ON THE SOCLE OF AN ENDOMORPHISM ALGEBRA

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ABSTRACT. The socle of an endomorphism algebra of a finite dimensional module of a finite dimensional algebra is described. The results are applied to the modular Hecke algebra of a finite group with a cyclic Sylow subgroup.

1. INTRODUCTION

The interplay between a module Y of a finite dimensional algebra $\mathbf{A}$ over an algebraically closed field k and the endomorphism ring $\mathbf{E}$ of Y has always been of special interest. The corresponding Hom-functor relates the module categories of $\mathbf{A}$ and $\mathbf{E}$. For example, this functor realizes the Fitting correspondence between the indecomposable direct summands of Y and the projective indecomposable modules of $\mathbf{E}$.

Our main interest is in the case where $\mathbf{E}$ is a modular Hecke algebra. By this we mean an algebra of the form $\mathbf{E} = \mathrm{End}_{kG}(\mathrm{Ind}_P^G(k))$, where G is a finite group, k is an algebraically closed field of characteristic p, and P is a Sylow p-subgroup of G. The successful use of the modular Hecke algebra in connection with Alperin's Weight Conjecture by Cabanes in [3] is a strong motivation for further study. The experimental part of [10, 11] has been focussing expecially on the structural meaning of the socle of a modular Hecke algebra $\mathbf{E}$. It is the purpose of this paper to shed light on some of the experimental results observed in [10, 11]. In particular, we can now explain the outcome of the experiments in case of a p-modular Hecke algebra of a finite group with a cyclic Sylow p-subgroup (see Theorem 1.3 below). The main device for achieving this is a convenient description of the socle of a (general) endomorphism algebra.

Before stating our theorems, let us fix some notation and assumptions.

Hypothesis 1.1. Let k be an algebraically closed field and $\mathbf{A}$ a finite dimensional k-algebra. Furthermore, let Y be a finitely generated right

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A-module with a decomposition

$$Y = \bigoplus_{j=1}^{n} Y_j$$

into indecomposable direct summands. Throughout the whole paper we assume that

(a) $Y_i \cong Y_j$ if and only if $i = j$,
(b) The head and socle of Y have the same composition factors (up to isomorphism, disregarding multiplicities).

We denote the endomorphism ring $\mathrm{End}_{\mathbf{A}}(Y)$ of Y by $\mathbf{E}$, and the covariant Hom-functor $\mathrm{Hom}_{\mathbf{A}}(Y, -)$ by F.

Condition (a) of the above hypothesis is only introduced for convenience. Without this "multiplicity freeness" one would obtain a Morita equivalent endomorphism ring.

In the first part of this paper we describe the socle of the right regular $\mathbf{E}$-module in terms of the structure of Y. Since the Hom-functor F is left exact, we may view $F(S)$ as a submodule of $\mathbf{E}_{\mathbf{E}}$ for every $S \leq Y$. It is easy to see that every simple submodule of $\mathbf{E}_{\mathbf{E}}$ is isomorphic to a submodule of $F(S)$ for some simple $S \leq Y$. Of particular interest are thus the socles of the modules $F(S)$ for simple $\mathbf{A}$-modules S. As a consequence of our description in Corollary 3.4 we obtain the following result.

Theorem 1.2. *Let the notation and assumptions be as in Hypothesis 1.1. Suppose in addition that one of the following conditions is satisfied.*

(a) *The head of each Y_j is simple, and for each simple module $S \leq \mathrm{hd}(Y)$, there is at most one non-projective direct summand Y_j of Y with head S.*

(b) *The algebra $\mathbf{A}$ is symmetric, and for each simple module $S \leq \mathrm{hd}(Y)$, the projective cover of S is isomorphic to one of the Y_j.*

Then the map $S \mapsto \mathrm{soc}(F(S))$ yields a bijection between the isomorphism classes of the simple $\mathbf{A}$-modules in the head of Y and the isomorphism classes of the simple submodules of $\mathbf{E}_{\mathbf{E}}$.

This theorem is related to the main results of Green in [7]. In this reference, Green assumes Hypothesis 1.1(b), and in addition that $\mathbf{E}$ is self-injective. Green shows that this latter condition forces the heads and socles of the Y_j to be simple. Moreover, the Y_j are determined by their heads and socles up to isomorphism. Green then obtains the stronger conclusion that $S \mapsto F(S)$ yields a bijection between the isomorphism classes of the simple $\mathbf{A}$-modules in the head of Y and the isomorphism classes of all simple $\mathbf{E}$-modules.

In the second part of this paper, we apply the previous theorem to special cases, in particular to the modular Hecke algebra of blocks with cyclic defect groups. Our main result is contained in the following theorem.

