppose that $s \in G$ is semisimple with multiplier α and characteristic polynomial $\tilde{\mu}^k$. Then $C_{[\mathbf{G},\mathbf{G}]}(s)$ is connected, unless μ has Type* (I) *or* (II). *The structure of the groups $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$ and $\tilde{A}_{\mathbf{G}}(s)^F$, as well as the action of the latter on the former, are as displayed in Tables* 2 *and* 3.

(c) *Let s be as in* (b) *and suppose that $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$. Let $S \in \hat{\mathbf{G}}$ denote the set of elements conjugating s to $-s$. Then S is F-stable and non-empty. Moreover, S^F is non-empty, unless μ has Type* (I) *or* (II)

TABLE 2. The critical elements in $\mathrm{CSp}_{2m}(q)$, q odd (explanations in the paragraph preceding Lemma 4.6)

Type	m	$C_{[\mathbf{G},\mathbf{G}]}(s)^F$	$\tilde{A}_{\mathbf{G}}(s)^F$	Action
I	even	$\mathrm{Sp}_m(q) \times \mathrm{Sp}_m(q)$	C_2	flip
II	even	$\mathrm{Sp}_m(q^2)$	C_2	f
III	any	$\mathrm{GU}_m(q)$	C_2	g
IV	$2ke$	$\mathrm{GU}_k(q^e) \times \mathrm{GU}_k(q^e)$	C_2	flip
V	kd	$\mathrm{GL}_k(q^d)$	C_2	g

TABLE 3. The critical elements in $\mathrm{CSO}^{\pm}_{2m}(q)$, q odd (explanations in the paragraph preceding Lemma 4.6)

Type	F	m	$C^{\circ}_{[\mathbf{G},\mathbf{G}]}(s)^F$	$\tilde{A}_{\mathbf{G}}(s)^F$	Action
I.1	F' F''	even	$\mathrm{SO}^+_m(q) \times \mathrm{SO}^+_m(q)$ $\mathrm{SO}^+_m(q) \times \mathrm{SO}^-_m(q)$	$C_2 \times C_2$ C_2	flip $\times$ sg sg
I.2	F' F''	even	$\mathrm{SO}^-_m(q) \times \mathrm{SO}^-_m(q)$ $\mathrm{SO}^-_m(q) \times \mathrm{SO}^+_m(q)$	$C_2 \times C_2$ C_2	flip $\times$ sg sg
II.1, 2	F' F''	even	$\mathrm{SO}^+_m(q^2)$ $\mathrm{SO}^-_m(q^2)$	$C_2 \times C_2$ C_2	f $\times$ g g
III	F' F''	even odd	$\mathrm{GU}_m(q)$	C_2	g
IV	F'	$2ke$	$\mathrm{GU}_k(q^e) \times \mathrm{GU}_k(q^e)$	C_2	flip
V	F'	kd	$\mathrm{GL}_k(q^d)$	C_2	g

and $F = F''$. If S^F is non-empty, then S^F contains an element with multiplier 1. *Table* 4 *displays the location of S in $\hat{\mathbf{G}}$.*

Proof. Let us begin with some preliminary remarks. Suppose that $s \in \mathbf{G}^F$ is semisimple with multiplier α and characteristic polynomial $\tilde{\mu}^k$ and let $t \in \mathbf{T}$ be conjugate to s in $\mathbf{G}$. Then s is conjugate to $-s$ in $\hat{\mathbf{G}}$, respectively $\mathbf{G}$, if and only if t and $-t$ lie in the same $\hat{W}$-orbit, respectively W-orbit, on $\mathbf{T}$. Assume now that $\tilde{\mu} \neq X^2 - \alpha$. Then

TABLE 4. The location of the set S of elements of $\hat{\mathbf{G}} = \mathrm{CO}_{2m}(\mathbb{F})$ conjugating s to $-s$

Type	F	m	S
I	F'	even	$S \cap \mathbf{G}^F \neq \emptyset \neq S \cap (\hat{\mathbf{G}}^F \setminus \mathbf{G}^F)$
	F''		$S \subset \mathbf{G} \setminus \hat{\mathbf{G}}^F$
II	F'	even	$S \subset \mathbf{G}, S^F \neq \emptyset$
	F''		$S \subset \mathbf{G} \setminus \hat{\mathbf{G}}^F$
III	F'	even	$S \subset \mathbf{G}, S^F \neq \emptyset$
	F''	odd	$S \subset \hat{\mathbf{G}} \setminus \mathbf{G}, S^F \neq \emptyset$
IV	F'	any	$S \subset \mathbf{G}, S^F \neq \emptyset$
V	F'	even	$S \subset \mathbf{G}, S^F \neq \emptyset$
		odd	$S \subset \hat{\mathbf{G}} \setminus \mathbf{G}, S^F \neq \emptyset$

$C_{[\mathbf{G},\mathbf{G}]}(s)$ is connected by Lemma 4.4. Moreover, $\mathrm{Stab}_{\hat{W}}(t) \leq W$, as $\xi \neq \alpha\xi^{-1}$ for all eigenvalues ξ of t. This implies $C_{\hat{\mathbf{G}}}(s) \leq C_{\mathbf{G}}(s)$. Then if s is conjugate to $-s$ by an element of $\hat{\mathbf{G}}$ with multiplier 1, there is such a conjugating element which is F-stable. Moreover, either S is contained in $\mathbf{G}$ or in $\hat{\mathbf{G}} \setminus \mathbf{G}$.

We prove (a), (b) and (c) simultaneously, distinguishing cases according to the type of $\tilde{\mu}$. For each of these types, we give representatives of the W-orbits of the elements of $\mathbf{T}$ with characteristic polynomial $\tilde{\mu}^k$. If t is such a representative, we give elements $w \in W$ with $F(t)^w = t$, one for each G-conjugacy class of F-stable elements conjugate to t in $\mathbf{G}$. For $\tilde{\mu} = X^2 - \alpha$ and $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ there are two such classes if m is even; in all other cases, there is at most one such class. The structure of $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$ has already been determined in Lemma 4.5. Finally, the action of $\tilde{A}_{\mathbf{G}}(s)^F$ on $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$ can be derived from the action of $\tilde{A}_{\mathbf{G}}(t)^{F\dot{w}} = \tilde{C}_{\mathbf{G}}(t)^{F\dot{w}}/C^\circ_{\mathbf{G}}(t)^{F\dot{w}}$ on $C^\circ_{[\mathbf{G},\mathbf{G}]}(t)^{F\dot{w}}$.

We put $a := \tau_1\tau_3\cdots\tau_{m-1} \in W$ and $b := \sigma_{m-1}\sigma_m \in W$ (if $m \geq 2$). We also choose commuting lifts $\dot{a}, \dot{b} \in N_{\mathbf{G}}(\mathbf{T})$ of a and b with multipliers 1 and $F'(\dot{a}) = \dot{a}$, $F(\dot{b}) = \dot{b}$. Finally, we put $c := \sigma_1\sigma_2\cdots\sigma_m \in \hat{W}$, and choose a lift $\dot{c} \in N_{\hat{\mathbf{G}}}(\mathbf{T})$ with multiplier 1 and $F(\dot{c}) = \dot{c}$. If $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$, we choose $\dot{a}$, $\dot{b}$ and $\dot{c}$ to be of order 2, and such that $F''(\dot{a})\dot{a}^{-1} = \dot{b}$.

Suppose first that μ has Type (I) or (II). Thus $\zeta \in \mathbb{F}_{q^2}$ and $\zeta^2 = \alpha$. As ζ and $-\zeta$ occur with even multiplicity as eigenvalues of any element of $\mathbf{G}$ (see 4.3.3), there is $s \in G^F$ with characteristic polynomial $\tilde{\mu}^m$ if and only if m is even. Suppose that m is even and let $s \in G$ have characteristic polynomial $\tilde{\mu}^m$. Then s is conjugate in $\mathbf{G}$ to

$$t = h(\zeta, -\zeta, \dots, \zeta, -\zeta; \alpha) \in \mathbf{T}.$$

Notice that $t^a = -t$, $t^b = t$ and that $\langle a, b \rangle$ is a Klein four group. We have $F(t) = t$ if $\zeta \in \mathbb{F}_q$, and $F(t)^a = t$, if not. Let $x, y, z \in \mathbf{G}$ with $F(x)x^{-1} = \dot{a}$, $F(y)y^{-1} = \dot{b}$ and $F(z)z^{-1} = \dot{a}\dot{b}$. If $\mathbf{G}$ is symplectic, $C_{\mathbf{G}}(t)$ is connected, and thus there is a unique G-conjugacy classes in the $\mathbf{G}$-class of t, with representative $s = t$ if $\zeta \in \mathbb{F}_q$ and $s = t^x$, otherwise. If $\mathbf{G}$ is is orthogonal, $C_{\mathbf{G}}(t) = \langle C^\circ_{\mathbf{G}}(t), \dot{b} \rangle$ by Lemma 4.4. Thus there are two G-conjugacy classes in the $\mathbf{G}$-class of t, with representatives $s \in \{t, t^y\}$ if $\zeta \in \mathbb{F}_q$, and $s \in \{t^x, t^z\}$, otherwise. We thus may take $w \in \{1, b\}$ if $\zeta \in \mathbb{F}_q$, and $w \in \{a, ab\}$, otherwise, and where $w = b$ and $w = ab$ only occur if $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$. We are done if $\mathbf{G}$ is symplectic. Suppose then that $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$. Lemma 4.4 implies that $\tilde{C}_{\mathbf{G}}(t) = \langle C^\circ_{\mathbf{G}}(t), \dot{a}, \dot{b} \rangle$. If $F = F'$, then the elements of $\langle \dot{a}, \dot{b} \rangle$ are fixed by $F'\dot{w}$ and thus $\tilde{A}_{\mathbf{G}}(t)^{F'\dot{w}} \cong \langle \dot{a}, \dot{b} \rangle$. In particular there is an element of $\mathbf{G}^{F'}$ with multiplier 1 conjugating s to $-s$. If μ has Type (I), then $\dot{\sigma}_m \in C_{\hat{\mathbf{G}}}(t)^{F'\dot{w}}$, which means that there is an F'-stable conjugate of $\dot{\sigma}_m$ centralizing s, and thus there is also an element in $\hat{\mathbf{G}}^{F'} \setminus \mathbf{G}^{F'}$ with multiplier 1 conjugating s to $-s$. If μ has Type (II), then $C_{\hat{\mathbf{G}}}(t)^{F'\dot{w}} \leq \mathbf{G}^{F'}$. Thus there is no element of $\hat{\mathbf{G}}^{F'} \setminus \mathbf{G}^{F'}$ conjugating s to $-s$. If $F = F''$, then $\tilde{A}_{\mathbf{G}}(t)^{F''\dot{w}} = C_{\mathbf{G}}(t)^{F''\dot{w}}/C^\circ_{\mathbf{G}}(t)^{F''\dot{w}}$, as $F''(\dot{a})\dot{a}^{-1} = \dot{b} \notin C^\circ_{\mathbf{G}}(t)$. In particular, there is no element in $\mathbf{G}^{F''}$ conjugating s to $-s$.

Suppose now that μ has Type (III). As $\tilde{\mu} = \mu$ in this case, the assertions of (a) are contained in Lemma 4.5. To prove part (b), assume that m is even in case $(\mathbf{G}, F) = (\mathrm{CSO}_{2m}(\mathbb{F}), F')$, and that m is odd in case $(\mathbf{G}, F) = (\mathrm{CSO}_{2m}(\mathbb{F}), F'')$. We have $\alpha\zeta^{-1} = -\zeta = \zeta^q$. Put

$$t_1 := h(\zeta, \dots, \zeta; \alpha) \in \mathbf{T}.$$

Any $s \in G$ with characteristic polynomial μ^m is conjugate in $\mathbf{G}$ to t_1 or to $t_2 := t_1^{\sigma_m}$. Let $t \in \{t_1, t_2\}$. Then $F(t) = -t$ if $F = F'$, and $F(t) = -t^{\sigma_m}$ if $F = F''$. As $t^c = -t$, we have $F(t)^w = t$ for $w = c$ or $w = \sigma_m \cdot c = \sigma_1\sigma_2 \cdots \sigma_{m-1}$. As $\dot{c} \in \hat{\mathbf{G}}$ has multiplier 1, we conclude that s and $-s$ are conjugate by an element of $\hat{G}$ by the preliminary remarks. This element can be chosen to lie in G, unless $(\mathbf{G}, F) = (\mathrm{CSO}_{2m}(\mathbb{F}), F'')$. This completes the proof of all claims in this case.

Suppose next that μ has Type (IV). Then $\mu = \mu^{*\alpha} \neq \mu'$. Assume that there is a semisimple element of G with characteristic polynomial $\tilde{\mu}^k$. Then $\mathbf{V} = \mathbf{V}_\mu(s) \oplus \mathbf{V}_{\mu'}(s)$ with non-degenerate, orthogonal F-stable subspaces $\mathbf{V}_\mu(s)$ and $\mathbf{V}_{\mu'}(s)$ of $\mathbf{V}$. Let s_1 denote the restriction of s to $\mathbf{V}_\mu(s)$ and s_1' the restriction of s to $\mathbf{V}_{\mu'}(s)$. Then the characteristic polynomials of s_1 and s_1' equal μ^k and $(\mu')^k$, respectively. Now apply Lemma 4.5 to $\mathbf{V}_\mu(s)$ and $\mathbf{V}_{\mu'}(s)$. If $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$, then $(\mathbf{G}_\mu(s), F)$ and $(\mathbf{G}_{\mu'}(s), F)$ are both of plus-type or both of minus-type and thus $(\mathbf{G}, F) = (\mathrm{CSO}_{2m}(\mathbb{F}), F')$ (recall the definition of $\mathbf{G}_\mu(s)$ at the beginning of 4.3.6). Suppose from now on that $(\mathbf{G}, F) \in \{(\mathrm{CSp}_{2m}(\mathbb{F}), F'), (\mathrm{CSO}_{2m}(\mathbb{F}), F')\}$. Again by Lemma 4.5, there is an element s as required. Put

$$t_1 := h(\zeta, -\zeta, \ldots, \zeta, -\zeta, \ldots, \zeta^{q^{e-1}}, -\zeta^{q^{e-1}}, \ldots, \zeta^{q^{e-1}}, -\zeta^{q^{e-1}}; \alpha) \in \mathbf{T},$$

where each eigenvalue of t occurs exactly k times. Then s is conjugate in $\mathbf{G}$ to t_1 or to $t_2 := t_1^{\sigma_m}$. Let $t \in \{t_1, t_2\}$. Then $F(t)^w = t$ for some $w \in W$, similar to the one taken in the proof of Lemma 4.5. Also $t^a = -t$. All assertions in this case follow from our preliminary considerations.

Suppose finally that μ has Type (V). Here, $\mu \neq \mu^{*\alpha} = \mu'$, and (a) follows from Lemma 4.5. Let $t \in \mathbf{T}$ be the element defined in (15), and let $w \in W$ with $F(t)^w = t$. As the sequence $(\alpha\zeta^{-q^{d-1}}, \ldots, \alpha\zeta^{-1})$ is a permutation of the sequence $(-\zeta, \ldots, -\zeta^{q^{d-1}})$, there is an element $\tau \in \langle \tau_1, \ldots, \tau_{m-1} \rangle$ such that $\delta := \tau \cdot a$ conjugates t to $-t$. Clearly, δ has an F'-stable lift $\dot{\delta}$ to $N_{\mathbf{G}}(\mathbf{T})$ with multiplier 1. Moreover, δ lies in W, if and only if $kd = m$ is even. Again we are done with the preliminary remarks. $\diamond$

4.3.7. *General semisimple elements.* Here we consider the most general semisimple elements of G relevant for our investigation. Let $s \in G$ be semisimple with multiplier α. Then the minimal polynomial of s lies in $\mathbb{F}_q[X]$, and we denote by $\mathcal{F}_s$ the set of irreducible monic factors in $\mathbb{F}_q[X]$ of this minimal polynomial. Assume now that q is odd and that every $\mu \in \mathcal{F}_s$ satisfies $\mu = \mu^{*\alpha}$ or $\mu' = \mu^{*\alpha}$. If $\mu = \mu'$, put $\tilde{\mu} := \mu$, otherwise put $\tilde{\mu} := \mu\mu'$. For $\mu \in \mathcal{F}_s$ write d_μ and $k_\mu(s)$ for the degree of μ and the multiplicity of μ as a factor of the characteristic polynomial of s, respectively.

As $\mathbf{V}_{\tilde{\mu}}(s)$ is a non-degenerate subspace of $\mathbf{V}$, the notation introduced at the beginning of 4.3.6 will be applied with $\nu = \tilde{\mu}$. As s is fixed in the following we put $\hat{\mathbf{G}}_{\tilde{\mu}} := \hat{\mathbf{G}}_{\tilde{\mu}}(s)$ and $\mathbf{G}_{\tilde{\mu}} := \mathbf{G}_{\tilde{\mu}}(s)$. Notice that if $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$, then $(\mathbf{G}_{\tilde{\mu}}, F)$ may be of plus-type or of minus-type (see 4.3.1).

Let $\mathcal{F}_s' \subseteq \mathcal{F}_s$ denote a set of representatives of the partition $\{\{\mu, \mu'\} \mid \mu \in \mathcal{F}_s\}$. Let $I_{(\mathrm{IV})}$ and $I_{(\mathrm{V})}$ be two disjoint index sets labelling the elements of $\mathcal{F}_s'$ of Type (IV) and of Type (V), respectively. Then for $j \in I_{(\mathrm{IV})}$ and $\mu = \mu_j$ we write $e_j := d_\mu/2$ and $\ell_j := k_\mu(s)$. Similarly, for $j \in I_{(\mathrm{V})}$ and $\mu = \mu_j$ we write $d_j := d_\mu$ and $k_j := k_\mu(s)$.

