NOTES ON THE DRINFELD DOUBLE OF FINITE-DIMENSIONAL GROUP ALGEBRAS

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Abstract. Let k be an algebraically closed field of characteristic $p > 0$. We characterize the finite groups G for which the Drinfeld double $D(kG)$ of the group algebra kG has the Chevalley property.

We also show that this is the case if and only if the tensor product of every simple $D(kG)$-module with its dual is semisimple. The analogous result for the group algebra kG is also true, but its proof requires the classification of the finite simple groups.

A further result concerns the largest Hopf ideal contained in the Jacobson radical of $D(kG)$). We prove that this generated by the augmentation ideal of $kO_p(Z(G))$, where $Z(G)$ is the center of G and $O_p(Z(G))$ the largest p-subgroup of this center.

1. Introduction

A Hopf algebra H is said to have the *Chevalley property*, if the tensor product of any two simple H-modules is semisimple. This notion was introduced by Andruskiewitsch, Etingof, and Gelaki in [2], and is motivated by a famous theorem of Chevalley which states that a group algebra kG does have this property if k is a field of characteristic 0.

In [12] Lorenz gives various characterizations of the Chevalley property for finite dimensional Hopf algebras H. Here, we add yet another such characterization in case $H = kG$ or $H = D(kG)$ for a finite group G and a field k of positive characteristic p. (As usual, $D(H)$ denotes the Drinfeld double of the finite dimensional Hopf algebra H.) Namely, we show that such a Hopf algebra has the Chevalley property if and only if $V \otimes V^*$ is semisimple for every simple H-module V. The proof of this result for group algebras requires the classification of the finite simple groups (except if $p = 2$), whereas the result for the Drinfeld double of a group algebra is elementary.

We also give a group theoretic characterization of the groups G such that $D(kG)$ has the Chevalley property. This is the case if and only if $G = S \times K$ where S is an abelian Sylow p-subgroup. The analogous

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characterization for a group algebra is due to Molnar [15]: The group algebra kG has the Chevalley property if and only if a Sylow p-subgroup of G is normal.

Let us write $J(A)$ for the Jacobson radical of an algebra A. Suppose that H is a finite dimensional Hopf algebra. It is well known that H has the Chevalley property if and only if $J(H)$ is a Hopf ideal of H. In general, there is a largest Hopf ideal $J_w(H)$ contained in $J(H)$ (see [5]). It is the annihilator of $(H/J(H))^{\otimes n}$ for all sufficiently large n. Suppose that G is a finite group and k is a field of characteristic p. It is well known that $J_w(kG)$ is generated by the augmentation ideal of $kO_p(G)$, where $O_p(G)$ denotes the largest normal p-subgroup of G. We give an analogous description for $J_w(D(kG))$: This is generated by the augmentation ideal of $kO_p(Z(G))$.

The smallest n such that the annihilator of $(H/J(H))^{\otimes n}$ equals $J_w(H)$ is denoted by $l_w(H)$. There are results indicating that $l_w(kG)$ is small if G is a finite group and k a field as above (see [5, Section 4]). We show that in contrast to this, $l_w(D(kG))$ can be arbitraily large if G runs through the class of finite p-groups. In fact we also give a purely group theoretical description of the number $l_w(D(kG))$ for such groups.

Our results indicate that in many respects the Drinfeld double of a group algebra behaves more smoothly than the group algebra itself.

2. The Chevalley property of the Drinfeld double of a group algebra

Let k be a field and H be a finite-dimensional Hopf algebra over k. The Drinfeld double of H, denoted by $D(H)$ as usual, is also a Hopf algebra, which is defined as follows (see [11, IX.4]). As a coalgebra, $D(H) = H^{*\mathrm{cop}} \otimes H$. (We always write $\otimes$ for $\otimes_k$.) Here, H^* is the dual Hopf algebra of H, and the comultiplication of $H^{*\mathrm{cop}}$ is opposite to that of H^*. Denote $f \otimes h$ by $f \bowtie h$ in $D(H)$ for all $f \in H^{*\mathrm{cop}} = H^*$ and $h \in H$. Then the multiplication of $D(H)$ is given by

$$(f \bowtie x)(g \bowtie y) = \sum f(x_2 \rightharpoonup g \leftharpoonup S^{-1}(x_3)) \bowtie x_2 y \ ,$$

where $f, g \in H^{*\mathrm{cop}} = H^*$, $x, y \in H$, S is the antipode of H, and $x \rightharpoonup g, g \leftharpoonup x \in H^*$ are defined by

$$\langle x \rightharpoonup g, y\rangle = \langle g, yx\rangle, \quad \langle g \leftharpoonup x, y\rangle = \langle g, xy\rangle.$$

Recall that a Hopf ideal of a Hopf algebra H is an ideal I of H which is also a coideal of H (i.e., $\Delta(I) \subseteq H \otimes I + I \otimes H$, $\varepsilon(I) = 0$) such that $S(I) \subseteq I$. In this case, H/I is a Hopf algebra with the structure inherited from H.

We begin with two preliminary results.

Lemma 2.1. *Let K and L be two finite dimensional Hopf algebras. Then we have a Hopf algebra isomorphism*

$$D(K \otimes L) \cong D(K) \otimes D(L).$$

Proof. Since K and L are finite dimensional Hopf algebras, there is a canonical Hopf algebra isomorphism $K^* \otimes L^* \cong (K \otimes L)^*$. Regarding $K^* \otimes L^* = (K \otimes L)^*$, one can get a coalgebra isomorphism ϕ from $D(K) \otimes D(L)$ to $D(K \otimes L)$ as follows

$$\begin{array}{rccc} \phi: & (K^{*\mathrm{cop}} \bowtie K) \otimes (L^{*\mathrm{cop}} \bowtie L) & \to & (K \otimes L)^{*\mathrm{cop}} \bowtie (K \otimes L) \\ & (f \bowtie x) \otimes (g \bowtie a) & \mapsto & (f \otimes g) \bowtie (x \otimes a) \end{array}$$

where $f \in K^{*\mathrm{cop}}$, $x \in K$, $g \in L^{*\mathrm{cop}}$ and $a \in L$. It is straightforward to check that ϕ is also an algebra homomorphism. Therefore, ϕ is a Hopf algebra isomorphism since a bialgebra homomorphism between two Hopf algebras must be a Hopf algebra homomorphism. □

Now suppose that G is a finite group and that k is algebraically closed and of characteristic $p > 0$. Put $H := kG$. Recall that $O_p(G)$ denotes the largest normal p-subgroup of G.

Lemma 2.2. *If G is abelian, then $J(D(H))$ is a Hopf ideal of $D(H)$, and $D(H)/J(D(H))$ is isomorphic, as an algebra, to k^n , where $n = |G|[G : O_p(G)]$.*

Proof. Suppose that G is abelian. Then it is easy to see that $D(H)$ is exactly the tensor product $H^* \otimes H$ as a Hopf algebra. Since G is abelian, $O_p(G)$ is a Sylow p-subgroup. It follows from [5, Theorem 4.4 and Corollary 4.2] that $J(H) = J_w(H) = H(kO_p(G))^+$ and $H/J(H) \cong k(G/O_p(G))$.

Note that $H^* \cong k^{|G|} = k \times k \times \cdots \times k$ as algebras. Hence $(H^* \otimes H)/(H^* \otimes J(H)) \cong H^* \otimes (H/J(H)) \cong (H/J(H))^{|G|}$ as algebras. Thus $J(D(H)) = J(H^* \otimes H) = H^* \otimes J(H)$ is a Hopf ideal of $D(H)$ since $H^* \otimes J(H)$ is a nilpotent Hopf ideal of $H^* \otimes H$ and $(H/J(H))^{|G|}$ is semisimple.

Now since $G/O_p(G)$ is an abelian group and $p \nmid |G/O_p(G)|$, we have $H/J(H) \cong k(G/O_p(G)) \cong k^m$ as algebras, where $m = |G/O_p(G)| = [G : O_p(G)]$. Therefore, $D(H) \cong k^n$ as algebras, where $n = |G|[G : O_p(G)]$. □

We now come to the main result of this section, characterizing the Chevalley property for $D(H)$ in different ways. Recall that a finite-dimensional Hopf algebra has the Chevalley property, if the tensor product of any two of its simple modules is semisimple. It is clear (see e.g., [12]) that this is the case if and only if its Jacobson radical is a Hopf ideal.