Proposition 4.7. *Let the notation and the assumptions be as above. In particular, q is odd. Then the following assertions hold.*

(a) *Suppose that $\mathbf{G} = \mathrm{CSp}_{2m}(\mathbb{F})$. Then s is conjugate to $-s$ in G, if and only if $k_\mu(s) = k_{\mu'}(s)$ for all $\mu \in \mathcal{F}_s$, and these multiplicities satisfy the conditions displayed in* Table 5. *(The part of this table after the vertical line is only relevant for Remark* 5.7.*)*

If these conditions are satisfied, $\tilde{A}_{\mathbf{G}}(s)^F$ has order 2.

(b) *Suppose that $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$. If s is conjugate to $-s$ in G, then $k_\mu(s) = k_{\mu'}(s)$ for all $\mu \in \mathcal{F}_s$, and one of the following cases occurs.*

(i) *If $F = F'$ and m is even, the conditions displayed in* Table 6 *are satisfied.*

(ii) *If $F = F'$ and m is odd, then $q \equiv 1\,(mod\ 4)$, all elements of $\mathcal{F}_s$ are of Types* (I), (IV) *or* (V), *and the conditions displayed in* Table 7 *are satisfied.*

(iii) *If $F = F''$, then m is odd, $q \equiv 3\,(mod\ 4)$, there are no elements in $\mathcal{F}_s$ of Type* (II), *$X^2 + \alpha \in \mathcal{F}_s$ and the conditions displayed in* Table 8 *are satisfied. (The respective parts of Tables* 6, 7 *and* 8 *after the vertical lines are only relevant for Remark* 5.9.*)*

Suppose that s is as in one of the cases (i), (ii) *or* (iii). *Then s is conjugate to $-s$ in G. If m is odd, then $\tilde{A}_{\mathbf{G}}(s)^F$ is cyclic of order* 4, *and if m is even, then $\tilde{A}_{\mathbf{G}}(s)^F$ is a Klein four group if $X^2 - \alpha$ divides the minimal polynomial of s, and is of order* 2, *otherwise.*

Proof. We only prove (b). The proof of (a) is similar but much simpler. Let, k and ℓ denote the multiplicity of $X^2 - \alpha$, respectively $X^2 + \alpha$, in the characteristic polynomial of s. Suppose that s is conjugate to $-s$ in G. Then $k_\mu(s) = k_{\mu'}(s)$ for all $\mu \in \mathcal{F}_s$. Now $\mathbf{V}$ is the orthogonal direct sum of the non-degenerate subspaces $\mathbf{V}_{\tilde{\mu}}(s)$, where μ runs through $\mathcal{F}_s'$. If $\tilde{\mu} \neq X^2 - \alpha$, then $s_{\tilde{\mu}} \in \mathbf{G}_{\tilde{\mu}} = \mathrm{CSO}(\mathbf{V}_{\tilde{\mu}}(s))$, as the eigenvalues of $s_{\tilde{\mu}}$ come in pairs $\xi, \alpha\xi^{-1}$ with $\xi \neq \alpha\xi^{-1}$. As $s \in \mathrm{CSO}(\mathbf{V})$, we also have $s_{\tilde{\mu}} \in \mathrm{CSO}(\mathbf{V}_{\tilde{\mu}}(s))$ for $\tilde{\mu} = X^2 - \alpha$ (if $k > 0$). Each of the pairs $(\mathbf{V}_{\tilde{\mu}}(s), s_{\tilde{\mu}})$ thus satisfies the hypotheses of Lemma 4.6. In particular, k is even. Moreover, an element of G conjugating s to $-s$ must fix the spaces $\mathbf{V}_{\tilde{\mu}}(s)$, and thus induces an element of $\hat{\mathbf{G}}_{\tilde{\mu}}^F$ conjugating $s_{\tilde{\mu}}$ to $-s_{\tilde{\mu}}$. If $\mu \in \mathcal{F}_s'$ is not of Type (III), then $(\mathbf{G}_{\tilde{\mu}}, F)$ is of plus-type by Lemma 4.6(c). If $\mu = X^2 + \alpha \in \mathcal{F}_s'$, then $(\mathbf{G}_{\tilde{\mu}}, F)$ is of plus-type or of minus-type if ℓ is even or odd, respectively. In particular, $(\mathbf{G}, F)$ is of plus-type if and only if ℓ is even. We now consider the individual cases.

(i) Suppose that $F = F'$ and that m is even. Then $\sum_{j\in I_{(\mathrm{V})}} d_jk_j$ is even, and thus all conditions of Table 6 are satisfied.

(ii) Now suppose that $F = F'$ and that m is odd. Then ℓ is even and $\sum_{j\in I_{(\mathrm{V})}} d_jk_j$ is odd, and thus an odd number of the $d_jk_j, j \in I_{(\mathrm{V})}$, are odd. Lemma 4.1(b) now implies that $-\alpha$ is a square in $\mathbb{F}_q$, and thus $X^2 + \alpha \notin \mathcal{F}_s$, as it is not irreducible. Lemma 4.6(c) implies that if μ is of Type (V) and $d_\mu k_\mu(s)$ is odd, then $s_{\tilde{\mu}}$ is conjugate to $-s_{\tilde{\mu}}$ by an element of $\hat{\mathbf{G}}_{\tilde{\mu}}^F$, but not by an element of $\mathbf{G}_{\tilde{\mu}}^F$. Thus there must exist $\mu \in \mathcal{F}_s'$ not of Type (V), such that an element of $\hat{\mathbf{G}}_{\tilde{\mu}}^F \setminus \mathbf{G}_{\tilde{\mu}}^F$ conjugates $s_{\tilde{\mu}}$ to $-s_{\tilde{\mu}}$. By Lemma 4.6(c), this μ must have Type (I). In particular, α is a square in $\mathbb{F}_q$. Hence -1 is a square in $\mathbb{F}_q$ and thus $q \equiv 1 \pmod 4$.

(iii) Suppose that $F = F''$. Then ℓ is odd, which implies that $-\alpha$ is not a square in $\mathbb{F}_q$. It follows that $\sum_{j\in I_{(\mathrm{V})}} d_jk_j$ is even by Lemma 4.1(b), and thus m is odd. As in the proof of (ii) we conclude that $\mathcal{F}_s$ contains an element of Type (I). Thus α is a square in $\mathbb{F}_q$ and -1 is not a square, hence $q \equiv 3 \pmod 4$.

Suppose now that, in the respective cases, the parameters are as in Tables 6, 7 and 8. Assume first that m is even. Then $F = F'$ and ℓ and $\sum_{j\in I_{(\mathrm{V})}} d_jk_j$ are even. By applying Lemma 4.6 to each of the pairs $(\mathbf{V}_{\tilde{\mu}}(s), s_{\tilde{\mu}})$, we find that s is conjugate to $-s$ in G and that $\tilde{A}_{\mathbf{G}}(s)^F$ is as asserted. Next assume that m is odd. Then α is a square in $\mathbb{F}_q$ and $\ell + \sum_j d_jk_j$ is odd. Moreover, $\ell = 0$ if $F = F'$, and ℓ is odd if $F = F''$. First consider the special case that $m = k + \ell + \sum_{j\in I_{(\mathrm{V})}} d_jk_j$, i.e. that there are no $\mu \in \mathcal{F}_s$ of Type (IV). Let $\zeta \in \mathbb{F}_q$ denote a root of $X^2 - \alpha$. Let $\zeta_{k+1}, \ldots, \zeta_{m-\ell}$ be the roots of $\prod_{j\in I_{(\mathrm{V})}} \mu_j^{k_j}$, counted with multiplicities. If $\ell \neq 0$, i.e. if $F = F''$, let $\zeta_{k+\ell+1}$ be a root of $X^2 + \alpha$, and put $\zeta_i := \zeta_{k+\ell+1}$ for $k + \ell + 2 \leq i \leq m$. Then s is conjugate in $\mathbf{G}$ to

$$t_1 := h(\zeta, -\zeta, \ldots, \zeta, -\zeta, \zeta_{k+1}, \ldots, \zeta_m; \alpha)$$

with $k/2$ occurrences of ζ, or to $t_2 := t_1^{\sigma_m}$ if $\ell \neq 0$. Let $t \in \{t_1, t_2\}$. Then there is $w \in \langle \tau_{k+1}, \ldots, \tau_{m-\ell-1} \rangle$ such that

$$u := (\sigma_1 \cdot \tau_1\tau_3 \cdots \tau_{k-1}) \cdot (\sigma_{k+1} \cdots \sigma_{m-\ell} \cdot w) \cdot (\sigma_{m-\ell+1} \cdots \sigma_m)$$

conjugates t to $-t$. Now $u \in W$ and lifts to an F-stable element $\dot{u}$ of $N_{\mathbf{G}}(\mathbf{T})$. Moreover, $\dot{u}^2 \notin C_{\mathbf{G}}^\circ(t)$ by Lemma 4.4, as $u^2 = \sigma_1\sigma_2 \cdot (\sigma_{k+1} \cdots \sigma_{m-\ell} \cdot w)^2$. This implies our claim in the special case. The general case follows from this with Lemma 4.6 applied to each $\mu \in \mathcal{F}_s'$ of Type (IV). $\diamond$

Table 5. The critical elements in $\mathrm{CSp}_{2m}(q)$, q odd (explanations in the paragraph preceding Proposition 4.7 and in Remark 5.7)

Type	$k_\mu(s)$	d_μ	$\dim(\mathbf{V}_{\tilde{\mu}}(s))$	Condt's	$C_{[\mathbf{G}_{\tilde{\mu}},\mathbf{G}_{\tilde{\mu}}]}(s_{\tilde{\mu}})^F$	λ	λ^a
I	k	1	$2k$	k even[1]	$\mathrm{Sp}_k(q) \times \mathrm{Sp}_k(q)$	(Λ_1, Λ_2)	(Λ_2, Λ_1)
II	k	2	$2k$	k even[1]	$\mathrm{Sp}_k(q^2)$	Λ	Λ
III	ℓ	2	2ℓ		$\mathrm{GU}_\ell(q)$	κ	κ
IV.j, $j \in I_{(\mathrm{IV})}$	ℓ_j	$2e_j$	$4e_j\ell_j$		$\mathrm{GU}_{\ell_j}(q^{e_j}) \times \mathrm{GU}_{\ell_j}(q^{e_j})$	$(\kappa_{1,j}, \kappa_{2,j})$	$(\kappa_{2,j}, \kappa_{1,j})$
V.j, $j \in I_{(\mathrm{V})}$	k_j	d_j	$2d_jk_j$		$\mathrm{GL}_{k_j}(q^{d_j})$	ω_j	ω_j

1 At most one of the multiplicities in Types I and II is non-zero.

TABLE 6. The critical elements in $\mathrm{CSO}_{2m}^+(q)$, m even, q odd (explanations in the paragraph preceding Proposition 4.7 and in Remark 5.9)

Type	$k_\mu(s)$	d_μ	$\dim(\mathbf{V}_{\tilde{\mu}}(s))$	Condt's	$C^\circ_{[\mathbf{G}_{\tilde{\mu}},\mathbf{G}_{\tilde{\mu}}]}(s_{\tilde{\mu}})^F$	λ	λ^a	λ^b
I.1	k	1	$2k$	k even[1]	$\mathrm{SO}_k^+(q)\times\mathrm{SO}_k^+(q)$	(Λ_1,Λ_2)	(Λ_2,Λ_1)	(Λ_1',Λ_2')
I.2	k	1	$2k$	k even[1]	$\mathrm{SO}_k^-(q)\times\mathrm{SO}_k^-(q)$	(Λ_1,Λ_2)	(Λ_2,Λ_1)	(Λ_1,Λ_2)
II.1, 2	k	2	$2k$	k even[1]	$\mathrm{SO}_k^+(q^2)$	Λ	Λ	Λ'
III	ℓ	2	2ℓ	ℓ even	$\mathrm{GU}_\ell(q)$	κ	κ	κ
IV.j, $j\in I_{(\mathrm{IV})}$	ℓ_j	$2e_j$	$4e_j\ell_j$		$\mathrm{GU}_{\ell_j}(q^{e_j})\times\mathrm{GU}_{\ell_j}(q^{e_j})$	$(\kappa_{1,j},\kappa_{2,j})$	$(\kappa_{2,j},\kappa_{1,j})$	$(\kappa_{1,j},\kappa_{2,j})$
V.j, $j\in I_{(\mathrm{V})}$	k_j	d_j	$2d_jk_j$	$\sum_j d_jk_j$ even	$\mathrm{GL}_{k_j}(q^{d_j})$	ω_j	ω_j	ω_j

1 At most one of the multiplicities in Types I.1, I.2, II.1 and II.2 is non-zero.

Table 7. The critical elements in $\mathrm{CSO}^+_{2m}(q)$, m odd, $q \equiv 1 \pmod 4$ (explanations in the paragraph preceding Proposition 4.7 and in Remark 5.9)

Type	$k_\mu(s)$	d_μ	$\dim(\mathbf{V}_{\tilde\mu}(s))$	Condt's	$C^\circ_{[\mathbf{G}_{\tilde\mu},\mathbf{G}_{\tilde\mu}]}(s_{\tilde\mu})^F$	λ	λ^a
I.1	k	1	$2k$	$k \geq 2$ even[1]	$\mathrm{SO}^+_k(q) \times \mathrm{SO}^+_k(q)$	(Λ_1, Λ_2)	(Λ_2, Λ_1')
I.2	k	1	$2k$	$k \geq 2$ even[1]	$\mathrm{SO}^-_k(q) \times \mathrm{SO}^-_k(q)$	(Λ_1, Λ_2)	(Λ_2, Λ_1)
IV.j, $j \in I_{(\mathrm{IV})}$	ℓ_j	$2e_j$	$4e_j\ell_j$		$\mathrm{GU}_{\ell_j}(q^{e_j}) \times \mathrm{GU}_{\ell_j}(q^{e_j})$	$(\kappa_{1,j}, \kappa_{2,j})$	$(\kappa_{2,j}, \kappa_{1,j})$
V.j, $j \in I_{(\mathrm{V})}$	k_j	d_j	$2d_jk_j$	$\sum_j d_jk_j$ odd	$\mathrm{GL}_{k_j}(q^{d_j})$	ω_j	ω_j

1 Exactly one of the multiplicities in Types I.1 and I.2 is non-zero.

TABLE 8. The critical elements in $\mathrm{CSO}_{2m}^{-}(q)$, m odd, $q \equiv 3 \pmod 4$ (explanations in the paragraph preceding Proposition 4.7 and in Remark 5.9)

Type	$k_\mu(s)$	d_μ	$\dim(\mathbf{V}_{\tilde{\mu}}(s))$	Condt's	$C^\circ_{[\mathbf{G}_{\tilde{\mu}},\mathbf{G}_{\tilde{\mu}}]}(s_{\tilde{\mu}})^F$	λ	λ^a
I.1	k	1	$2k$	$k \geq 2$ even[1]	$\mathrm{SO}_k^+(q) \times \mathrm{SO}_k^+(q)$	(Λ_1, Λ_2)	(Λ_2, Λ_1')
I.2	k	1	$2k$	$k \geq 2$ even[1]	$\mathrm{SO}_k^-(q) \times \mathrm{SO}_k^-(q)$	(Λ_1, Λ_2)	(Λ_2, Λ_1)
III	ℓ	2	2ℓ	ℓ odd	$\mathrm{GU}_\ell(q)$	κ	κ
IV.$j, j \in I_{(\mathrm{IV})}$	ℓ_j	$2e_j$	$4e_j\ell_j$		$\mathrm{GU}_{\ell_j}(q^{e_j}) \times \mathrm{GU}_{\ell_j}(q^{e_j})$	$(\kappa_{1,j}, \kappa_{2,j})$	$(\kappa_{2,j}, \kappa_{1,j})$
V.$j, j \in I_{(\mathrm{V})}$	k_j	d_j	$2d_jk_j$	$\sum_j d_jk_j$ even	$\mathrm{GL}_{k_j}(q^{d_j})$	ω_j	ω_j

1 Exactly one of the multiplicities in Types I.1, I.2 is non-zero.

4.4. **Centralizers and Levi subgroups.** In this section we let $(\mathbf{G}, F)$, be one of the pairs of a reductive algebraic group $\mathbf{G}$ and a Frobenius morphism F introduced in 4.2 and 4.3. We also include the case that $\mathbf{G} = \mathrm{GL}_n(\mathbb{F})$ and F is the standard Frobenius map raising every entry of an element of $\mathbf{G}$ to its qth power. We let $\mathbf{V}$ and V denote the natural vector spaces for $\mathbf{G}$ respectively $G = \mathbf{G}^F$. We study the containment of centralizers of semisimple elements of G in Levi subgroups of $\mathbf{G}$ via the action of G on V. Notice that we do not, in general, assume q to be odd here.

Lemma 4.8. *Let $s \in G$ be semisimple, and let μ be an irreducible factor of the minimal polynomial of s. Then $C^\circ_{\mathbf{G}}(s)^F$ acts irreducibly on $V_\mu(s)$, unless $\mathbf{G}$ is the conformal special orthogonal group and $V_\mu(s)$ is 2-dimensional and μ is of Type* (I)*, or $V_\mu(s)$ is 4-dimensional and μ is of Type* (II) *(in which case q is odd).*

Proof. If $G = \mathrm{GL}_n(q)$, the result is trivial, as then the restriction of $C_{\mathbf{G}}(s)^F$ to $\mathrm{GL}(V_\mu(s))$ contains a Singer cycle. Thus assume that G is one of the other groups. The same argument as above works if μ has Type (V). We may thus assume that $\mu = \mu^\dagger$ in the unitary case or that $\mu = \mu^{*\alpha}$ in the other cases (where α denotes the multiplier of s). We may further assume that $V = V_\mu(s)$. The claim then follows from Lemmas 4.2 and 4.5, as the given centralizers, respectively their subgroups $C^\circ_{[\mathbf{G},\mathbf{G}]}(s)^F$, do indeed act irreducibly on the spaces $V_\mu(s)$. $\diamond$

In case G is a general linear or a unitary group, the statement of Lemma 4.8 follows from the results presented, without proof, in [10, Proposition (1A)]. In the other cases, if q is odd, one may use [11, (1.13)]. As we have shown, analogous results also hold for q even, but there does not seem to be a convenient reference in the literature.

Lemma 4.9. *Let $s \in G$ be semisimple and let $H \leq G$ fixing a totally isotropic subspace $U \leq V$. Suppose that there is $\mu \in \mathcal{F}_s$ such that $V_\mu(s)$ is H-invariant and that H acts irreducibly on $V_\mu(s)$. If $U_\mu(s) \neq \{0\}$, then $V_\mu(s) \leq U$. In particular, $V_\mu(s)$ is totally isotropic.*

Proof. As $U_\mu(s) \leq V_\mu(s)$, we obtain $V_\mu(s) = \langle U_\mu(s)H \rangle \leq U$. $\diamond$

Lemma 4.10. *Let $(\mathbf{G}, F)$ be unitary as in* 4.2 *and let $s \in G$ be semisimple. Then the following statements hold.*

(a) *Suppose that $C_{\mathbf{G}}(s)^F$ is contained in a proper split F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$. Then there exists $\mu \in \mathcal{F}_s$ such that $\mu^\dagger$ does not lie in the $\tilde{A}_{\mathbf{L}}(s)^F$-orbit of μ. In particular, $\mu \neq \mu^\dagger$.*

(b) *Suppose that there is $\mu \in \mathcal{F}_s$ such that $\mu^\dagger$ does not lie in the $\tilde{A}_{\mathbf{G}}(s)^F$-orbit of μ. Then there is a proper split F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ with $\tilde{C}_{\mathbf{G}}(s)^F\, C_{\mathbf{G}}(s) \leq \mathbf{L}$.*

(c) *The centralizer $C_{\mathbf{G}}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}$ if and only if there is $\mu \in \mathcal{F}_s$ with $\mu \neq \mu^\dagger$.*

Proof. (a) Let U be a non-zero totally isotropic subspace of V stabilized by L. Let $\mu \in \mathcal{F}_s$ with $U_\mu(s) \neq 0$. It follows from Lemmas 4.8 and 4.9, the latter applied with $H = C_{\mathbf{G}}(s)^F \leq L$, that $V_\mu(s) \leq U$. In particular, $\mu \neq \mu^\dagger$. As $V_\mu(s)x \leq U$ for all $x \in \tilde{C}_{\mathbf{L}}(s)^F \leq L$, the claim follows.

(b) Let $g \in \tilde{C}_{\mathbf{G}}(s)^F$ such that its image in $\tilde{A}_{\mathbf{G}}(s)^F$ generates this group (see the first paragraph of the proof of Lemma 4.3). Suppose that the $\tilde{A}_{\mathbf{G}}(s)^F$-orbit $\mathcal{O}$ of μ does not contain $\mu^\dagger$. Then $\mathbf{V}_1 := \sum_{\nu \in \mathcal{O}} \mathbf{V}_\nu(s)$ and $\mathbf{V}_2 := \sum_{\nu \in \mathcal{O}} \mathbf{V}_{\nu^\dagger}(s)$ are a pair of non-zero, complementary, totally isotropic subspaces of $\mathbf{V}$ fixed by g. Let $\mathbf{L}$ denote the stabilizer in $\mathbf{G}$ of $\mathbf{V}_1$ and $\mathbf{V}_2$. Then $\mathbf{L}$ is a proper split F-stable Levi subgroup of $\mathbf{G}$ containing g and $C_{\mathbf{G}}(s)$. It follows that $\tilde{C}_{\mathbf{G}}(s)^F = \langle C_{\mathbf{G}}(s)^F, g \rangle \leq \mathbf{L}$.

(c) This is a direct consequence of (a) and (b). $\diamond$

We single out F-stable semisimple elements of $\mathrm{CSO}_{2m}(\mathbb{F})$ of a special kind.

Definition 4.11. Let $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and let $s \in G$ be semisimple with multiplier α. We call s *exceptional*, if the following conditions are satisfied: $m \geq 2$, every $\nu \in \mathcal{F}_s$ satisfies $\nu = \nu^{*\alpha}$, there is $\nu \in \mathcal{F}_s$ of Type (I) with $V_\nu(s)$ of dimension 2 and $(\mathbf{G}_\nu(s), F)$ of plus-type, and if q is odd, $X^2 - \alpha$ does not divide the minimal polynomial of s. (Here, $\mathbf{G}_\nu(s)$ is as defined in 4.3.6.)

The latter four conditions can also be phrased as follows: the multiplier α has a square root $\zeta \in \mathbb{F}_q^*$, the ζ-eigenspace $V_{X-\zeta}(s)$ of s on V is 2-dimensional and contains a non-zero isotropic vector, and if q is odd, $-\zeta$ is not an eigenvalue of s.

Lemma 4.12. *Let $(\mathbf{G}, F)$ be one of the groups considered in* 4.3. *Assume that $m \geq 2$ if $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$. Let $s \in G$ be semisimple with multiplier α. Then the following statements hold.*

(a) (i) *Suppose that $C^\circ_{\mathbf{G}}(s)^F$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}$. Then one of the following cases occurs.*

(i.1) *There is $\mu \in \mathcal{F}_s$ with $\mu \neq \mu^{*\alpha}$.*

(i.2) *We have $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$, there is $\nu \in \mathcal{F}_s$ of Type* (I) *or Type* (II) *with multiplicity* 2 *in the characteristic polynomial of s and $(\mathbf{G}_\nu(s), F)$ is of plus-type. Moreover, $C^\circ_{\mathbf{G}}(s)$ is contained in proper split F-stable Levi subgroup of $\mathbf{G}$ stabilizing a pair of complementary, totally isotropic subspaces $\mathbf{U}', \mathbf{U}'' \leq \mathbf{V}$ such that $\mathbf{U}' \oplus \mathbf{U}'' = \mathbf{V}_\nu(s)$. Finally, $C_{\mathbf{G}}(s)^F$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}$ if and only if s is exceptional.*

(a) (ii) *Suppose that q is odd, and that there is $h \in G$ with $hsh^{-1} = -s$. If s is as in* (i.1) *and $\langle C^\circ_{\mathbf{G}}(s)^F, h\rangle$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}$, then there is $\mu \in \mathcal{F}_s$ with $\mu \neq \mu^{*\alpha} \neq \mu'$. If s is as in* (i.2) *and ν is of Type* (I)*, then there is no proper split F-stable Levi subgroup of $\mathbf{G}$ containing $\langle C^\circ_{\mathbf{G}}(s)^F, h\rangle$.*

(b) *Suppose that there is $\mu \in \mathcal{F}_s$ with $\mu \neq \mu^{*\alpha}$. Then $C_{\mathbf{G}}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}$. The analogous conclusion holds for $\tilde{C}_{\mathbf{G}}(s)$ if $\mu \neq \mu^{*\alpha} \neq \mu'$.*

(c) *The centralizer $C_{\mathbf{G}}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}$ if and only if there is $\mu \in \mathcal{F}_s$ with $\mu \neq \mu^{*\alpha}$, or $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and s is exceptional.*

Proof. (a) For the proof of (i) let $h = 1$, and for the proof of (ii) let h be as in the assertion. Suppose that U is a non-zero totally isotropic subspace of V fixed by $\langle C^\circ_{\mathbf{G}}(s)^F, h\rangle$. Suppose first that there is $\mu \in \mathcal{F}_s$ such that $U_\mu(s) \neq \{0\}$ and such that $C^\circ_{\mathbf{G}}(s)^F$ acts irreducibly on $V_\mu(s)$. Then $V_\mu(s)$ is totally isotropic by Lemma 4.9. In particular, $\mu \neq \mu^{*\alpha}$, yielding (i.1). To prove (ii) in this case, assume that $\mu' = \mu^{*\alpha}$. Then $V_{\mu^{*\alpha}}(s) = V_{\mu'}(s) = V_\mu(s)h \leq Uh \leq U$. It follows that U contains the non-degenerate subspace $V_\mu(s) \oplus V_{\mu^{*\alpha}}(s)$, a contradiction.

Now assume that $C^\circ_{\mathbf{G}}(s)^F$ acts reducibly on $V_\mu(s)$ for all $\mu \in \mathcal{F}_s$ with $U_\mu(s) \neq \{0\}$. Let $\nu \in \mathcal{F}_s$ with $U_\nu(s) \neq \{0\}$. Lemma 4.8 implies that $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and that ν is of Type (I) or (II) and occurs with multiplicity 2 in the characteristic polynomial of s. It follows that $U = U_\nu(s) \lneq V_\nu(s)$.

Let us prove the other assertions of (i.2). Put $\mathbf{G}_\nu := \mathbf{G}_\nu(s)$, and write s_ν for the restriction of s to $\mathbf{V}_\nu(s)$. Our assumption implies that $C^\circ_{\mathbf{G}_\nu}(s_\nu)^F$ fixes U. We claim that $(\mathbf{G}_\nu, F)$ is of plus-type. Indeed, if ν has Type (I) and $(\mathbf{G}_\nu, F)$ is of minus type, then $V_\nu(s)$ does not have any non-zero totally isotropic subspace. Now suppose that ν has Type (II). Then α is not a square in $\mathbb{F}_q^*$, and in particular q is odd. Assume that F acts as F'' on $\mathbf{G}_\nu \cong \mathrm{CSO}_4(\mathbb{F})$. The F-stable Borel subgroups are the only proper split F-stable Levi subgroups of $\mathbf{G}_\nu$. The p'-part of the order of a Borel subgroup of $\mathbf{G}_\nu^F$ equals $(q-1)(q^2-1)$. On the other hand, the order of $C^\circ_{\mathbf{G}_\nu}(s_\nu)^F$ is divisible by q^2+1 (see Lemma 4.5). Thus $C^\circ_{\mathbf{G}_\nu}(s_\nu)^F$ cannot fix U, a contradiction. Our claim is proved. Any non-trivial minimal s-invariant subspace of $V_\nu(s)$ is totally isotropic by Lemma 4.5. Thus s_ν lies in a split Levi subgroup of $\mathbf{G}_\nu$. It is now easy to see that $C^\circ_{\mathbf{G}_\nu}(s_\nu)$ fixes a pair of complementary, totally isotropic subspace $\mathbf{U}', \mathbf{U}'' \leq \mathbf{V}_\nu(s)$ with $\mathbf{U}' \oplus \mathbf{U}'' = \mathbf{V}_\nu(s)$ whose stabilizer in $\mathbf{G}$ is F-stable. Then $C^\circ_{\mathbf{G}}(s)$ also fixes $\mathbf{U}'$ and $\mathbf{U}''$. It remains to prove the last assertion. If s is exceptional, then $C_{\mathbf{G}}(s)$ is connected by

Lemma 4.4 and we are done. If s is not exceptional, q is odd and $X^2-\alpha$ divides the minimal polynomial of s. Let $\tilde{\nu} := \nu\nu'$ if ν is of Type (I), and let $\tilde{\nu} := \nu$, otherwise. We claim that there exists $g_{\tilde{\nu}} \in C_{\mathbf{G}_{\tilde{\nu}}}(s_{\tilde{\nu}})^F$ mapping some non-zero vector $v \in U$ to a vector v' such that (v, v') is a hyperbolic pair in $V_\nu(s)$. Indeed, if ν has Type (I), there is an F-stable element in $\hat{\mathbf{G}}_\nu^F \setminus \mathbf{G}_\nu^F$ achieving this; take any element in $\hat{\mathbf{G}}_{\nu'}^F \setminus \mathbf{G}_{\nu'}^F$ and multiply the two elements to obtain $g_{\tilde{\nu}}$. If ν has Type (II), identify $(\mathbf{G}_\nu, F)$ with $(\mathrm{CSO}_4(\mathbb{F}), F')$ and s_ν with a suitable $\mathrm{CSO}_4(\mathbb{F})$-conjugate of $t := h(\zeta, -\zeta; \alpha)$, where ζ is a square root of α. Then some conjugate of the element $\dot{b}$ introduced in the proof of Lemma 4.6 satisfies the requirement, and our claim follows. This implies that $g_{\tilde{\nu}}$ does not fix U. Clearly, $g_{\tilde{\nu}}$ can be extended to an element $g \in C_{\mathbf{G}}(s)^F$, proving the last assertion of (i.2).

We turn to the proof of (ii) in this case. Assume that the assertion is false. As $\nu \neq \nu' = \nu^{*\alpha}$, we obtain a contradiction as in the first paragraph of the proof of (a).

(b) Let $\mu \in \mathcal{F}_s$ with $\mu \neq \mu^{*\alpha}$. For the proof of the first statement, put $\tilde{\mu} := \mu$. For the proof of the second statement, set $\tilde{\mu}$ to be equal to μ if $\mu = \mu'$, and to $\mu\mu'$ if $\mu \neq \mu'$. Then $\mathbf{V}_{\tilde{\mu}}(s)$ and $\mathbf{V}_{\tilde{\mu}^{*\alpha}}(s)$ form a pair of complementary, non-zero, totally isotropic subspaces of $\mathbf{V}$ which are fixed by $C_{\mathbf{G}}(s)$, and also by $\tilde{C}_{\mathbf{G}}(s)$ if $\mu' \neq \mu^{*\alpha}$.

(c) By Lemma 4.4, $C_{\mathbf{G}}(s)$ is connected if $\mathbf{G} = \mathrm{CSp}_{2m}(\mathbb{F})$ or $\mathbf{G} = \mathrm{CSO}_{2m}(\mathbb{F})$ and s is exceptional, so that the *if* part of the assertion follows from (b) and (a)(i.2). For the *only if* part use (a)(i). $\diamond$

5. The quasisimple groups of Lie type

5.1. **Introduction.** We are now going to describe the Harish-Chandra imprimitive ordinary absolutely irreducible characters of all finite quasisimple groups of Lie type. Except for finitely many such groups, an absolutely irreducible character is Harish-Chandra imprimitive if and only if it is imprimitive (see [15, Theorem 6.1]). As we do not exclude the groups providing exceptions right away, we will stick to the notion of Harish-Chandra imprimitivity. The above task has already been achieved for groups with an exceptional Schur multiplier and for groups with two distinct defining characteristics in [15, Chapter 5], for the Tits simple group in [15, Chapter 3], and for the exceptional series $G_2(q)$, ${}^3D_4(q)$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$ and ${}^2B_2(2^{2m+1})$ in [15, Section 10.2]. The groups $F_4(q)$ and $E_8(q)$ arise from algebraic groups with connected centers, and are treated in [15, Section 10.1].

We are thus left with the groups G listed in Subsection 3.5, as well as with the symplectic and orthogonal groups $\mathrm{Sp}_{2m}(q)$ and $\Omega_{2m}^{\pm}(q)$, where q is even. The latter groups arise from algebraic groups with connected center and will not be considered here, as their Harish-Chandra imprimitive characters have been classified in terms of centralizers of semisimple elements of the dual groups (see [15, Proposition 9.5]).

We take this opportunity to correct some inaccuracies in the statement of [15, Proposition 9.5] and its proof. Firstly, to establish the equivalence of (b) and (c) in [15, Proposition 9.5], we cite Fong and Srinivasan [10, Proposition (1A)] and [11, (1.13)] for the structure of the centralizers in classical groups. In the latter reference, the results are only formulated for odd q. However, analogous statements also hold for even q, as we have now sketched in Lemmas 4.2 and 4.5 above. Secondly, and more seriously, statements (a) and (b) of [15, Proposition 9.5] are not equivalent if G is an orthogonal group. In (b) one also has to allow for special elements. Also, in order to show that (b) implies (a), we implicitly assumed a result which we have formulated as Lemma 4.9 here. The proof of this implication as well as the correct statement (b) follow at once from our Lemmas 4.10(c) and 4.12(c) above. Finally, if G is a conformal group in [15, Proposition 9.5], one should also take into account the multipliers of the semisimple elements of G and replace μ^* by $\mu^{*\alpha}$.

To remedy these deficiencies, we present the following corrected version of the relevant parts of [15, Proposition 9.5].

Proposition 5.1. *Let $p = 2$ and let $\mathbf{G}$ be one of the groups $\mathrm{Sp}_{2m}(\mathbb{F})$ with $m \geq 1$ or $\mathrm{SO}_{2m}(\mathbb{F})$ with $m \geq 2$, where $\mathrm{SO}_{2m}(\mathbb{F})$ is as defined in* [24, p. 160]*. Let F denote a Frobenius endomorphism of $\mathbf{G}$ arising from an $\mathbb{F}_q$-structure on $\mathbf{G}$, and let $(\mathbf{G}^*, F)$ be dual to $(\mathbf{G}, F)$. Then $G = \mathbf{G}^F$ is one of $\mathrm{Sp}_{2m}(q)$, $m \geq 1$ or $\mathrm{SO}^{\pm}_{2m}(q)$, $m \geq 2$. In particular, G is quasisimple except for $G \in \{\mathrm{Sp}_2(2), \mathrm{Sp}_4(2), \mathrm{SO}^+_4(2)\}$. If $\mathbf{G}$ is orthogonal and $G \neq \mathrm{SO}^+_4(2)$, then $G = \Omega^{\pm}_{2m}(q)$.*

Let $s \in G^*$ be semisimple. Then the following statements are equivalent.