Theorem 2.3. *The following statements are equivalent.*

(1) *$J(D(H))$ is a Hopf ideal.*

(2) *$V \otimes V^*$ is semisimple for all simple $D(H)$-modules V.*

(3) *$P(k) \otimes V \cong P(V)$ for all simple $D(H)$-modules V.*

(4) *The trivial module is a direct summand of $V \otimes V^*$ for all simple $D(H)$-modules V.*

(5) *The dimension of each simple $D(H)$-module is not divisible by p.*

(6) *$G = S \times K$ with S an abelian Sylow p-subgroup of G.*

Proof. We show (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3): See [12, Section 3.3].

(2) $\Rightarrow$ (4): It follows from

$$\mathrm{Hom}_{D(H)}(V \otimes V^*, k) \cong \mathrm{Hom}_{D(H)}(V, V)$$

that k is a composition factor of $V \otimes V^*$. Since the latter module is semisimple, it follows that k is a direct summand of $V \otimes V^*$.

(3) $\Rightarrow$ (4): We deduce from

$$\begin{aligned} k &\cong \mathrm{Hom}_{D(H)}(P(V), V) \\ &\cong \mathrm{Hom}_{D(H)}(P(k) \otimes V, V) \\ &\cong \mathrm{Hom}_{D(H)}(P(k), V \otimes V^*), \end{aligned}$$

that the multiplicity of k as a composition factor of $V \otimes V^*$ equals 1.

By the arguments used in the proof (2) $\Rightarrow$ (4), the head of $V \otimes V^*$ contains k as a composition factor. But $V \otimes V^* \cong (V^*)^* \otimes V^* \cong (V \otimes V^*)^*$, and so k is a composition factor of the socle and of the head of $V \otimes V^*$. It follows that k is a direct summand of $V \otimes V^*$.

(4) $\Rightarrow$ (5): See [3, Theorem 3.1.9].

(5) $\Rightarrow$ (6): By the description of the simple $D(H)$-modules (see [7, 8, 13, 18] or Section 4 below), our assumptions imply that $p \nmid |C|$ for every conjugacy class C of G. It is well known (see, e.g. [6, Proposition 4]) that this forces a Sylow p-subgroup of G, say S, to be contained in the center of G. In particular S is a normal, abelian subgroup of G. By the Schur-Zassenhaus theorem S has a complement K, say, in G. Since S is central, K is also a normal subgroup of G. Thus $G = S \times K$.

(6) $\Rightarrow$ (1): Suppose that $G = S \times K$ with S abelian and $p \nmid |K|$. Then $H = kG \cong kS \otimes kK$ as Hopf algebras. Thus it follows from Lemma 2.1 that

$$D(H) \cong D(kS \otimes kK) \cong D(kS) \otimes D(kK)$$

By Lemma 2.2, $J(D(kS))$ is a Hopf ideal of $D(kS)$ and, as algebras, $D(kS)/J(D(kS)) \cong k^n$ with $n = |S|[S : O_p(S)]$. Hence $J(D(kS)) \otimes D(kK)$ is a nilpotent Hopf ideal of $D(kS) \otimes D(kK)$, and

$$\begin{aligned} & (D(kS) \otimes D(kK))/(J(D(kS)) \otimes D(kK)) \\ \cong\ & (D(kS)/J(D(kS))) \otimes D(kK) \\ \cong\ & D(kK)^n \end{aligned}$$

as algebras. Since $p \nmid |K|$, kK is a semisimple and cosemisimple Hopf algebra. It follows from [17, Proposition 7] that $D(kK)$ is a semisimple Hopf algebra. Hence $(D(kS) \otimes D(kK))/(J(D(kS)) \otimes D(kK))$ is semisimple, and so $J(D(kS) \otimes D(kK)) = J(D(kS)) \otimes D(kK)$. This implies that $J(D(H))$ is a Hopf ideal of $D(H)$. □

3. The Chevalley property of a group algebra

The characterizations of the Chevalley property in Theorem 2.3 also hold for group algebras. The equivalences given in Theorem 3.1 below, except (2) and (7), are long known and have been published before (see [15] and [12, Section 3.3]). The implication (3) ⇒ (1) is due to Brockhaus [4]; its proof requires the classification of the finite simple groups. We repeat these results just to emphasize the analogy with the Drinfeld double of a group algebra.

The essential ingredient of the implication (2) ⇒ (1) is a theorem of Michler [14, Theorem 2.4], which states that a group without a normal Sylow p-subgroup has a simple module in characteristic p, whose dimension is divisble by p. The theorem of Michler uses the classification of the finite simple groups. For $p = 2$, a result of Okuyama can be applied which does not require the classification and which furthermore proves that Statement (7) implies the Chevalley property.

The investigations in this section were inspired by a question of Külshammer, who asked, whether the fact that the tensor square of any two simple modules is semisimple would imply the Chaevalley property. This is not the case. We give an example of a group algebra not having the Chevalley property, for which the tensor square of every simple module is semisimple.

Theorem 3.1. *Let G be a finite group and let k be an algebraically closed field of characteristic $p > 0$. Then the following statements are equivalent:*

(1) *$J(kG)$ is a Hopf ideal.*
(2) *$V \otimes V^*$ is semisimple for all simple kG-modules V.*
(3) *$P(k) \otimes V \cong P(V)$ for all simple kG-modules V.*

(4) *The trivial module is a direct summand of $V \otimes V^*$ for all simple kG-modules V.*

(5) *The dimension of each simple kG-module is not divisible by p.*

(6) *G has a normal Sylow p-subgroup.*

Furthermore, if $p = 2$, then these statements are equivalent to

(7) *$V \otimes V$ is semisimple for every simple kG-module V.*

Proof.

(1) ⇒ (2), (1) ⇒ (3) and (1) ⇒ (7): See [12, Section 3.3].

(3) ⇒ (4): Similar to (3) ⇒ (4) of Theorem 2.3.

(4) ⇒ (5): See [3, Theorem 3.1.9].

(2) ⇒ (6), (5) ⇒ (6) and (7) ⇒ (6): Suppose that G does not have a normal Sylow p-subgroup. If $p = 2$, a result of Okuyama (see [16, Theorem 2.33]) shows that there is a non-trivial, self-dual simple kG-module V. By Fong's Lemma (see [10, Theorem VII.8.13]), V has even dimension.

Now let p be odd. By a result of Michler [14, Theorem 2.4], which uses the classification of the finite simple groups, there is a simple kG-module V with $p \mid \dim_k(V)$.

Now $V \otimes V^*$ is not semisimple if the dimension of V is divisible by p (see [3, Theorem 3.1.9]).

(6) ⇒ (1): This follows from [12, 15]. □

The following example was found with the help of GAP (see [9]). It shows that Statement (7) of the above theorem does not imply the other statements if p is odd.

Example 3.2. Let G be the non-abelian group of order 21, and let k be an algebraically closed field of characteristic 3. Then kG has three simple modules (up to isomorphism). Apart from the trivial kG-module, there is a pair S, S^* of dual simple kG-modules of dimension 3. The Brauer character φ of S has the following three values:

$$3, \quad \frac{-1+\sqrt{-7}}{2}, \quad \frac{-1-\sqrt{-7}}{2}.$$

The Brauer character of S^* equals $\bar{\varphi}$, the complex conjugate of φ. We have $\varphi \cdot \varphi = \varphi + 2\bar{\varphi}$. Since S and S^* are projective, this implies $S \otimes S \cong S \oplus S^* \oplus S^*$. Dually, $S^* \otimes S^* \cong S \oplus S \oplus S^*$. Hence $V \otimes V$ is semisimple for every simple kG-module V. However, G does not have a normal Sylow 3-subgroup.

4. The ideal $J_w(D(H))$

Let G be a finite group, k be an algebraically closed field of characteristic $p > 0$, and put $H = kG$ as in Section 2.