(a) *Every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive.*

(b) *Every $\mu \in \mathcal{F}_s$ satisfies $\mu = \mu^*$, and if $\mathbf{G} = \mathrm{SO}_{2m}(\mathbb{F})$, then s is not exceptional.*

Proof. The stated facts about G are standard (see [24, Section 11]). The equivalence of (a) and (b) follows from [15, Theorem 7.3, Theorem 8.4] in conjunction with 4.12(c), using the fact that the conformal groups corresponding to $\mathbf{G}^*$ are direct products of their derived subgroups (isomorphic to $\mathbf{G}^*$) with their centers. $\diamond$

5.2. **The special linear groups.** Let $G = \mathrm{SL}_n(q)$, $n \geq 2$. In this case we take $\mathbf{G} = \mathrm{SL}_n(\mathbb{F})$ with the standard Frobenius map F raising every matrix entry of an element of $\mathbf{G}$ to its qth power. We also take $\tilde{\mathbf{G}} = \tilde{\mathbf{G}}^* = \mathrm{GL}_n(\mathbb{F})$, acting on the natural vector space $\mathbf{V} = \mathbb{F}^n$. Finally, $\mathbf{G}^* = \mathrm{PGL}_n(\mathbb{F})$ and i^* is the canonical epimorphism.

Let $s \in \mathrm{PGL}_n(q) = G^*$ be semisimple and let $\chi \in \mathcal{E}(G, [s])$. Our aim is to decide whether χ is Harish-Chandra primitive, in terms of the $A_{\mathbf{G}^*}(s)^F$-orbit $[\lambda]$ of unipotent characters of $C^\circ_{G^*}(s)$ associated to the $\tilde{G}$-orbit $[\chi]$. For this purpose choose $\tilde{s} \in \tilde{G}^* = \mathrm{GL}_n(q)$ with $i^*(\tilde{s}) = s$ and let $\tilde{\chi} \in \mathcal{E}(\tilde{G}, [\tilde{s}])$ such that χ is a constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi})$.

Let $\mathcal{F}_{\tilde{s}} = \{\mu_0, \ldots, \mu_{e-1}\}$, and for $0 \leq i \leq e-1$, let d_i denote the degree of μ_i and k_i the multiplicity of μ_i in the characteristic polynomial of $\tilde{s}$. (For the definition of $\mathcal{F}_{\tilde{s}}$ see Subsection 4.1.2.) Then

$$C_{\tilde{G}^*}(\tilde{s}) \cong \mathrm{GL}_{k_0}(q^{d_0}) \times \cdots \times \mathrm{GL}_{k_{e-1}}(q^{d_{e-1}}).$$

Let λ be the unipotent character of $C_{\tilde{G}^*}(\tilde{s})$ such that $\tilde{\chi}$ corresponds to $(\tilde{s}, \lambda)$ in Lusztig's Jordan decomposition of characters. Then $[\chi] \leftrightarrow [\lambda]$ (if λ is viewed as a character of $C^\circ_{G^*}(s)$). Let λ be labelled by the e-tuple $(\pi_0, \ldots, \pi_{e-1})$ of partitions $\pi_i \vdash k_i$ for $0 \leq i \leq e-1$.

Let $\beta \in \mathbb{F}_q^*$ denote a generator of the setwise stabilizer of $\mathcal{F}_{\tilde{s}}$ in $\mathbb{F}_q^*$ (see 4.1.1).

Theorem 5.2. *Assume the notation introduced in* 5.2. *Then χ is Harish-Chandra primitive if and only if $\langle\beta\rangle$ permutes $\mathcal{F}_{\tilde{s}}$ transitively, $k_0 = k_1 = \cdots = k_{e-1}$ and $\pi_0 = \pi_1 = \cdots = \pi_{e-1}$.*

In particular, Theorem 1.1(b) *is satisfied for $G = \mathrm{SL}_n(q)$.*

Proof. Let us use the notation introduced in 3.3. In particular, $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})$ is the inverse image under i^* of $C_{\mathbf{G}^*}(s)$, i.e.

$$\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}) = \{\tilde{g} \in \tilde{\mathbf{G}}^* \mid [\tilde{g}, \tilde{s}] = \gamma I_n \text{ for some } \gamma \in \mathbb{F}^*\}.$$

Recall that i^* induces an equivariant isomorphism between $A_{\mathbf{G}^*}(s)^F$ and $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F / C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$, the former group acting on $\mathcal{E}(C^\circ_{G^*}(s), [1])$, the latter group on $\mathcal{E}(C_{\tilde{G}^*}(\tilde{s}), [1])$. It follows from 4.1.2 that $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})$ permutes the set $\{\mathbf{V}_{\mu_i}(\tilde{s}) \mid 0 \leq i \leq e-1\}$.

Suppose first that either the action of $\langle\beta\rangle$ on $\mathcal{F}_{\tilde{s}}$ is intransitive, or that $k_i \neq k_j$ for some $0 \leq i \neq j \leq e-1$. Then $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ does not permute the spaces $\mathbf{V}_{\mu_i}(\tilde{s})$ transitively. Hence $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F\, C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ is contained in a proper split F-stable Levi subgroup $\tilde{\mathbf{L}}^*$ of $\tilde{\mathbf{G}}^*$, which implies that $C_{\mathbf{G}^*}(s)^F\, C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$. This is exactly Condition (1) of Theorem 1.1. It follows from Corollary 3.6 that χ is Harish-Chandra induced from L.

Now suppose that $\langle\beta\rangle$ permutes $\mathcal{F}_{\tilde{s}}$ transitively, and that $k_0 = k_1 = \cdots = k_{e-1}$. We claim that in this case $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ acts transitively on the set $\{\mathbf{V}_{\mu_i}(\tilde{s}) \mid 0 \leq i \leq e-1\}$. To see this, let k be the common value of the k_i's. Then the characteristic polynomial of $\tilde{s}$ equals $(\mu_0 \cdots \mu_{e-1})^k$. As β stabilizes $\{\mu_0, \ldots, \mu_{e-1}\}$, the element $\beta\tilde{s}$ has the same characteristic polynomial as $\tilde{s}$, and thus $\tilde{s}$ and $\beta\tilde{s}$ are conjugate by an element of $\tilde{\mathbf{G}}^*$. As $\tilde{s}$ and $\beta\tilde{s}$ are F-stable and the centralizers of semisimple elements in $\tilde{\mathbf{G}}^*$ are connected, there is $\tilde{g} \in \tilde{G}^*$ with $\tilde{g}\tilde{s}\tilde{g}^{-1} = \beta\tilde{s}$. By definition, $\tilde{g} \in \tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ and our claim follows from 4.1.2. As a consequence, $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ is not contained in any proper split F-stable Levi subgroup of $\tilde{\mathbf{G}}^*$ (use the fact that $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ acts irreducibly on each $V_{\mu_i}(\tilde{s})$; see Lemma 4.8). In turn, the same statement holds for $C_{\mathbf{G}^*}(s)^F$ by 3.1.

If two entries of the label $(\pi_0, \ldots, \pi_{e-1})$ of λ are distinct, the stabilizer $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_\lambda$ of λ in $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ is not transitive on $\{\mathbf{V}_{\mu_i}(\tilde{s}) \mid 0 \leq i \leq e-1\}$ any longer (as a proper subgroup of a transitive cyclic group acting on a set with more than 1 element is not transitive any more), and thus lies in a proper split F-stable Levi subgroup $\tilde{\mathbf{L}}^*$ of $\tilde{\mathbf{G}}^*$, such that $\tilde{\mathbf{L}}^*$ also contains $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Again, Condition (1) of Theorem 1.1 is satisfied. It follows from Corollary 3.6 that χ is Harish-Chandra induced from L.

Finally suppose that $\pi_0 = \pi_1 = \cdots = \pi_{e-1}$. Then $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ stabilizes λ, and Corollary 3.7(b) implies that χ is Harish-Chandra primitive. $\diamond$

5.3. **The special unitary groups.** As in the case of the special linear groups, we take $\tilde{\mathbf{G}} = \tilde{\mathbf{G}}^* = \mathrm{GL}_n(\mathbb{F})$, acting on the natural vector space $\mathbf{V} = \mathbb{F}^n$. Also, $\mathbf{G}^* = \mathrm{PGL}_n(\mathbb{F})$ and i^* is the canonical epimorphism. We let F denote the Frobenius endomorphism of $\mathbf{G}$ introduced in 4.2.1. Then $\tilde{G} = \tilde{\mathbf{G}}^F = (\tilde{\mathbf{G}}^*)^F = \tilde{G}^* = \mathrm{GU}_n(q) \leq \mathrm{GL}_n(q^2)$, $G = \mathbf{G}^F = \mathrm{SU}_n(q)$, and $G^* = \mathbf{G}^{*F} = \mathrm{PGU}_n(q)$. We assume that $n \geq 3$.

To present the main theorem of this subsection, we use the notation and concepts introduced in Lemmas 4.2, 4.3. Suppose that $s \in G^*$ and $\tilde{s} \in \tilde{G}^*$ are semisimple with $i^*(\tilde{s}) = s$. As in 4.2.1 we write $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}) := (i^*)^{-1}(C_{\mathbf{G}^*}(s)) = \{\tilde{g} \in \tilde{\mathbf{G}}^* \mid \tilde{g}\tilde{s}\tilde{g}^{-1} = \gamma s \text{ for some } \gamma \in \mathbb{F}^*\}$ and put $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s}) := \tilde{C}_{\mathbf{G}^*}(\tilde{s})/C_{\tilde{\mathbf{G}}^*}(\tilde{s})$. Recall from 3.3, that i^* induces an isomorphism $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F \to A_{\mathbf{G}^*}(s)^F$ which commutes with the actions of $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ on $\mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1])$ and of $A_{\mathbf{G}^*}(s)^F$ on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$, respectively, and induces, by transport of structure, an action of $A_{\mathbf{G}^*}(s)^F$ on $\mathcal{F}_{\tilde{s}}$.

Notice that the structure of the centralizer $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ and the action of $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ on this centralizer can be determined with Lemma 4.3(b).

The automorphism group of a finite general unitary group fixes every unipotent character of the group (see [21, Remarks on p. 159]). Therefore, the stabilizer $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_{\tilde{\lambda}}$ of a unipotent character $\tilde{\lambda}$ of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ can easily be found by inspection. By the remarks above, $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F_{\tilde{\lambda}} \cong A_{\mathbf{G}^*}(s)^F_{\lambda}$, if $\tilde{\lambda}$ is obtained from $\lambda \in \mathcal{E}(C_{\mathbf{G}^*}(s)^F, [1])$ by inflation.

Theorem 5.3. *Let $s \in G^* = \mathrm{PGU}_n(q)$ be semisimple and choose $\tilde{s} \in \mathrm{GU}_n(q)$ with $i^*(\tilde{s}) = s$. Let $\chi \in \mathcal{E}(G, [s])$ and $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ such that $[\chi] \leftrightarrow [\lambda]$.*

(a) *If there are $A_{\mathbf{G}^*}(s)^F$-orbits on $\mathcal{F}_{\tilde{s}}$ of type* (ls)*, then χ is Harish-Chandra imprimitive.*

(b) *If every $A_{\mathbf{G}^*}(s)^F$-orbit on $\mathcal{F}_{\tilde{s}}$ is of type* (u)*, then χ is Harish-Chandra primitive.*

(c) *Suppose that there is at least one $A_{\mathbf{G}^*}(s)^F$-orbit of type* (lt) *on $\mathcal{F}_{\tilde{s}}$. Then q is odd. Suppose further that there are no $A_{\mathbf{G}^*}(s)^F$-orbits on $\mathcal{F}_{\tilde{s}}$ of type* (ls)*. Define e such that $2e$ is the length of an orbit of type* (lt) *on $\mathcal{F}_{\tilde{s}}$, and such that $\nu_2(2e) \leq \nu_2(2e')$ for all other orbit lengths $2e'$ of this type. Then χ is Harish-Chandra primitive if and only if*

$$\nu_2\left(\frac{2e|A_{\mathbf{G}^*}(s)^F_{\lambda}|}{|A_{\mathbf{G}^*}(s)^F|}\right) > 0$$

(here, ν_2 denotes the 2-adic valuation of $\mathbb{Q}$).

(d) Theorem 1.1(b) *is satisfied for $G = \mathrm{SU}_n(q)$.*

Proof. Part (d) will be proved as we go along.

(a) This is an immediate consequence of Lemma 4.10(b) and Corollary 3.6. Notice that in this case Condition (1) of Theorem 1.1 is satisfied.

(b) Suppose that χ is Harish-Chandra imprimitive. Let $\mathbf{L}^*$ be a proper split F-stable Levi subgroup of $\mathbf{G}^*$ and let ϑ be a character of L such that $R_L^G(\vartheta) = \chi$. We may assume that $s \in L^*$ (see [4, Proposition 15.7]). Let ν denote the unipotent character of $C^\circ_{\mathbf{L}^*}(s)^F$ such that the $\tilde{L}$-orbit of ϑ corresponds to the $A_{\mathbf{L}^*}(s)^F$-orbit of ν under Lusztig's generalized Jordan decomposition of characters. It follows from (3) and Theorem 3.5 that $|A_{\mathbf{G}^*}(s)^F_{\lambda}| \leq |A_{\mathbf{L}^*}(s)^F_{\nu}|$.

We claim that $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F = C_{\tilde{\mathbf{L}}^*}(\tilde{s})^F$ and that $\nu \in [\lambda]$. (The second assertion of this claim will only be used in the proof of (c).) Suppose this claim has been proved. Then $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F \leq \tilde{\mathbf{L}}^*$. By Lemma 4.10(a), there is $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^\dagger$. This, however, contradicts our hypothesis.

It remains to prove the claim. Let $\tilde{\vartheta} \in \mathcal{E}(\tilde{L}, [\tilde{s}])$ denote an irreducible constituent of $\mathrm{Ind}_L^{\tilde{L}}(\vartheta)$ (see [1, Proposition 11.7(b)]). Put

$C_{\tilde{G}} := C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ and $C_{\tilde{L}} := C_{\tilde{\mathbf{L}}^*}(\tilde{s})^F$. We let $\tilde{\lambda}$ and $\tilde{\nu}$ denote the unipotent characters of $C_{\tilde{G}}$ and $C_{\tilde{L}}$, respectively, which are obtained from λ, respectively ν, by inflation. As $\tilde{\mathbf{G}}^* = \mathrm{GL}_n(\mathbb{F})$, the centralizers $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ and $C_{\tilde{\mathbf{L}}^*}(\tilde{s})$ are regular subgroups of $\tilde{\mathbf{G}}^*$. Hence there is a linear character $\hat{s}$ of $C_{\tilde{G}}$, associated to $\tilde{s}$ as in [4, (8.19)], such that

$$\tilde{\vartheta} = \pm R_{C_{\tilde{L}}}^{\tilde{L}}(\hat{s} \cdot \tilde{\nu}),$$

where $\hat{s}$ is viewed as a character of $C_{\tilde{L}} \leq C_{\tilde{G}}$ through restriction (see [4, Proposition 15.10(ii)]). By the transitivity of twisted induction and by [4, (8.20)], we find that

$$\begin{aligned} R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta}) &= \pm R_{\tilde{L}}^{\tilde{G}}(R_{C_{\tilde{L}}}^{\tilde{L}}(\hat{s} \cdot \tilde{\nu})) \\ &= \pm R_{C_{\tilde{L}}}^{\tilde{G}}(\hat{s} \cdot \tilde{\nu}) \\ &= \pm R_{C_{\tilde{G}}}^{\tilde{G}}(R_{C_{\tilde{L}}}^{C_{\tilde{G}}}(\hat{s} \cdot \tilde{\nu})) \\ &= \pm R_{C_{\tilde{G}}}^{\tilde{G}}(\hat{s} \cdot R_{C_{\tilde{L}}}^{C_{\tilde{G}}}(\tilde{\nu})). \end{aligned}$$

Let $\tilde{\rho}$ be an irreducible constituent of $R_{C_{\tilde{L}}}^{C_{\tilde{G}}}(\tilde{\nu})$. By the above equation,

$$\pm R_{C_{\tilde{G}}}^{\tilde{G}}(\hat{s} \cdot \tilde{\rho})$$

is an irreducible constituent of $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$, and hence, by Lemma 3.3, of $\mathrm{Ind}_G^{\tilde{G}}(\chi)$. As $[\chi] \leftrightarrow [\lambda]$, we have $\rho \in [\lambda]$, where ρ is the unipotent character of $C_{\mathbf{G}^*}(s)^F$ such that $\tilde{\rho}$ is inflated from ρ (cf. 3.3). Now $C_{\tilde{\mathbf{L}}^*}(\tilde{s})$ is a split Levi subgroup of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$, so that the map $R_{C_{\tilde{L}}}^{C_{\tilde{G}}}$ is Harish-Chandra induction. As $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$ is multiplicity free by (5) and Lusztig's result (see [21, Section 10] and [4, Proposition 15.11]), the same is true for $R_{C_{\tilde{L}}}^{C_{\tilde{G}}}(\tilde{\nu})$. Now $C_{\tilde{G}}$ is a direct product $C_1 \times \cdots \times C_l$, where the C_i correspond to the $A_{\mathbf{G}^*}(s)^F$-orbits on $\mathcal{F}_{\tilde{s}}$. Each of the C_i is of the form (11) or (10), and $A_{\mathbf{G}^*}(s)^F$ acts on $C_{\tilde{G}}$ by transitively permuting the direct factors of the C_i. Apply Lemma 2.1 to each of the C_i. The claim follows from this in conjunction with [15, Lemma 8.2].