Since $H \to D(H), h \mapsto \varepsilon \bowtie h$ and $H^{*\mathrm{cop}} \to D(H), f \mapsto f \bowtie 1$ are Hopf algebra monomorphisms, we may regard H and $H^{*\mathrm{cop}}$ as Hopf subalgebras of $D(H)$. In this case, $D(H) = H^{*\mathrm{cop}}H$ and $f \bowtie h = fh$ for any $f \in H^{*\mathrm{cop}}$ and $h \in H$. Let $p_g \in H^{*\mathrm{cop}}$ be given by

$$p_g(h) := \delta_{g,h},\ \ g, h \in G.$$

Then $\{p_g h \mid g, h \in G\}$ is a k-basis of $D(H)$. Moreover, $hp_g = p_{hgh^{-1}}h$ for all $g, h \in G$. The comultiplication of $D(H)$ is given by

$$\Delta(p_g h) = \sum_{x \in G} p_x h \otimes p_{gx^{-1}} h,$$

for $g, h \in G$ (see [11, IX.4.3]).

Recall that $J_w(D(H))$ denotes the largest Hopf ideal contained in $J(D(H))$.

Lemma 4.1. *We have*

$$J_w(D(H)) \supseteq D(H)(kO_p(Z(G)))^+ = H^{*\mathrm{cop}}H(kO_p(Z(G)))^+.$$

Proof. Note that $H(kO_p(Z(G)))^+ = (kO_p(Z(G)))^+H$ is a nilpotent Hopf ideal of H. Since $HH^{*\mathrm{cop}} = H^{*\mathrm{cop}}H$ and $H^{*\mathrm{cop}}(kO_p(Z(G)))^+ = (kO_p(Z(G)))^+H^{*\mathrm{cop}}$, $H^{*\mathrm{cop}}H(kO_p(Z(G)))^+$ is a nilpotent Hopf ideal of $D(H)$. Since $J_w(D(H))$ is the largest nilpotent Hopf ideal of $D(H)$, the result follows. $\square$

Let K be a subgroup of $Z(G)$, where $Z(G)$ denotes the center of G. Then K is a normal subgroup of G, and $hf = fh$ in $D(H)$ for all $h \in K$ and $f \in H^{*\mathrm{cop}}$. It follows that $D(H)(kK)^+ = H^{*\mathrm{cop}}H(kK)^+$ is a Hopf ideal of $D(H)$. Let $\overline{G}$ be the quotient group G/K and $\pi : G \to \overline{G}$ be the natural epimorphism. Denote $\pi(g)$ by $\overline{g}$ for all $g \in G$. Then the quotient Hopf algebra $D(H)/(D(H)(kK)^+)$ can be described as follows.

Now $H^{*\mathrm{cop}}$ and $k\overline{G}$ are two Hopf subalgebras of $D(H)/(D(H)(kK)^+)$, and
$D(H)/(D(H)(kK)^+) = H^{*\mathrm{cop}}(k\overline{G})$. Moreover, $\{p_g\overline{h} \mid g \in G, \overline{h} \in \overline{G}\}$ is a basis of $D(H)/(D(H)(kK)^+)$ over k with multiplication given by

$$\overline{h}p_g = p_{hgh^{-1}}\overline{h},\ \ g, h \in G.$$

For any module M over $D(H)/(D(H)(kK)^+)$ and $g \in G$, let $M_g = p_g \cdot M$. Then $M = \bigoplus_{g \in G} M_g$ and $\overline{h} \cdot M_g = M_{hgh^{-1}}$ for all $g, h \in G$. Hence for any $g \in G$, M_g is a $k\overline{C_G(g)}$-module, where $\overline{C_G(g)} = C_G(g)/K$ is a subgroup of $\overline{G}$. Thus M_g is a $kC_G(g)$-module. Moreover, for any conjugacy C of G, $\bigoplus_{g \in C} M_g$ is a submodule of M.

On the other hand, let $g \in G$ and N be any $k\overline{C_G(g)}$-module. Then the induced $k\overline{G}$-module

$$\mathrm{Ind}_{\overline{C_G(g)}}^{\overline{G}}(N) := k\overline{G} \otimes_{k\overline{C_G(g)}} N$$

becomes a $D(H)/(D(H)(kK)^+)$-module with the $H^{*\mathrm{cop}}$-action given by

$$p_x \cdot (\overline{h} \otimes n) = \delta_{x,hgh^{-1}}(\overline{h} \otimes n), \ x, h \in G, n \in N.$$

That is, $\overline{h} \otimes_{k\overline{C_G(g)}} N = (\mathrm{Ind}_{\overline{C_G(g)}}^{\overline{G}}(N))_{hgh^{-1}}$.

With the above notations, for any $D(H)/(D(H)(kK)^+)$-module M, an argument similar to [18] shows that there is a $D(H)/(D(H)(kK)^+)$-module isomorphism $\bigoplus_{h\in C} M_h \cong \mathrm{Ind}_{\overline{C_G(g)}}^{\overline{G}}(M_g)$, where C is a conjugacy class of G and $g \in C$.

Let $\mathcal{O}$ denote the set of conjugacy classes of G. For any $C \in \mathcal{O}$, choose an element $g_C \in C$. Then by an argument similar to [18, Theorem 2.2], one can get the following theorem.

Theorem 4.2. *Let K be a subgroup of $Z(G)$. Then there is an equivalence between the category ${}_{\overline{D(H)}}\mathcal{M}$ of $\overline{D(H)}$-modules and the Cartesian product category $\Pi_{C\in\mathcal{O}}\ {}_{k\overline{C_G(g_C)}}\mathcal{M}$ of the categories ${}_{k\overline{C_G(g_C)}}\mathcal{M}$ of $k\overline{C_G(g_C)}$-modules, where $\overline{D(H)} = D(H)/(D(H)(kK)^+)$ and $\overline{C_G(g_C)} = C_G(g_C)/K$.*

Putting $K = O_p(Z(G))$ in the above theorem, one gets the following corollary.

Corollary 4.3. *Up to isomorphism, the indecomposable (respectively, simple) $D(H)/(D(H)(kO_p(Z(G)))^+)$-modules are parametrized by pairs (N, g), where g is a representative of a conjugacy class of G, and N is an indecomposable (respectively, simple) $k(C_G(g)/O_p(Z(G)))$-module.*

Putting $K = \{1\}$ in Theorem 4.2, one recovers a known result of [18] as follows.

Corollary 4.4. *Up to isomorphism, the indecomposable (respectively, simple) $D(H)$-modules are parametrized by pairs (N, g), where g is a representative of a conjugacy class of G, and N is an indecomposable (respectively, simple) $kC_G(g)$-module.*

Remark 4.5. (1) $D(H)$ has a projective, simple module if and only if there exists $g \in G$ such that $kC_G(g)$ has a projective, simple module.

In fact, Putting $K = \{1\}$ in Theorem 4.2, one knows that the category ${}_{D(H)}\mathcal{M}$ of $D(H)$-modules is equivalent to the Cartesian product category $\Pi_{C\in\mathcal{O}}\ {}_{kC_G(g_C)}\mathcal{M}$ of the categories ${}_{kC_G(g_C)}\mathcal{M}$ of $kC_G(g_C)$-modules. Then the result follows immediately.

(2) If $p \nmid |C_G(g)|$ for some $g \in G$, then $D(H)$ has a projective, simple module.

(3) If $p \mid |Z(G)|$, then $D(H)$ does not have a projective, simple module. (Indeed, by [5, Proposition 2.6(3)], the existence of such a module would imply $J_w(D(H)) = \{0\}$, contradicting Lemma 4.1.)