(c) The fact that q is odd follows from Lemma 4.3(b). Let $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^\dagger$ and let $2e'$ denote the length of the $A_{\mathbf{G}^*}(s)^F$-orbit of μ. Thus $2e' = |A_{\mathbf{G}^*}(s)^F : S|$, where S denotes the stabilizer of μ in $A_{\mathbf{G}^*}(s)^F$. Therefore,

$$\frac{2e'|A_{\mathbf{G}^*}(s)^F_\lambda|}{|A_{\mathbf{G}^*}(s)^F|} = \frac{|A_{\mathbf{G}^*}(s)^F_\lambda|}{|S|}.$$

As $A_{\mathbf{G}^*}(s)^F$ is cyclic, $\nu_2(|A_{\mathbf{G}^*}(s)^F_\lambda|/|S|) \leq 0$, if and only if S contains a Sylow 2-subgroup of $A_{\mathbf{G}^*}(s)^F_\lambda$. This is the case if and only if the length $|A_{\mathbf{G}^*}(s)^F_\lambda|/|S \cap A_{\mathbf{G}^*}(s)^F_\lambda|$ of the $A_{\mathbf{G}^*}(s)^F_\lambda$-orbit of μ is odd. In turn, this is equivalent to the statement that this orbit does not contain $\mu^\dagger$.

Suppose first that $\nu_2(2e|A_{\mathbf{G}^*}(s)^F_\lambda|/|A_{\mathbf{G}^*}(s)^F|) \leq 0$ and choose μ such that its orbit has length $2e$. By the above, the $A_{\mathbf{G}^*}(s)^F_\lambda$-orbit of μ does not contain $\mu^\dagger$. Let $\tilde{\mathbf{L}}^*$ denote the stabilizer of the corresponding pair of totally isotropic subspaces of $\mathbf{V}$. Then $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$ and $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$, i.e. $C_{\mathbf{G}^*}(s)^F_\lambda \leq \mathbf{L}^*$. If follows from Corollary 3.6, that χ is Harish-Chandra induced from L. Again, Condition (1) of Theorem 1.1 is satisfied.

Now suppose that $\nu_2(2e|A_{\mathbf{G}^*}(s)^F_\lambda|/|A_{\mathbf{G}^*}(s)^F|) > 0$. If follows from our choice of e and the above considerations, that for every $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^\dagger$ there is an element in $A_{\mathbf{G}^*}(s)^F_\lambda$ mapping μ to $\mu^\dagger$. We aim to show that under these conditions χ is Harish-Chandra primitive.

Suppose that this is not the case and adopt the notation of the proof of (b). By the claim in that proof, $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F = C_{\tilde{\mathbf{L}}^*}(\tilde{s})^F$ and $\nu \in [\lambda]$. This implies that $|A_{\mathbf{L}^*}(s)^F_\nu| \leq |A_{\mathbf{G}^*}(s)^F_\nu| = |A_{\mathbf{G}^*}(s)^F_\lambda|$. As we also have $|A_{\mathbf{G}^*}(s)^F_\lambda| \leq |A_{\mathbf{L}^*}(s)^F_\nu|$, it follows that $|A_{\mathbf{G}^*}(s)^F_\lambda| = |A_{\mathbf{L}^*}(s)^F_\nu|$ and this number divides $|A_{\mathbf{L}^*}(s)^F|$. Hence $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$, as each of these groups is a subgroup of the cyclic group $A_{\mathbf{G}^*}(s)^F$. Now $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F \leq \tilde{\mathbf{L}}^*$, and $\mu^\dagger$ lies in the $A_{\mathbf{G}^*}(s)^F_\lambda$-orbit of μ for every $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^\dagger$. This contradicts Lemma 4.10(a). $\diamond$

5.4. **The symplectic groups.** Let $G = \mathrm{Sp}_{2m}(q)$ with $m \geq 2$ and q odd (see Proposition 5.1). In this case we take $\mathbf{G} = \mathrm{Sp}_{2m}(\mathbb{F})$ and $\mathbf{G}^* = \mathrm{SO}_{2m+1}(\mathbb{F})$ with the standard Frobenius map F raising every matrix entry of an element of $\mathbf{G}$, respectively $\mathbf{G}^*$, to its qth power. By $\mathbf{V}$ we denote the natural $\mathbb{F}$-vector space for $\mathbf{G}^*$, i.e. $\mathbf{V} = \mathbb{F}^{2m+1}$.

Proposition 5.4. *Let $s \in G^* = \mathrm{SO}_{2m+1}(q)$ be semisimple. Then the following statements are equivalent.*

(1) *The centralizer $C_{\mathbf{G}^*}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$.*

(2) *The connected centralizer $C^\circ_{\mathbf{G}^*}(s)$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$.*

(3) *The minimal polynomial of s (viewed as a linear transformation on $\mathbb{F}_q^{2m+1}$) has an irreducible factor μ with $\mu \neq \mu^*$.*

Proof. Trivially, (1) implies (2).

Suppose that (2) holds. It follows from [11, (1.13) and subsequent remarks] that $H := C^\circ_{\mathbf{G}^*}(s)^F$ satisfies the hypotheses of Lemma 4.9. Hence there is an irreducible factor μ of the minimal polynomial of s

such that $V_\mu(s)$ is totally isotropic. In particular, $\mu \neq \mu^*$. Thus (2) implies (3).

Now assume (3). Then the subspaces $\mathbf{V}_\mu(s)$ and $\mathbf{V}_{\mu^*}(s)$ are totally isotropic and fixed by $C_{\mathbf{G}^*}(s)$. As the stabilizer of this pair of subspaces is a proper split F-stable Levi subgroup of $\mathbf{G}^*$, this implies (1). $\diamond$

It is well known and easy to see that the centralizer $C_{\mathbf{G}^*}(s)$ of a semisimple element $s \in G^*$ is connected if and only if -1 is not an eigenvalue of s (in its action on $\mathbf{V}$).

Corollary 5.5. *Let $s \in \mathrm{SO}_{2m+1}(q)$ be semisimple. Then every element of $\mathcal{E}(\mathrm{Sp}_{2m}(q), [s])$ is Harish-Chandra imprimitive, if and only if the minimal polynomial of s has an irreducible factor μ with $\mu \neq \mu^*$.*

Otherwise, every element of $\mathcal{E}(\mathrm{Sp}_{2m}(q), [s])$ is Harish-Chandra primitive.

Proof. This follows from Proposition 5.4 with the help of Corollaries 3.6 and 3.11. $\diamond$

5.5. **The odd dimensional spin groups.** Let $G = \mathrm{Spin}_{2m+1}(q)$, $m \geq 2$, and assume that q is odd. (As $\mathrm{Spin}_{2m+1}(q) \cong \Omega_{2m+1}(q) \cong \mathrm{Sp}_{2m}(q)$ if q is even, we may assume that q is odd here; see Proposition 5.1.) In this case we take $\mathbf{G} = \mathrm{Spin}_{2m+1}(\mathbb{F})$ and $\tilde{\mathbf{G}}$ the connected component of the corresponding Clifford group (see [2, §9, n°5] for the definition of the Clifford group). Then $\tilde{\mathbf{G}}^* = \mathrm{CSp}_{2m}(\mathbb{F})$ (see [19, 8.1]) and $\mathbf{G}^* = \mathrm{PCSp}_{2m}(\mathbb{F})$ with i^* the natural epimorphism. We write $\mathbf{V} := \mathbb{F}^{2m}$ for the natural vector space of $\tilde{\mathbf{G}}^*$, and assume that $\mathbf{V}$ is equipped with a non-degenerate symplectic form defined over $\mathbb{F}_q$. Thus $\tilde{\mathbf{G}}^* \leq \mathrm{GL}_{2m}(\mathbb{F})$ is invariant under the standard Frobenius map F raising every matrix entry of an element of $\tilde{\mathbf{G}}^*$ to its qth power; we let the Frobenius map on $\tilde{\mathbf{G}}$ be dual to the latter.

Theorem 5.6. *Let $s \in G^*$ be semisimple and let $\tilde{s} \in \tilde{G}^*$ with $s = i^*(\tilde{s})$. Suppose that $\tilde{s}$ has multiplier α. Then one of the following occurs.*

(a) *For all $\mu \in \mathcal{F}_{\tilde{s}}$ we have $\mu = \mu^{*\alpha}$; in this case every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive.*

(b) *There exists $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^{*\alpha} \neq \mu'$; in this case every element of $\mathcal{E}(G, [s])$ is Harish-Chandra imprimitive.*

(c) *Every $\mu \in \mathcal{F}_{\tilde{s}}$ satisfies $\mu = \mu^{*\alpha}$ or $\mu' = \mu^{*\alpha}$, and there exists $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^{*\alpha}$. Here, we distinguish two cases.*

(i) *If $\tilde{s}$ is not conjugate to $-\tilde{s}$ in $\tilde{G}^*$, then every element of $\mathcal{E}(G, [s])$ is Harish-Chandra imprimitive.*

(ii) *Suppose that $\tilde{s}$ is conjugate to $-\tilde{s}$ in $\tilde{G}^*$. Then $|A_{\mathbf{G}^*}(s)^F| = 2$. Let $\chi \in \mathcal{E}(G, [s])$, and let $\lambda \in \mathcal{E}(C^\circ_{G^*}(s), [1])$ be such that $[\chi] \leftrightarrow$*

$[\lambda]$ *under Lusztig's generalized Jordan decomposition of characters (see 3.3). Then χ is Harish-Chandra imprimitive if and only if $A_{\mathbf{G}^*}(s)^F_\lambda$ is trivial.*

Proof. (a) By Lemmas 4.12(c) and 3.1, the hypothesis implies that $C^\circ_{\mathbf{G}^*}(s) = i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s}))$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$. It follows from Corollary 3.11 that every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive.

(b) In this case, $C_{\mathbf{G}^*}(s) = i^*(\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}))$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$ by Lemmas 4.12(b) and 3.1. The claim follows from [15, Theorem 7.3].

(c) By assumption, there is $\mu \in \mathcal{F}_{\tilde{s}}$ with $\mu \neq \mu^{*\alpha}$. Hence $C^\circ_{\mathbf{G}^*}(s) = i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s}))$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$ by Lemmas 4.12(c) and 3.1. If $\tilde{s}$ is not conjugate to $-\tilde{s}$ in $\tilde{G}^*$, then $\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s}) = C_{\tilde{\mathbf{G}}^*}(\tilde{s})$, and thus $C_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s)$, as $(i^*)^{-1}(C_{\mathbf{G}^*}(s)) = \tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})$. The result follows as in (b).

Now suppose that $\tilde{s}$ is conjugate to $-\tilde{s}$ by some $\tilde{h} \in \tilde{G}^*$. Then $\tilde{s}$ satisfies the conditions of Proposition 4.7(a). In particular, $A_{\mathbf{G}^*}(s)^F$ has order 2 (see 3.3). Moreover, Lemma 4.12(a)(ii) implies that $C_{\mathbf{G}^*}(s)^F = i^*(\tilde{C}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F)$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$. The assertion follows from Corollaries 3.7(b) and 3.6. $\diamond$

We finally explain how to determine $A_{\mathbf{G}^*}(s)^F_\lambda$ in the situation of Theorem 5.6(c)(ii).

Remark 5.7. Assume the hypotheses of Theorem 5.6(c)(ii). Table 5 lists the various possibilities for $\tilde{s}$ by the types of the elements of $\mathcal{F}_{\tilde{s}}$ (the elements are denoted by s in that table). The table also gives the labels for the unipotent characters $\lambda \in \mathcal{E}(C_{[\tilde{\mathbf{G}}^*,\tilde{\mathbf{G}}^*]}(\tilde{s})^F, [1])$ and the labels of their conjugates λ^a where a is a generator of $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ (for the definition of the latter group see 4.3.3). The conjugates are determined from the action given in Lemma 4.6(b), and the fact that the unipotent character of the nearly simple components involved in $C_{[\tilde{\mathbf{G}}^*,\tilde{\mathbf{G}}^*]}(\tilde{s})^F$ are invariant under automorphisms (see [21, Remarks on p. 159] and [22, Theorem 2.5]). As labels for the unipotent characters we use symbols as defined in [20, Appendix] (where the condition $k \geq 4$ is imposed; but the symbols can also be defined and used for $k = 2$), respectively partitions.

From this information it is easy to read off $A_{\mathbf{G}^*}(s)^F_\lambda$ as follows. The unipotent characters of $C^\circ_{\mathbf{G}^*}(s)^F$ may be identified with those of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$, and the latter with those of $C_{[\tilde{\mathbf{G}}^*,\tilde{\mathbf{G}}^*]}(\tilde{s})^F$. This identification is compatible with the action of $A_{\mathbf{G}^*}(\tilde{s})^F$ on the first of these sets, and with the action of $\tilde{A}_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ on the latter.

5.6. **The even dimensional spin groups.** Assume that $\mathrm{char}(\mathbb{F})$ is odd (see Proposition 5.1). Let $\mathbf{G} = \mathrm{Spin}_{2m}(\mathbb{F})$, $m \geq 2$, defined with respect to a non-degenerate quadratic form on $\mathbf{V} = \mathbb{F}^{2m}$. Let $\check{\mathbf{G}}$ denote the connected component of the Clifford group with respect to this form (see [2, §9, n°5]), and choose a regular embedding $k : \check{\mathbf{G}} \to \tilde{\mathbf{G}}$. Denote by j the embedding $j : \mathbf{G} \to \check{\mathbf{G}}$. Then $i := k \circ j : \mathbf{G} \to \tilde{\mathbf{G}}$ is a regular embedding. We have $\check{\mathbf{G}}^* = \mathrm{CSO}_{2m}(\mathbb{F})$ and $\mathbf{G}^* = \mathrm{PCSO}_{2m}(\mathbb{F})$, the quotient group of $\mathrm{CSO}_{2m}(\mathbb{F})$ modulo its center. Moreover, $j^* : \check{\mathbf{G}}^* \to \mathbf{G}^*$ is the canonical epimorphism, and i^* factors as

$$\begin{tikzcd} \tilde{\mathbf{G}}^* \arrow[r, "k^*"] \arrow[dr, "i^*"'] & \check{\mathbf{G}}^* \arrow[d, "j^*"] \\ & \mathbf{G}^* \end{tikzcd}$$

Let F' and F'' denote Frobenius morphisms of $\tilde{\mathbf{G}}$ such that $\mathbf{G}^{F'} = \mathrm{Spin}^{+}_{2m}(q)$ and $\mathbf{G}^{F''} = \mathrm{Spin}^{-}_{2m}(q)$ and such that the induced morphisms on $\check{\mathbf{G}}^*$ are as in 4.3.1. Let F be one of F' or F''. The groups $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{G}}^*$ are only used in the proof, but not in the statement of Theorem 5.10 below.

Lemma 5.8. *Let $s \in G^*$ be semisimple and let $\check{s} \in \check{G}^*$ with $s = j^*(\check{s})$. Then*

$$C^{\circ}_{\mathbf{G}^*}(s) = j^*(C^{\circ}_{\check{\mathbf{G}}^*}(\check{s}))$$

and

$$C^{\circ}_{\check{\mathbf{G}}^*}(\check{s}) = (j^*)^{-1}(C^{\circ}_{\mathbf{G}^*}(s)).$$

Moreover, putting

$$\tilde{C}_{\check{\mathbf{G}}^*}(\check{s}) := \{\check{g} \in \check{\mathbf{G}}^* \mid \check{g}\check{s}\check{g}^{-1} = \pm\check{s}\} \tag{16}$$

(cf. 4.3.3*) and*

$$\tilde{A}_{\check{\mathbf{G}}^*}(\check{s}) := \tilde{C}_{\check{\mathbf{G}}^*}(\check{s})/C^{\circ}_{\check{\mathbf{G}}^*}(\check{s}), \tag{17}$$

then j^* induces an F-equivariant isomorphism $j^* : \tilde{A}_{\check{\mathbf{G}}^*}(\check{s}) \to A_{\mathbf{G}^*}(s)$, compatible with the actions of these groups on the unipotent characters of $C^{\circ}_{\check{\mathbf{G}}^*}(\check{s})^F$, respectively $C^{\circ}_{\mathbf{G}^*}(s)^F$.