Theorem 4.6. *Suppose G is a non-abelian finite simple group. Then $D(H)$ has a projective, simple module. In particular, if $p \mid |G|$, then $l_w(D(H)) = 2$.*

Proof. By [6, Proposition 3], there is $g \in G$ such that $p \nmid |C_G(g)|$. (Note that this result is based on a theorem of Michler using the classification of the finite simple groups.) The first claim thus follows from Remark 4.5. The second claim follows from [5, Corollary 3.10]. $\square$

Remark 4.7. *Let G be a non-abelian finite simple group. Then by* [5, Corollary 4.5] *and its proof, it follows that kG has a projective, simple module if $p \geq 5$. In particular, if $p \mid |G|$ and $p \geq 5$, then $l_w(kG) = 2$. We do not know whether a global bound for $l_w(kG)$ exists for simple groups G and $p = 2$ or 3. If G is the alternating group on 7 letters and $p = 2$ or 3, then $l_w(kG) = 3$. We do not know any example with $l_w(kG) = 4$.*

We now aim at proving that in fact equality holds in Lemma 4.1, i.e., $J_w(D(H)) = D(H)(kO_p(Z(G)))^+$. For a subgroup U of G we write $\mathbf{1}_U$ for the trivial representation of U. We begin with a lemma.

Lemma 4.8. *Let $g_1, \ldots, g_n$ be elements of G such that $C_G(g_1, \ldots, g_n) = Z(G)$ and $g_n \cdots g_1 = 1$. Furthermore, let V_i denote the simple $D(H)$-module corresponding to the pair $(\mathbf{1}_{C_G(g_i)}, g_i)$ (notation of* [18, Corollary 2.3]*). Then*

$$\mathrm{Ind}_{Z(G)}^{G}(\mathbf{1}_{Z(G)}) \mid \mathrm{Res}_{kG}^{D(H)}(V_1 \otimes \cdots \otimes V_n). \tag{1}$$

Proof. Using the remarks in [13, Section 3], in particular Formula (3.4) (or the arguments in the proof of [18, Lemma 3.2]) and Mackey's theorem, we find that $V_1 \otimes V_2$ contains a direct summand of the form

$$\left(\mathrm{Ind}_{C_G(g_1,g_2)}^{C_G(g_2g_1)}(\mathbf{1}_{C_G(g_1,g_2)}), g_2g_1\right).$$

Using induction on n and Mackey's theorem once more, we find that

$$\mathrm{Ind}_{C_G(g_1,\ldots,g_n)}^{C_G(g_n\cdots g_1)}(\mathbf{1}_{C_G(g_1,\ldots,g_n)})$$

is a direct summand of

$$\mathrm{Res}_{kC_G(g_n\cdots g_1)}^{D(H)}(V_1 \otimes \cdots \otimes V_n),$$

from which the claim follows. □

Proposition 4.9. *We have $J_w(D(H)) = D(H)(kO_p(Z(G)))^+$.*

Proof. Let us write $\overline{D(H)} := D(H)/(D(H)(kO_p(Z(G)))^+)$. By Lemma 4.1 we have

$$J_w(\overline{D(H)}) = J_w(D(H))/(D(H)(kO_p(Z(G)))^+).$$

Thus it suffices to prove

$$J_w(\overline{D(H)}) = 0. \tag{2}$$

It follows from [5, Proposition 2.4] and the proof of [5, Proposition 2.6(2)], that (2) is implied by

$$P(k_\varepsilon) \mid \Big((\overline{D(H)})/J(\overline{D(H)})\Big)^{\otimes n} \text{ for some } n. \tag{3}$$

Here, k_ε is the 1-dimensional module of $\overline{D(H)}$ corresponding to the counit ε of $\overline{D(H)}$, and $P(k_\varepsilon)$ its projective cover.

According to Theorem 4.3, the projective module $P(k_\varepsilon)$ corresponds to the pair $(P(\mathbf{1}_{\overline{G}}), 1)$, the first component denoting the projective cover of the trivial $k\overline{G}$-module. Now $\overline{Z(G)}$ is a group whose order is not divisible by p. Hence $\mathbf{1}_{\overline{Z(G)}}$ is a projective $k\overline{Z(G)}$-module, and hence so is $\mathrm{Ind}_{\overline{Z(G)}}^{\overline{G}}(\mathbf{1}_{\overline{Z(G)}})$. Since the latter maps onto the trivial $k\overline{G}$-module by Frobenius reciprocity, it contains $P(\mathbf{1}_{\overline{G}})$ as a direct summand. The result now follows from Lemma 4.8 and the discussion preceding Theorem 4.2. □

We can apply similar ideas to determine the invariant $l_w(D(H))$ if $H = kG$ is the group algebra of a finite p-group G.

Proposition 4.10. *Let G be a finite p-group. Define $z(G)$ to be the smallest positive integer n such that there exist $g_1, \dots, g_n$ in G with $g_n \cdots g_1 = 1$ and $C_G(g_1, \dots, g_n) = Z(G)$. (Note that $z(G) = 1$ if and only if G is abelian.) Then $l_w(D(kG)) = z(G)$.*

Proof. By Proposition 4.9, we have $J_w(D(H)) = D(H)(kZ(G))^+$. By definition, $l_w(D(H)) = l_w(\overline{D(H)})$ with $\overline{D(H)} = D(H)/J_w(D(H))$. Thus $l_w(D(H))$ is the smallest positive integer m such that $P(\mathbf{1}_{kG/Z(G)})$ is a direct summand of $\mathrm{Res}^{\overline{D(H)}}_{kG/Z(G)}(V_1 \otimes \cdots \otimes V_m)$ for some simple $\overline{D(H)}$-modules $V_1, \dots, V_m$. Using the fact that G is a p-group and the considerations preceding Theorem 4.2, we see that $l_w(D(H))$ is the smallest integer m such that $V_1 \otimes \cdots \otimes V_m$ contains the direct summand parametrized by $(\mathrm{Ind}_{Z(G)}^{G}(\mathbf{1}_{Z(G)}), 1)$, for some simple $D(H)$-modules $V_1, \ldots, V_m$.

It follows from Lemma 4.8 and [5, Corollary 3.11] that $l_w(D(H)) \leq z(G)$.

To show the reverse inequality, put $m := l_w(D(H))$ and let $h_1, \ldots, h_m$ be elements of G such that $V_1 \otimes \cdots \otimes V_m$ contains a direct summand P corresponding to $(\mathrm{Ind}_{Z(G)}^{G}(\mathbf{1}_{Z(G)}), 1)$, where V_i denotes the simple $D(H)$-module corresponding to h_i. (Since G is a p-group, there is just one simple $D(H)$-module for each conjugacy class of G.)

Then there is an indecomposable direct summand V of $V_2 \otimes \cdots \otimes V_m$ such that P is a direct summand of $V_1 \otimes V$. Since V is indecomposable, there is some $g \in G$ such that V corresponds to (W, g) for an indecomposable $C_G(g)$-module W. Since $V_1 \otimes V$ has a non-zero component beloning to the conjugacy class of the 1-element of G, we conclude that g is conjugate to h_1^{-1}. Replacing g by a conjugate, we may assume that $gh_1 = 1$.

Since V is a component of $V_2 \otimes \cdots \otimes V_m$, the element g is of the form $g_m \cdots g_2$ with elements g_i conjugate to h_i for $2 \leq i \leq m$. Moreoever, $W \cong \mathrm{Ind}_{C_G(g_2,\ldots,g_m)}^{C_G(g)}(\mathbf{1}_{C_G(g_2,\ldots,g_m)})$. Now if $C_G(g_2, \ldots, g_m) \neq Z(G)$, then the 1-component of $V_1 \otimes V$ does not contain P as a direct summand. It follows that $z(G) \leq m = l_w(D(H))$. $\square$

Corollary 4.11. *The invariant $l_w(D(kG))$ can become arbitrarily large as G varies through the finite p-groups.*

Proof. Let G be an extraspecial p group of order p^{2m+1} (see [1, Section 23]). We claim that $z(G) = 2m + 1$. On the one hand, if U is a proper subgroup of G, then $C_G(U)$ properly contains $Z(G)$. To see this, we may assume that U is a maximal subgroup of G. Then, with respect to the commutator form, $U/Z(G)$ is a symplectic vector space of dimension $2m - 1$ (cf. [1, 23.10]). But then $U/Z(G)$ is degenerate which means that there is an element of $U \setminus Z(G)$ centralizing U.

On the other hand, any set of less then $2m$ elements generates a proper subgroup of G. This is the case since $G/\Phi(G)$ is elementary abelian of order p^{2m}, where $\Phi(G)$ denotes the Frattini subgroup of G. $\square$

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