Proof. Let $\tilde{s} \in \tilde{G}^*$ with $k^*(\tilde{s}) = \check{s}$. We have $j^*(k^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s}))) = i^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C^{\circ}_{\mathbf{G}^*}(s)$, and $k^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C^{\circ}_{\check{\mathbf{G}}^*}(\check{s})$ (see [1, p. 36]). Hence $j^*(C^{\circ}_{\check{\mathbf{G}}^*}(\check{s})) = C^{\circ}_{\mathbf{G}^*}(s)$. As the kernel of j^* is contained in $C^{\circ}_{\check{\mathbf{G}}^*}(\check{s})$, we obtain the second claim. The third follows from the second, as $(j^*)^{-1}(C_{\mathbf{G}^*}(s)) = \tilde{C}_{\check{\mathbf{G}}^*}(\check{s})$. ◇

In the theorem below and its proof we will use the following notation. Let $s \in G^*$, choose $\check{s} \in \check{G}^*$ with $j^*(\check{s}) = 2$, and choose $\tilde{s} \in \tilde{G}^*$ with

$k^*(\tilde{s}) = \check{s}$. By [1, p. 36] and Lemma 5.8, we have surjective homomorphisms

$$C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F \xrightarrow{k^*} C^\circ_{\check{\mathbf{G}}^*}(\check{s})^F \xrightarrow{j^*} C^\circ_{\mathbf{G}^*}(s)^F.$$

Next, for $\chi \in \mathcal{E}(G, [s])$, let $\tilde{\chi}$ denote an irreducible constituent of $\mathrm{Ind}_G^{\tilde{G}}(\chi)$ lying in $\mathcal{E}(\tilde{G}, [\tilde{s}])$, and let $\check{\chi}$ denote an irreducible constituent of $\mathrm{Ind}_G^{\check{G}}(\chi)$ such that $\tilde{\chi}$ occurs in $\mathrm{Ind}_{\check{G}}^{\tilde{G}}(\check{\chi})$. Then $\check{\chi} \in \mathcal{E}(\check{G}, [\check{s}])$ (see the results summarized in 3.3). Fix a Jordan decomposition of characters between $\mathcal{E}(\tilde{G}, [\tilde{s}])$ and $\mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1])$ and let $\tilde{\lambda}$ denote the unipotent character of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ corresponding to $\tilde{\chi}$. Then the kernel of $\tilde{\lambda}$ contains the kernels of $k^*|_{\tilde{G}^*}$ and of $i^*|_{\tilde{G}^*}$, and we let $\check{\lambda}$ und λ denote the unipotent characters of $C^\circ_{\check{\mathbf{G}}^*}(\check{s})^F$, respectively $C^\circ_{\mathbf{G}^*}(s)^F$, obtained from $\tilde{\lambda}$ by deflation. Then $[\check{\chi}] \leftrightarrow [\check{\lambda}]$ and $[\chi] \leftrightarrow [\lambda]$ in Lusztig's generalized Jordan decomposition of characters. Notice that $\check{\lambda}$ is uniquely determined by its restriction to $C^\circ_{[\check{\mathbf{G}}^*, \check{\mathbf{G}}^*]}(\check{s})^F$, and we will usually identify $\check{\lambda}$ with this restriction. The configuration considered is depicted in Table 9.

Table 9. The notation in Theorem 5.10 (explanations in the paragraph following Lemma 5.8)

$$\begin{tikzcd}
\tilde{G} & \mathcal{E}(\tilde{G}, [\tilde{s}]) \ni \tilde{\chi} \arrow[r, leftrightarrow] & \tilde{\lambda} \in \mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F, [1]) \arrow[d] & \tilde{G}^* \arrow[d, "k^*"] \\
\check{G} \arrow[u, "k"] & \mathcal{E}(\check{G}, [\check{s}]) \supseteq [\check{\chi}] \arrow[u] \arrow[r, leftrightarrow] & {[\check{\lambda}]} \subseteq \mathcal{E}(C_{\check{\mathbf{G}}^*}(\check{s})^F, [1]) \arrow[d] & \check{G}^* \arrow[d, "j^*"] \\
G \arrow[u, "j"] & \mathcal{E}(G, [s]) \supseteq [\chi] \arrow[u] \arrow[r, leftrightarrow] & {[\lambda]} \subseteq \mathcal{E}(C_{\mathbf{G}^*}(s)^F, [1]) & G^*
\end{tikzcd}$$

The main result of this subsection is formulated in terms of the F-stable semisimple elements of the conformal group $\check{\mathbf{G}}^*$, and we will make use of the notation introduced in 4.3.6 and 4.3.7 with respect to this group and the action on its natural vector space. We will, of course, also use the notation of Lemma 5.8. The following remark describes how to determime the action of $A_{\mathbf{G}^*}(s)^F$ on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$. This information will be used at various places in the proof without further comment.

Remark 5.9. Tables 6, 7 and 8 list the various possibilities for $\check{s}$ by the types of the elements of $\mathcal{F}_{\check{s}}$ (the elements are denoted by s there). These tables also give the labels for the elements of $\mathcal{E}(C^\circ_{[\check{\mathbf{G}}^*, \check{\mathbf{G}}^*]}(\check{s})^F, [1])$ and the labels of their conjugates under $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$ (for the definition of the latter group see (16) and (17)). Again, the labels are symbols as in [20, Appendix]. (There, the condition $k \geq 8$ is imposed; but the symbols can also be defined and used for $k = 2, 4, 6$. For example, if $k = 2$, there is exactly one relevant symbol, corresponding to the fact that the unique unipotent character of $\mathrm{SO}_2^{\pm}(q)$ is the trivial character.) A pair of symbols labels the two factors of an outer tensor product of unipotent characters in a direct product of groups such as, e.g. $\mathrm{SO}_k^+(q) \times \mathrm{SO}_k^+(q)$.

In the situation of Table 6 we have $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F = \langle a, b\rangle$ if $X^2 - \alpha$ divides the minimal polynomial of $\check{s}$ (i.e. if $k \neq 0$), and $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F = \langle a\rangle$, otherwise. In Tables 7 and 8 we have $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F = \langle a\rangle$. The conjugates are determined from the action given in Lemma 4.6(b), and the fact that the unipotent characters of the nearly simple components involved in $C^\circ_{[\tilde{\mathbf{G}}^*,\check{\mathbf{G}}^*]}(\check{s})^F$ are invariant under automorphisms, except for graph automorphisms of components equal to $\mathrm{SO}_k^+(q)$ for k divisible by 4 (see [21, Remarks on p. 159] and [22, Theorem 2.5], which also gives the action of the graph automorphisms in the latter case). The unipotent characters of $\mathrm{SO}_k^+(q)$ for even k (including $k = 2$, where only the trivial character is unipotent) are labelled by symbols Λ and copies Λ' thereof, where we use the convention that Λ and Λ' label the same unipotent character unless Λ is degenerate (in which case k is divisible by 4). Then the graph automorphism of $\mathrm{SO}_k^+(q)$ swaps the characters labelled by Λ and Λ' and fixes the other unipotent characters (see [22, Theorem 2.5]). From this information it is easy to read off $A_{\mathbf{G}^*}(s)^F_\lambda$ as explained in Remark 5.7.

We now come to our main result for the even dimensional spin groups.

Theorem 5.10. *Let $s \in G^*$ be semisimple and let $\check{s} \in \check{G}^*$ with $s = j^*(\check{s})$. Suppose that $\check{s}$ has multiplier α. Let $\chi \in \mathcal{E}(G, [s])$, and let $\lambda \in \mathcal{E}(C^\circ_{G^*}(s), [1])$ be such that $[\chi] \leftrightarrow [\lambda]$ under Lusztig's generalized Jordan decomposition of characters (see* 3.3*). Let $\check{\lambda} \in \mathcal{E}(C^\circ_{\check{\mathbf{G}}^*}(\check{s})^F, [1])$ denote the inflation of λ over the kernel of $j^*|_{\check{G}^*}$. Then one of the following occurs.*

(a) *For all $\mu \in \mathcal{F}_{\check{s}}$ we have $\mu = \mu^{*\alpha}$ and $\check{s}$ is not as in Lemma* 4.12(a)(i.2) *Then χ is Harish-Chandra primitive.*

(b) *There exists $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha} \neq \mu'$; in this case χ is Harish-Chandra imprimitive.*

(c) *Either $\check{s}$ is as in* Lemma 4.12(a)(i.2) *or every $\mu \in \mathcal{F}_{\check{s}}$ satisfies $\mu = \mu^{*\alpha}$ or $\mu' = \mu^{*\alpha}$, and there exists $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha}$. Then there is a proper split F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ such that $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$. Let $\mathbf{L}$ denote a split F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$.*

Then χ is Harish-Chandra induced from L if and only if $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$. To investigate this latter condition more closely, we distinguish three cases.

(i) *Suppose that $\check{s}$ is not conjugate to $-\check{s}$ in $\check{G}^*$. If $\check{s}$ is as in Lemma* 4.12(a)(i.2)*, then χ is Harish-Chandra induced from L if either $\check{s}$ is exceptional, or if the following conditions are satisfied: There is $\zeta \in \mathbb{F}_q$ with $\alpha = \zeta^2$ such that $\nu := X - \zeta$ and $\nu' := X + \zeta$ occur with multiplicity 2 and $4k' > 0$, respectively, in the characteristic polynomial of $\check{s}$. Moreover, $(\check{\mathbf{G}}^*_\nu(\check{s}), F)$ and $(\check{\mathbf{G}}^*_{\nu'}(\check{s}), F)$ are of plus-type, and the factor of $\check{\lambda}$ corresponding to $V_{\nu'}(\check{s})$ is labelled by a degenerate symbol. Otherwise, χ is Harish-Chandra primitive.*

If $\check{s}$ is not as in Lemma 4.12(a)(i.2)*, then $C_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$ and thus $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$. Hence χ is Harish-Chandra induced from L.*

(ii) *Suppose that $\check{s}$ is conjugate to $-\check{s}$ in $\check{G}^*$ and that $X^2 - \alpha$ does not divide the minimal polynomial of $\check{s}$. Then $|A_{\mathbf{G}^*}(s)^F| = 2$ and $|A_{\mathbf{L}^*}(s)^F| = 1$. In particular, χ is Harish-Chandra induced from L if $|A_{\mathbf{G}^*}(s)^F_\lambda| = 1$, and is Harish-Chandra primitive, otherwise.*

(iii) *Suppose that $\check{s}$ is conjugate to $-\check{s}$ in $\check{G}^*$ and that $X^2 - \alpha$ divides the minimal polynomial of $\check{s}$. Then $|A_{\mathbf{G}^*}(s)^F| = 4$.*

If $\mu = \mu^{*\alpha}$ for all $\mu \in \mathcal{F}_{\check{s}}$, then χ is not Harish-Chandra induced from L. If there is $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha}$, then $|A_{\mathbf{L}^*}(s)^F| = 2$ and the following statements hold.

(iii.1) *If m is odd, then χ is Harish-Chandra induced from L if and only if λ is not fixed by $|A_{\mathbf{G}^*}(s)^F|$.*

(iii.2) *Suppose that m is even. Then $G = \mathrm{Spin}^+_{2m}(q)$ and $A_{\mathbf{G}^*}(s)^F = \langle a, b\rangle$ is a Klein four group. We may choose notation such that $b \in A_{\mathbf{L}^*}(s)^F$, and a denotes the image of an element of $\check{G}^*$ which conjugates $\check{s}$ to $-\check{s}$. Then χ is Harish-Chandra induced from L, if and only if neither a nor ab fix λ.*

(d) *Suppose that $\check{s}$ and $\mathbf{L}^*$ are as in* (c) *and that χ is not Harish-Chandra induced from L. Then χ is Harish-Chandra primitive. In particular,* Theorem 1.1(b) *holds for G.*

Proof. (a) Suppose that χ is Harish-Chandra imprimitive. Let $\mathbf{L}^*$ be a proper split F-stable Levi subgroup of $\mathbf{G}^*$, and let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$. Let ϑ be a character of L such that $R_L^G(\vartheta) = \chi$. We may assume that $s \in L^*$ and that $\vartheta \in \mathcal{E}(L, [s])$ (see [4, Proposition 15.7]). Using the notation and the statements of Theorem 3.5, we find that $c(\vartheta) \geq c(\chi)$. Suppose first that $c(\vartheta) = c(\chi)$. Then $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$ by Theorem 3.5, and we conclude with Lemmas 5.8 and 3.1 that $C^\circ_{\check{\mathbf{G}}^*}(\check{s}) = (j^*)^{-1}(C^\circ_{\mathbf{G}^*}(s))$ is contained in a proper split

F-stable Levi subgroup of $\check{\mathbf{G}}^*$. As $\check{s}$ is not as in Lemma 4.12(a)(i.2), we have arrived at a contradiction.

Hence $c(\vartheta) > c(\chi)$. Now $c(\chi) \leq |A_{\mathbf{G}^*}(s)^F| \leq |Z(\mathbf{G})/Z^\circ(\mathbf{G})| \leq 4$, where the first inequality follows from (3), and by Corollary 3.9 and Theorem 3.5 we cannot have $c(\vartheta)/c(\chi) = 2$. Thus $c(\chi) = 1$ and $c(\vartheta) = 4$. As $c(\vartheta) \leq |A_{\mathbf{L}^*}(s)^F| \leq |A_{\mathbf{G}^*}(s)^F|$, we have in particular that $|A_{\mathbf{G}^*}(s)^F| = 4$. Applying Lemma 5.8, we find that $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$ has order 4, so that $\check{s}$ is conjugate to $-\check{s}$ in $\check{G}$. By (3) we have $1 = c(\chi) = |A_{\mathbf{G}^*}(s)^F_\lambda|$, i.e. the orbit of λ under $A_{\mathbf{G}^*}(s)^F$ has length 4. Transferring the situation to $\check{\mathbf{G}}^*$ with the help of Lemma 5.8, we see that the orbit of $\check{\lambda}$ under $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$ has length 4. With the notation introduced in 4.3.6, put $\check{\mathbf{G}}^*_{\tilde{\mu}} := \check{\mathbf{G}}^*_{\tilde{\mu}}(\check{s})$ and $\check{C}_{\tilde{\mu}} := C^\circ_{\check{\mathbf{G}}^*_{\tilde{\mu}}}(\check{s}_{\tilde{\mu}})^F$, where $\mu \in \mathcal{F}_{\check{s}}$, and $\tilde{\mu} = \mu$ if $\mu = \mu'$, and $\tilde{\mu} = \mu\mu'$, otherwise. Then $C^\circ_{\check{\mathbf{G}}^*}(\check{s})^F$ is the direct product of the groups $\check{C}_{\tilde{\mu}}$, and $\check{\lambda}$ is the outer tensor product of unipotent characters $\check{\lambda}_{\tilde{\mu}}$ of $\check{C}_{\tilde{\mu}}$. Moreover, $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$ fixes each of the groups $\check{C}_{\tilde{\mu}}$. By Poposition 4.7, the characters $\check{\lambda}_{\tilde{\mu}}$ lie in $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$ orbits of lenghts 1, 2 or 4. As $\check{\lambda}$ lies in an orbit of length 4, one of the $\check{\lambda}_{\tilde{\mu}}$ must lie in an orbit of length 4. We are thus in the following situation: $F = F'$ and $\mathcal{F}_{\check{s}}$ contains an element $\nu = X - \zeta$ of Type (I), which occurs with even multiplicity $k \geq 2$ in the characteristic polynomial of $\check{s}$ (in particular, $\alpha = \zeta^2$ is a square in $\mathbb{F}_q$), and the restriction of $\check{\lambda}_{\tilde{\nu}}$ to $C^\circ_{[\check{\mathbf{G}}^*_{\tilde{\nu}},\check{\mathbf{G}}^*_{\tilde{\nu}}]}(\check{s}_{\tilde{\nu}})^F \cong \mathrm{SO}_k^+(q) \times \mathrm{SO}_k^+(q)$ is labelled by a pair of symbols (Λ_1, Λ_2), where Λ_1 and Λ_2 are both degenerate and label two different unipotent characters of $\mathrm{SO}_k^+(q)$ (in particular, $k \geq 4$). Here, of course, the pair (Λ_1, Λ_2) labels the outer tensor product of the characters of $\mathrm{SO}_k^+(q)$ corresponding to Λ_1 and Λ_2, respectively.

As $c(\vartheta)/c(\chi) = 4$, we conclude from Theorem 3.5 that there is an irreducible constituent $\tilde{\vartheta} \in \mathcal{E}(\tilde{L}, [\tilde{s}])$ of $\mathrm{Ind}_L^{\tilde{L}}(\vartheta)$ such that $R_{\tilde{L}}^{\tilde{G}}(\tilde{\vartheta})$ is a sum of four irreducible characters of equal degrees. Now put $C_{\tilde{G}} := C_{\tilde{\mathbf{G}}^*}(\tilde{s})^F$ and $C_{\tilde{L}} := C_{\tilde{\mathbf{L}}^*}(\tilde{s})^F$ Let $\tilde{\kappa} \in \mathcal{E}(C_{\tilde{L}}, [1])$ correspond to $\tilde{\vartheta}$ under Lusztig's Jordan decomposition of characters. The latter commutes with Harish-Chandra induction (see [9, p. 1049–1050]) and thus $R_{C_{\tilde{L}}}^{C_{\tilde{G}}}(\tilde{\kappa})$ is a sum of four irreducible characters, and all of these have the same degree. Now $k^*(C_{\tilde{G}}) = C^\circ_{\check{\mathbf{G}}^*}(\check{s})^F =: C_{\check{G}}$ by Lemma 5.8. Similarly, $k^*(C_{\tilde{L}}) = C^\circ_{\check{\mathbf{L}}^*}(\check{s})^F =: C_{\check{L}}$. Let $\check{\kappa}$ denote the unipotent character of $C_{\check{L}}$ such that $\tilde{\kappa}$ is the inflation of $\check{\kappa}$ over the kernel of $k^*|_{\tilde{G}}$. As the unipotent characters of $\tilde{G}$ and $C_{\tilde{G}}$ have this kernel in their kernels, we find that $R_{C_{\check{L}}}^{C_{\check{G}}}(\check{\kappa})$ is a sum of four unipotent characters, one of which is $\check{\lambda}$. As $c(\vartheta) = 4$, we conclude from (3) that $\check{\kappa}$ is invariant under the action of $\tilde{A}_{\check{\mathbf{L}}^*}(\check{s})^F = \tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$. It follows that the four constituents of $R_{C_{\check{L}}}^{C_{\check{G}}}(\check{\kappa})$ are exactly the four characters in the $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$-orbit of $\check{\lambda}$. Now $C_{\check{L}}$ decomposes into direct summands $C_{\check{L},\tilde{\mu}}$, and each of these is a Levi subgroup of $\check{C}_{\tilde{\mu}}$. As Harish-Chandra induction is compatible with this direct product decomposition, we conclude that for each $\tilde{\mu}$, the orbit sums of the $\check{\lambda}_{\tilde{\mu}}$ are Harish-Chandra induced. This applies in particular for $\tilde{\nu} = X^2 - \alpha$. To show that the latter is impossible, we may assume that $\mathcal{F}_{\check{s}} = \{X - \zeta, X + \zeta\}$.

Assuming this, we put $\mathbf{C} := C^\circ_{\check{\mathbf{G}}^*}(\check{s})$. We then have two embeddings $\mathbf{C}_1' \times \mathbf{C}_2' \xrightarrow{\gamma} \mathbf{C} \xrightarrow{\delta} \mathbf{C}_1 \times \mathbf{C}_2$, where $\mathbf{C}_1$ and $\mathbf{C}_2$ denote the special conformal orthogonal groups acting on the ζ eigenspace and the $-\zeta$ eigenspace, respectively, of $\check{s}$. Also, $\mathbf{C}_i'$ denotes the set of elements of $\mathbf{C}_i$ with multipliers 1, $i = 1, 2$, and $\mathbf{C} = \{(g_1, g_2) \in \mathbf{C}_1 \times \mathbf{C}_2 \mid \alpha_{g_1} = \alpha_{g_2}\}$. As $\mathbf{C}_1 \times \mathbf{C}_2$ has connected center, the maps δ and $\delta \circ \gamma$ are regular embeddings. Thus the Levi subgroups of $\mathbf{C}$ are the intersections of the Levi subgroups of $\mathbf{C}_1 \times \mathbf{C}_2$ with $\mathbf{C}$, and similarly for the Levi subgroups of $\mathbf{C}_1' \times \mathbf{C}_2'$. Now let $\mathbf{M}$ be an F-stable split Levi subgroup of $\mathbf{C}$. By the above considerations, $\mathbf{M}' := \mathbf{M} \cap (\mathbf{C}_1' \times \mathbf{C}_2')$ is a split F-stable Levi subgroup of $\mathbf{C}_1' \times \mathbf{C}_2'$. We also find that $C = (C_1' \times C_2')M$, as the Levi subgroups of $\mathbf{C_i}$ contain elements with arbitrary multipliers. It follows that $\mathrm{Res}^C_{C_1' \times C_2'}(R^C_M(\beta)) = R^{C_1' \times C_2'}_{M'}(\mathrm{Res}^M_{M'}(\beta))$ for $\beta \in \mathrm{Irr}(M)$. Applying this to our situation above, we find that the $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$-orbit sum of the restriction of $\check{\lambda}$ to $\mathrm{SO}_k^+(q) \times \mathrm{SO}_k^+(q)$ is Harish-Chandra induced. As this orbit contains the four different characters labelled by $(\Lambda_1, \Lambda_2), (\Lambda_2, \Lambda_1), (\Lambda_1', \Lambda_2'), (\Lambda_2', \Lambda_1')$, this is impossible, a contradiction.

(b) The proof is the same as the one of Theorem 5.6(b).

(c) Let us begin with some preliminary remarks. Lemma 4.12 implies that there is a proper split F-stable Levi subgroup $\check{\mathbf{L}}^*$ of $\check{\mathbf{G}}^*$ containing $C^\circ_{\check{\mathbf{G}}^*}(\check{s})$, and even $C_{\check{\mathbf{G}}^*}(\check{s})$ if there is $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha}$ or if $\check{s}$ is exceptional. Putting $\mathbf{L}^* := j^*(\check{\mathbf{L}}^*)$, we obtain $C^\circ_{\mathbf{G}^*}(s) = j^*(C^\circ_{\check{\mathbf{G}}^*}(\check{s})) \leq \mathbf{L}^*$ from Lemma 5.8. This implies in particular $C^\circ_{\mathbf{G}^*}(s) = C^\circ_{\mathbf{G}^*}(s) \cap \mathbf{L}^* = C^\circ_{\mathbf{L}^*}(s)$ (see 2.5), a fact that will be used throughout the proof. As $\mathbf{L}^*$ is a proper split F-stable Levi subgroup of $\mathbf{G}^*$ by Lemma 3.1, our claim about χ follows from Corollary 3.6.

(i) Suppose that $\check{s}$ is not conjugate to $-\check{s}$ in $\check{G}^*$. Then $\tilde{C}_{\check{\mathbf{G}}^*}(\check{s}) = C_{\check{\mathbf{G}}^*}(\check{s})$. Assume first that there is $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha}$ or that $\check{s}$ is exceptional. By Lemma 5.8 and the preliminary remarks, we conclude that $C_{\mathbf{G}^*}(s) = j^*(\tilde{C}_{\check{\mathbf{G}}^*}(\check{s})) = j^*(C_{\check{\mathbf{G}}^*}(\check{s})) \leq \mathbf{L}^*$. In particular, $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{G}^*}(s)^F = A_{\mathbf{L}^*}(s)^F$, proving our claim.

Now assume that $\check{s}$ is as in Lemmas 4.12(a)(i.2) and that $\check{s}$ is not exceptional. In particular, $X^2 - \alpha$ divides the minimal polynomial of $\check{s}$. It follows from Lemma 4.4 that $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F$ has order 2 and acts as a graph automorphism on the components of $C^\circ_{\check{\mathbf{G}}^*}(\check{s})^F$ corresponding to the Type (I) or Type (II) elements of $\mathcal{F}_{\check{s}}$, and acts trivially on the other components of this centralizer. From this it is easy to see (cf. Remark 5.9) that $\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F = \tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F_{\check{\lambda}}$, unless $\check{s}$ and $\check{\lambda}$ are as in the statement, in which case $A_{\mathbf{G}^*}(s)^F_\lambda$ is trivial. Corollary 3.6 implies that χ is Harish Chandra induced from L if $A_{\mathbf{G}^*}(s)^F_\lambda$ is trivial. Otherwise, χ is Harish-Chandra primitive by Corollary 3.7 and Lemma 4.12(a)(i.2). This completes the proof of (i).

(ii) Now suppose that $\check{s}$ is not exceptional, that $\check{s}$ is conjugate to $-\check{s}$ in $\check{G}^*$ and that $X^2 - \alpha$ does not divide the minimal polynomial of $\check{s}$. This implies, first of all, that $\check{s}$ is not as in Lemma 4.12(a)(i.2). Also, $C^\circ_{\check{\mathbf{G}}^*}(\check{s}) = C_{\check{\mathbf{G}}^*}(\check{s}) \lneq \tilde{C}_{\check{\mathbf{G}}^*}(\check{s})$ by Lemma 4.4 and the definition of $\tilde{C}_{\check{\mathbf{G}}^*}(\check{s})$ in (16). Thus $|A_{\mathbf{G}^*}(s)^F| = 2$ by Lemma 5.8. Lemmas 4.12(a)(ii) and 3.1 imply that $C_{\mathbf{G}^*}(s)^F = j^*(\tilde{C}_{\check{\mathbf{G}}^*}(\check{s})^F)$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$. In particular, $|A_{\mathbf{L}^*}(s)^F| = 1$. The last two assertions follow from Corollary 3.6, respectively Corollary 3.7.

(iii) Now suppose that $\check{s}$ is conjugate to $-\check{s}$ in $\check{G}^*$ and that $X^2 - \alpha$ divides the minimal polynomial of $\check{s}$. Then $|A_{\mathbf{G}^*}(s)^F| = |\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F| = 4$ by Lemma 5.8 and Proposition 4.7.

Suppose first that $\mu = \mu^{*\alpha}$ for all $\mu \in \mathcal{F}_{\check{s}}$, i.e. that $\check{s}$ is as in Lemma 4.12(a)(i.2). As the multiplicity of ν in the characteristic polynomial of $\check{s}$ is 2, the orbits of $A_{\mathbf{G}^*}(s)^F$ on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$ have lengths 1 or 2 by Proposition 4.7. Hence $|A_{\mathbf{G}^*}(s)^F_\lambda| \in \{4, 2\}$. Also, $\check{\mathbf{L}}^*$ is the stabilizer of two complementary, totally isotropic subspaces of $\mathbf{V}_\nu(s)$ whose sum equals $\mathbf{V}_\nu(s)$. If ν is of Type (I), this easily implies that $C_{\check{\mathbf{L}}^*}(\check{s}) = C^\circ_{\check{\mathbf{G}}^*}(\check{s})$, hence $A_{\mathbf{L}^*}(s)^F$ is trivial by Lemma 5.8. In particular, $A_{\mathbf{G}^*}(s)^F_\lambda \lneq A_{\mathbf{L}^*}(s)^F$, proving our assertion. If ν has Type (II), then $\tilde{A}_{\check{\mathbf{L}}^*}(\check{s})^F$ has order 2 and acts by conjugating $\check{s}$ to $-\check{s}$. Now if $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$, an element of $\check{G}^*$ conjugating $\check{s}$ to $-\check{s}$ would stabilize $\check{\lambda}$. But then $|A_{\mathbf{G}^*}(s)^F_\lambda| = |\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F_{\check{\lambda}}| = 4$ by Proposition 4.7, a contradiction.

Now suppose that there is $\mu \in \mathcal{F}_{\check{s}}$ such that $\mu \neq \mu^{*\alpha}$. By the preliminary remark, $C_{\check{\mathbf{G}}^*}(\check{s}) \leq \check{\mathbf{L}}^*$. We have $|(C_{\check{\mathbf{G}}^*}(\check{s})/C^\circ_{\check{\mathbf{G}}^*}(\check{s}))^F| = 2$, as $|\tilde{A}_{\check{\mathbf{G}}^*}(\check{s})^F| = 4$. In turn, $|(j^*(C_{\check{\mathbf{G}}^*}(\check{s}))/C^\circ_{\mathbf{G}^*}(s))^F| = 2$. Now $j^*(C_{\check{\mathbf{G}}^*}(\check{s})) \leq C_{\mathbf{G}^*}(s) \cap \mathbf{L}^* = C_{\mathbf{L}^*}(s)$, and thus $A_{\mathbf{L}^*}(s)^F$ is nontrivial. We cannot have $|A_{\mathbf{L}^*}(s)^F| = 4$ by Lemma 4.12(a)(ii), hence $|A_{\mathbf{L}^*}(s)^F| = 2$ as claimed, and we may assume that $b \in A_{\mathbf{L}^*}(s)^F$.

(iii.1) If m is odd, $|A_{\mathbf{G}^*}(s)^F|$ is cyclic by Proposition 4.7. In particular, $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$, if and only if $A_{\mathbf{G}^*}(s)^F_\lambda \lneq A_{\mathbf{G}^*}(s)^F$.

(iii.2) Now suppose that m is even. The structure of $A_{\mathbf{G}^*}(s)$ and the description of its elements follow from Proposition 4.7. We have $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{L}^*}(s)^F$, if and only if a does not fix λ.

(d) The assertion has already been proved for elements $\check{s}$ as in (c)(i) or (c)(ii). Thus let $\check{s}$ is as in (c)(iii) and assume that χ is not Harish-Chandra induced from L. Then $|A_{\mathbf{G}^*}(s)^F_\lambda| > 1$. If $|A_{\mathbf{G}^*}(s)^F_\lambda| = 4$, our assertion follows from Corollary 3.7(b), as $C_{\mathbf{G}^*}(s)^F$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$ by Lemmas 4.12(a), 5.8 and 3.1. Hence $|A_{\mathbf{G}^*}(s)^F_\lambda| = 2$, and by (iii.2) we have $A_{\mathbf{G}^*}(s)^F_\lambda = \langle a \rangle$ or $A_{\mathbf{G}^*}(s)^F_\lambda = \langle ab \rangle$. Proposition 4.7 excludes the case that $\check{s}$ is as in Lemma 4.12(a)(i.2) and ν is of Type (II) (recall that the multiplicity of ν in the characteristic polynomial of $\check{s}$ in this case is 2, and that $\mathrm{SO}_2^+(q^2)$ has only one unipotent character).

Suppose that χ is Harish-Chandra induced from $M = \mathbf{M}^F$, where $\mathbf{M}$ is a proper split Levi subgroup of $\mathbf{G}$. Let $\mathbf{M}^*$ be an F-stable Levi subgroup of $\mathbf{G}^*$ dual to $\mathbf{M}$. Let ϑ be an irreducible character of M such that $\chi = R_M^G(\vartheta)$. By Theorem 3.5 (with $\tilde{\mathbf{L}}$ replaced by $\tilde{\mathbf{M}}$), we have $c(\chi) \leq c(\vartheta)$. Now $c(\chi) = |A_{\mathbf{G}^*}(s)^F_\lambda| = 2$ by (3), and $c(\vartheta) \leq |A_{\mathbf{M}^*}(s)^F|$, where $|A_{\mathbf{M}^*}(s)^F|$ divides $|A_{\mathbf{G}^*}(s)^F| = 4$.

If $c(\chi) = c(\vartheta)$, Theorem 3.5 implies that $C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{M}^*$. But then $A_{\mathbf{G}^*}(s)^F_\lambda \leq A_{\mathbf{M}^*}(s)^F$ by Corollary 3.6. As $A_{\mathbf{G}^*}(s)^F_\lambda$ contains a or ab, it follows that $(j^*)^{-1}(C_{\mathbf{G}^*}(s)^F_\lambda) \leq (j^*)^{-1}(\mathbf{M}^*)$ contains an element that conjugates $\check{s}$ to $-\check{s}$. As $(j^*)^{-1}(\mathbf{M}^*)$ is a proper split F-stable Levi subgroup of $\check{\mathbf{G}}^*$ by Lemma 3.1, this contradicts Lemma 4.12(a)(ii).

Thus suppose that $c(\vartheta) = 4$. By Theorem 3.5, there is an irreducible character $\tilde{\vartheta}$ of $\tilde{M}$ such that $R_{\tilde{M}}^{\tilde{G}}(\tilde{\vartheta})$ has exactly two irreducible constituents of the same degree. This contradicts Corollary 3.9(a).

Condition (1) of Theorem 1.1 is trivially satisfied under the hypothesis on $\check{s}$ in (b). We have just shown that if $\check{s}$ is as in (c), then either Condition (1) is satisfied for $\mathbf{L}^*$, or χ is Harish-Chandra primitive. Thus Theorem 1.1(b) holds for G. ◇

We summarize the conditions for Harish-Chandra imprimitivity of an irreducible character of $\mathrm{Spin}^{\pm}_{2m}(q)$.

Corollary 5.11. *Let the notation be as in* Theorem 5.10 *and* Tables 6, 7 *and* 8. *Then χ is Harish-Chandra imprimitive exactly in the following cases.*

(a) *For all $\mu \in \mathcal{F}_{\check{s}}$ we have $\mu = \mu^{*\alpha}$, there is $\zeta \in \mathbb{F}_q$ with $\zeta^2 = \alpha$ and $X - \zeta$, respectively $X + \zeta$, occur with multiplicity 2, respectively $4k'$, in the characteristic polynomial of $\check{s}$. Moreover, the eigenspaces $V_{X\pm\zeta}(\check{s})$*

have maximal Witt index and if $k' > 0$, the component of $\check{\lambda}$ corresponding to $V_{X+\zeta}(\check{s})$ is labelled by a degenerate symbol.

(b) *There exists $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha} \neq \mu'$.*

(c) *Every $\mu \in \mathcal{F}_{\check{s}}$ satisfies $\mu = \mu^{*\alpha}$ or $\mu' = \mu^{*\alpha}$, there exists $\mu \in \mathcal{F}_{\check{s}}$ with $\mu \neq \mu^{*\alpha}$, and if $\check{s}$ is conjugate to $-\check{s}$ in $\check{G}^*$, the following conditions are satisfied: There is j such that $\kappa_{1,j} \neq \kappa_{2,j}$ or m is even and $\Lambda_1 \neq \Lambda_2, \Lambda_2'$ or m is odd and $\Lambda_1 \neq \Lambda_2$.*

5.7. **The exceptional groups.** Here, we present the results for the quasisimple exceptional groups of Lie type arising from algebraic groups with non-connected centers. These are the groups $E_6(\mathbb{F})_{\mathrm{sc}}$ for $\mathrm{char}(\mathbb{F}) \neq 3$ and $E_7(\mathbb{F})_{\mathrm{sc}}$ for $\mathrm{char}(\mathbb{F}) \neq 2$. Let $\mathbf{G}$ denote one of these groups. Then $\mathbf{G}^* = E_6(\mathbb{F})_{\mathrm{ad}}$, respectively $\mathbf{G}^* = E_7(\mathbb{F})_{\mathrm{ad}}$. In the first case, $|Z(\mathbf{G})| = 3$ and in the second case, $|Z(\mathbf{G})| = 2$. If $\mathbf{G} = E_6(\mathbb{F})_{\mathrm{sc}}$, let F' and F'' denote Frobenius morphisms of $\mathbf{G}$ such that $\mathbf{G}^{F'} = E_6(q)_{\mathrm{sc}}$ and $\mathbf{G}^{F''} = {}^2E_6(q)_{\mathrm{sc}}$. Let F be one of F' or F''. Then $G^* = E_6(q)_{\mathrm{ad}}$ if $F = F'$, and $G^* = {}^2E_6(q)_{\mathrm{ad}}$, otherwise. If $\mathbf{G} = E_7(\mathbb{F})_{\mathrm{sc}}$, let F denote a Frobenius morphism of $\mathbf{G}$ such that $\mathbf{G}^F = E_7(q)_{\mathrm{sc}}$. Then $G^* = E_7(q)_{\mathrm{ad}}$.

In our situation Theorem 1.1(b) holds by Corollary 3.12. We thus have to decide, for semisimple elements $s \in G^*$, the containment of $C_{\mathbf{G}^*}(s)^F_\lambda C^\circ_{\mathbf{G}^*}(s)$ in proper split F-stable Levi subgroups of $\mathbf{G}^*$ (in the notation of Theorem 1.1). For the purpose of this investigation we introduce one further piece of notation and recall some facts from a paper by Broué and Malle [3]. For a positive integer i, we let $\Phi_i \in \mathbb{Z}[X]$ denote the ith cyclotomic polynomial. In [3, Définition 1.9, Lemma 3.1], Broué and Malle associate to an F-stable torus $\mathbf{T}^*$ of $\mathbf{G}^*$ an order polynomial $f \in \mathbb{Z}[X]$, such that $|(\mathbf{T}^*)^F| = f(q)$. Moreover, f is a product of Φ_i's for certain values of i, and $\mathbf{T}^*$ contains a nontrivial split F-stable subtorus, if and only if Φ_1 divides f. Suppose that $\mathbf{H}^*$ is a closed F-stable subgroup of $\mathbf{G}^*$ satisfying $C_{\mathbf{G}^*}(\mathbf{H}^*) \leq \mathbf{H}^*$. (The latter condition is satisfied if $\mathbf{H}^*$ contains a maximal torus of $\mathbf{G}^*$.) Then $\mathbf{H}^*$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$, if and only if the order polynomial of $Z^\circ(\mathbf{H}^*)$ is divisible by Φ_1. Indeed, $Z^\circ(\mathbf{H}^*)$ is an F-stable torus of $\mathbf{G}^*$, and $\mathbf{H}^*$ is contained in a proper split F-stable Levi subgroup of $\mathbf{G}^*$, if and only if $\mathbf{H}^*$ centralizes a nontrivial split F-stable torus.

In the following remark we introduce the cases that may arise, the notation and labels used in the tables below.

Remark 5.12. Let $s \in G^*$ be semisimple and let $\chi \in \mathcal{E}(G, [s])$. Let $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F, [1])$, such that $[\chi]$ corresponds to $[\lambda]$ under Lusztig's generalized Jordan decomposition of characters. Put $\mathbf{C} := C_{\mathbf{G}^*}(s)$ and $\mathbf{Z} := Z(\mathbf{C})$. The following cases arise.

Case $*$: Here, $\mathbf{C}^\circ$ is not contained in any proper split F-stable Levi subgroup of $\mathbf{G}^*$. By the remarks above, this happens if and only if the order polynomial of $Z^\circ(\mathbf{C}^\circ)$ is not divisible by Φ_1. In this case, every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive by Corollary 3.11.

Case $\dagger$: Here, $\mathbf{C}^\circ$ is contained in some proper split F-stable Levi subgroup of $\mathbf{G}^*$ but $\mathbf{C}$ is not contained in any such subgroup. This is the case if and only if the order polynomial of $Z^\circ(\mathbf{C}^\circ)$ is divisible by Φ_1, while the order polynomial of $\mathbf{Z}^\circ$ is not divisible by Φ_1. In this case, all elements of $\mathcal{E}(G, [s])$ are Harish-Chandra primitive if $|A_{\mathbf{G}^*}(s)_\lambda^F| \neq 1$. Otherwise, all elements of $\mathcal{E}(G, [s])$ are Harish-Chandra imprimitive. Indeed, if $|A_{\mathbf{G}^*}(s)_\lambda^F| = 1$, then $\mathbf{C}_\lambda^F \leq \mathbf{C}^\circ$, and the claim follows from Theorem 1.1(a). Conversely, suppose that χ is Harish-Chandra imprimitive. Then, by Theorem 1.1(b), there is some proper split F-stable Levi subgroup $\mathbf{L}^* \leq \mathbf{G}^*$ such that $\mathbf{C}_\lambda^F \mathbf{C}^\circ \leq \mathbf{L}^*$. Aiming at a contradiction, we assume that $|A_{\mathbf{G}^*}(s)_\lambda^F| \neq 1$. Then $A_{\mathbf{G}^*}(s)^F = A_{\mathbf{G}^*}(s)$, and thus $\mathbf{C} = \mathbf{C}^F\mathbf{C}^\circ = \mathbf{C}_\lambda^F\mathbf{C}^\circ$, as every coset of $\mathbf{C}/\mathbf{C}^\circ$ contains an F-stable element. This is the desired contradiction.

Case ✓: Here, $\mathbf{C}$ is contained in some proper split F-stable Levi subgroup of $\mathbf{G}^*$. This is the case, if and only if the order polynomial of $\mathbf{Z}^\circ$ is divisible by Φ_1. In this case, every element of $\mathcal{E}(G, [s])$ is Harish-Chandra imprimitive by [15, Theorem 7.3].

Our results for the groups G considered here rely on a classification of the semisimple class types of G^* and their centralizers. (Recall that two semisimple elements of G^* belong to the same class type, if their centralizers in $\mathbf{G}^*$ are conjugate in the finite group G^*.) In the adjoint case we are considering, this classification is due to Frank Lübeck and is given in the Tables of [18]. There, each class type is labelled by a triple of natural numbers. Suppose that $s, s' \in G^*$ represent class types. Put $\mathbf{C} := C_{\mathbf{G}^*}(s)$ and $\mathbf{C}' := C_{\mathbf{G}^*}(s')$. Then the first index in the triples labelling the class types of s and s' are equal, if and only if $\mathbf{C}^\circ$ and $(\mathbf{C}')^\circ$ are conjugate in $\mathbf{G}^*$. The first two indices of these triples are equal, if and only if $\mathbf{C}$ and $\mathbf{C}'$ are conjugate in $\mathbf{G}^*$. Put $\mathbf{Z} := Z(\mathbf{C})$. Then the entry corresponding to the class type of s gives $|\mathbf{Z}^F|$, the latter in a form reflecting the order polynomial of the torus $\mathbf{Z}^\circ$: the φ_j in the table (there written as `phi1, phi2, phi3,` ...) stands for $\Phi_j(q)$, and φ_j^b is given as a factor in the entry for $|\mathbf{Z}^F|$, if and only if Φ_j^b divides the order polynomial of $\mathbf{Z}^\circ$. (This follows from the fact that the order formulae in [18] for $(\mathbf{Z}^\circ)^F$ are valid for all q; they are in particular valid for all powers of F.) The extra factor for $|\mathbf{Z}^F|$, if present, gives $|(\mathbf{Z}/\mathbf{Z}^\circ)^F|$. In particular, $\mathbf{Z}^\circ$ contains a nontrivial split F-stable torus, if and only if φ_1 occurs as a factor in $|(\mathbf{Z}^\circ)^F|$. In case $\mathbf{C}$ is not connected, the entry corresponding to s also gives the order polynomial of $Z^\circ(\mathbf{C}^\circ)$, following the same conventions as for the order polynomial of $\mathbf{Z}^\circ$. The tables in [18] thus easily allow to assign s to one of the three cases of Remark 5.12.

5.7.1. Now let $\mathbf{G} := E_6(\mathbb{F})_{\mathrm{sc}}$, and let F be one of F' or F''. Table 10 contains a list of those semisimple elements $s \in G^*$ such that the $\mathbf{G}^*$-conjugacy class of s contains at least one F-stable element s' with $A_{\mathbf{G}^*}(s')^F \neq 1$ and such that $\mathbf{C}^\circ_{\mathbf{G}^*}(s)$ has semisimple rank at least 2.

Let us explain the notation used in Table 10. First of all, we define $\varepsilon \in \{\pm 1\}$ by $\varepsilon = 1$ if $F = F'$, and $\varepsilon = -1$ if $F = F''$. Now let s be such an element for which there is an entry in the table. We then put $\mathbf{C} := C_{\mathbf{G}^*}(s)$ and $\mathbf{Z} := Z(\mathbf{C})$. We also write $C := \mathbf{C}^F$ and $C^\circ := (\mathbf{C}^\circ)^F$; similarly, we put $Z := \mathbf{Z}^F$ and $Z^\circ := (\mathbf{Z}^\circ)^F$. The first column of Table 10 just numbers the cases, and the second column gives the label of the class type of s according to [18]. The third column gives the Dynkin type of $[\mathbf{C}^\circ, \mathbf{C}^\circ]$, where A_2^3 denotes three copies of Type A_2, etc. The fourth column describes $[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$. As we are only interested in the unipotent characters of C°, which are insensitive to the center of C° and to isogeny, the information here is given in a Chevalley group type of notation, just presenting the simple components of $[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$. Again, exponents denote the number of copies of a specific group, and juxtaposition indicates direct products of groups. Moreover, a notation such as $A_2(-q)$ stands for a twisted group of type A_2, defined over the field with q^2 elements, i.e. a group with the same unipotent characters as $\mathrm{SU}_3(q)$. The fifth column gives the order of $C/C^\circ \cong (\mathbf{C}/\mathbf{C}^\circ)^F = A_{\mathbf{G}^*}(s)^F$. The sixth column gives the order of $Z/Z^\circ \cong (\mathbf{Z}/\mathbf{Z}^\circ)^F$. The seventh column describes the torus $\mathbf{Z}^\circ$ by its order polynomial (see the remarks in the introduction to 5.7). The next column gives the conditions for the existence of the elements in each row. Finally, the last column yields information about the containment of $\mathbf{C}^\circ$ and $\mathbf{C}$ in split F-stable Levi subgroups of $\mathbf{G}^*$, where we use the symbols introduced in Remark 5.12 to label the cases.

Remark 5.13. As discussed in Remark 5.12, the question about primitivity of the characters in $\mathcal{E}(G, [s])$ for the classes in Table 10 can be read off the "Notes" column of that table, provided the entry is one of $*$ or ✓. In the three cases where the entry is a †, we have to determine $A_{\mathbf{G}^*}(s)^F_\lambda$. For this we need to know the action of $A_{\mathbf{G}^*}(s)^F = C/C^\circ$ on the components of $[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$. In each of the three cases, the Frobenius map F fixes the simple roots of $[\mathbf{C}^\circ, \mathbf{C}^\circ]$, as indicated in the tables in [18] (or as follows from the structure of $[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$).

TABLE 10. Some F-stable semisimple elements in $\mathbf{G}^* = E_6(\mathbb{F})_{\mathrm{ad}}$, $F \in \{F', F''\}$; explanations in 5.7.1

No.	Label in [18]	Dynkin Type	$[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$	$\lvert C/C^\circ\rvert$	$\lvert Z/Z^\circ\rvert$	$\mathbf{Z}^\circ$	Condition	Notes F'	Notes F''
1	[3, 2, 1]	A_2^3	$A_2(\varepsilon q)^3$	3	3	1	$3 \mid q-\varepsilon$	$*$	$*$
2	[3, 2, 2]		$A_2(q^2)A_2(-\varepsilon q)$	1	3	1	$3 \mid q+\varepsilon$	$*$	$*$
3	[3, 2, 3]		$A_2(\varepsilon q^3)$	3	3	1	$3 \mid q-\varepsilon$	$*$	$*$
4	[13, 2, 1]	A_1^4	$A_1(q)^4$	3	6	1	$6 \mid q-\varepsilon$	†	$*$
5	[13, 2, 2]		$A_1(q)A_1(q^3)$	3	6	1	$6 \mid q-\varepsilon$	$*$	$*$
6	[13, 2, 3]		$A_1(q)^2A_1(q^2)$	1	6	1	$6 \mid q+\varepsilon$	✓	✓
7	[14, 2, 1]	D_4	$D_4(q)$	3	3	1	$3 \mid q-\varepsilon$	†	$*$
8	[14, 2, 2]		${}^3D_4(q)$	3	3	1	$3 \mid q-\varepsilon$	$*$	$*$
9	[14, 2, 3]		${}^2D_4(q)$	1	3	1	$3 \mid q+\varepsilon$	✓	✓
10	[16, 2, 1]	A_1^3	$A_1(q)^3$	3	3	Φ_1	$3 \mid q-\varepsilon$	✓	✓
11	[16, 2, 2]		$A_1(q)^3$	3	3	Φ_2	$3 \mid q-\varepsilon$	†	$*$
12	[16, 2, 3]		$A_1(q)A_1(q^2)$	1	3	Φ_1	$3 \mid q+\varepsilon$	✓	✓
13	[16, 2, 4]		$A_1(q)A_1(q^2)$	1	3	Φ_2	$3 \mid q+\varepsilon$	✓	✓
14	[16, 2, 5]		$A_1(q^3)$	3	3	Φ_2	$3 \mid q-\varepsilon$	$*$	$*$
15	[16, 2, 6]		$A_1(q^3)$	3	3	Φ_1	$3 \mid q-\varepsilon$	✓	✓

The action of C/C° on the components of $[\mathbf{C}^\circ, \mathbf{C}^\circ]$ is described in the tables of [18]. We find that C/C° acts as follows: In Case 4 it fixes one component $A_1(q)$ and acts as a three-cycle on the other components; in Case 7 it acts as the graph automorphism; in Case 11 it acts again by a three-cycle. For the action of the graph automorphisms on the unipotent characters on a group of type $D_4(q)$ see [22, Theorem 2.5].

The classes with $|A_{\mathbf{G}^*}(s)^F| = 3$ not contained in Table 10 can be treated as follows. For these classes, we always have $C_\lambda = C$. Thus, as discussed in Remark 5.12, either all elements of $\mathcal{E}(G, [s])$ are Harish-Chandra primitive, or all of them are Harish-Chandra imprimitive. The latter occurs if and only if s is in Case ✓.

5.7.2. We now consider the case that $G = E_7(q)_{\mathrm{sc}}$. In Table 11 we only display those semisimple class types, for which there is potentially a nontrivial action of $A_{\mathbf{G}^*}(s)^F$ on the unipotent characters of $[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$, and which fall into Case † of Remark 5.12. The columns have the same meaning as in Table 10, except that we have omitted the column for the order of $A_{\mathbf{G}^*}(s)^F$, since this order is always equal to 2. We have added a column headed "Action" which describes the action of $A_{\mathbf{G}^*}(s)^F = C/C^\circ$ on the components of $[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$. These components are permuted by the action, and we give the cycle lengts of this permutation. If one of these components is fixed, so are its unipotent characters, except in the two instances where this is a component of type $D_4(q)$. In this case we replace the 1 for the corresponding cycle length by "g" to indicate that C/C° induces the graph automorphism on this component.

Again, the entries of Table 11 can be extracted from Lübeck's tables [18]. For the entries in the "Action" column one uses the information on the F-action on the set of simple roots of $[\mathbf{C}^\circ, \mathbf{C}^\circ]$, together with the given information on the action of C/C° on this set.

Remark 5.14. From the information contained in Table 11 we can decide which irreducible characters of $\mathcal{E}(G, [s])$ are Harish-Chandra imprimitive (see the explanations in Remark 5.12). The classes of semisimple elements $s \in G^*$ with $|A_{\mathbf{G}^*}(s)^F| = 2$ not contained in Table 11 can be treated as in the analogous cases for $\mathbf{G} = E_6(\mathbb{F})_{\mathrm{sc}}$.

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Table 11. Some F-stable semisimple elements in $\mathbf{G}^* = E_7(\mathbb{F})_{\mathrm{ad}}$; explanations in 5.7.2

No.	Case	Dynkin Type	$[\mathbf{C}^\circ, \mathbf{C}^\circ]^F$	Action	$\lvert Z/Z^\circ \rvert$	$\lvert \mathbf{Z}^\circ \rvert$	Condition
1	$[12, 2, 1]$	A_2^3	$A_2(q)^3$	$(1, 2)$	6	1	$6 \mid q-1$
2	$[17, 2, 1]$	$D_4A_1^2$	$D_4(q)A_1(q)^2$	$(\mathrm{g}, 2)$	4	1	$4 \mid q-1$
3	$[25, 4, 4]$	$A_3A_1^2$	$A_3(-q)A_1(q)^2$	$(1, 2)$	2	Φ_2	$2 \mid q-1$
4	$[27, 3, 3]$	A_1^5	$A_1(q)A_1(q^2)^2$	$(1, 2)$	4	Φ_2	$4 \mid q-1$
5	$[27, 3, 4]$		$A_1(q)A_1(q^2)^2$	$(1, 1, 1)$	4	Φ_2	$4 \mid q+1$
6	$[27, 3, 9]$		$A_1(q)^5$	$(1, 2, 2)$	4	Φ_2	$4 \mid q-1$
7	$[27, 3, 10]$		$A_1(q)A_1(q^2)^2$	$(1, 2)$	4	Φ_2	$4 \mid q+1$
8	$[30, 2, 6]$	$A_2A_1^2$	$A_2(-q)A_1(q)^2$	$(1, 2)$	2	Φ_2	$2 \mid q-1$
9	$[33, 4, 2)$	A_2^2	$A_2(q)^2$	2	2	Φ_3	$2 \mid q-1$
10	$[34, 3, 6]$	D_4	$D_4(q)$	g	2	Φ_2	$2 \mid q-1$
11	$[36, 3, 13]$	A_1^4	$A_1(q^2)A_1(q)^2$	$(1, 2)$	2	Φ_4	$2 \mid q-1$
12	$[36, 3, 20]$		$A_1(q^2)^2$	2	2	Φ_2^2	$2 \mid q-1$
13	$[36, 3, 23]$		$A_1(q)^4$	$(2, 2)$	2	Φ_2^2	$2 \mid q-1$
14	$[36, 3, 24]$		$A_1(q^2)^2$	$(1, 1)$	2	Φ_2^2	$2 \mid q-1$
15	$[38, 3, 6]$	A_1^3	$A_1(q)^3$	$(1, 2)$	2	Φ_2^2	$2 \mid q-1$
16	$[41, 5, 6]$	A_1^2	$A_1(q)^2$	2	2	$\Phi_2\Phi_4$	$2 \mid q-1$
17	$[41, 5, 17]$		$A_1(q)^2$	2	2	$\Phi_2\Phi_6$	$2 \mid q-1$
18	$[41, 5, 18]$		$A_1(q)^2$	2	2	$\Phi_2\Phi_4$	$2 \mid q-1$
19	$[41, 5, 20]$		$A_1(q)^2$	2	2	Φ_2^3	$2 \mid q-1$

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