Theorem 1.3. *Let kG be the group ring over k for a finite group G and let P be a Sylow p-subgroup of G. Let $\mathbf{A}$ be a sum of blocks of kG with cyclic defect groups. Assume one of the following conditions.*
(a) *G is p-solvable;*
(b) *Each block in $\mathbf{A}$ has defect at most 1;*
(c) *$\mathbf{A}$ is the principal block. (Thus P is cyclic in this case.)*
Let Y be the $\mathbf{A}$-component of the permutation module $\mathrm{Ind}_P^G(k)$ and put $\mathbf{E} = \mathrm{End}_{\mathbf{A}}(Y)$. Then each non-projective indecomposable direct summand of Y is uniserial. Moreover, in Cases (b) and (c), the hypothesis of Theorem 1.2 is satisfied, the PIMs of $\mathbf{E}$ have simple socles, and for each simple $\mathbf{E}$-module T, there are at most two non-isomorphic PIMs of $\mathbf{E}$ with T as socle.

We conclude our paper with some examples demonstrating the relevance of the hypotheses of Theorem 1.3 (Subsection 4.4).

2. Preliminaries on basic algebras

Throughout this section, k is an algebraically closed field and $\mathbf{A}$ a finite dimensional basic k-algebra. The group of units of $\mathbf{A}$ is denoted by $\mathbf{A}^*$. Let $1_{\mathbf{A}} = \sum_{i=1}^n e_i$ be a decomposition of $1_{\mathbf{A}}$ into pairwise orthogonal, primitive idempotents.

We are interested in describing the simple $\mathbf{A}$-submodules of $e_j\mathbf{A}$ for $j \in \{1, \ldots, n\}$.

Lemma 2.1. *Fix $1 \leq j \leq n$ and let $a \in e_j\mathbf{A}$ such that $\langle a \rangle_k$ is a simple submodule of $e_j\mathbf{A}$. Then there is some i, $1 \leq i \leq n$, such that $\langle a \rangle_k = \langle e_j a e_i \rangle_k$.*

Proof. Indeed, we have $a = \sum_{i=1}^n ae_i$, and the set of non-zero summands is linearly independent. Since $ae_i \in \langle a \rangle_k$ for all i, it follows that there is some i with $\langle a \rangle_k = \langle ae_i \rangle_k$. By assumption $ae_i = e_j ae_i$. □

Lemma 2.2. *Let $i, i', j \in \{1, \ldots, n\}$ and $a \in e_j\mathbf{A}e_i$, $a' \in e_j\mathbf{A}e_{i'}$. Then the following hold.*
(a) *$a'\mathbf{A} \subseteq a\mathbf{A}$ if and only if there exists $b \in e_i\mathbf{A}e_{i'}$ with $a' = ab$.*
(b) *Suppose that $a \neq 0$. Then $a'\mathbf{A} = a\mathbf{A}$ if and only if $i = i'$ and there exist $b \in (e_i\mathbf{A}e_i)^*$ with $a' = ab$.*

Proof. (a) This is trivial. (b) By (a), there is $b \in e_i\mathbf{A}e_{i'}$ and $c \in e_{i'}\mathbf{A}e_i$ with $a' = ab$ and $a = a'c$. Thus $a = a(bc)$ and $a(bc)^m = a$ for all positive integers m. Hence $bc \notin J(\mathbf{A})$ as $a \neq 0$. Since $\mathbf{A}$ is basic, this implies $i' = i$ and $bc \in (e_i\mathbf{A}e_i)^*$. □

Definition 2.3. (a) Let $i, j \in \{1, \ldots, n\}$. For $a, a' \in e_j\mathbf{A}e_i$ we write $a \sim a'$ if and only if $a\mathbf{A} = a'\mathbf{A}$. This is an equivalence relation on $e_j\mathbf{A}e_i$ and the equivalence classes are denoted by $[a]$, $a \in e_j\mathbf{A}e_i$.

(b) Put $\mathcal{K}_j := \{[a'] \mid a' \in e_j\mathbf{A}e_{i'} \text{ for some } 1 \leq i' \leq n\}$.

(c) For $[a], [a'] \in \mathcal{K}_j$ we write $[a] \leq [a']$ if and only if $a'\mathbf{A} \subseteq a\mathbf{A}$. (By Lemma 2.2, this is well defined and gives a partial order on $\mathcal{K}_j$.)

By definition, the maximal elements of $\mathcal{K}_j$ correspond to the simple $\mathbf{A}$-submodules of $e_j\mathbf{A}$. The following lemma provides a characterisation of the minimal and maximal elements of $\mathcal{K}_j$.

Lemma 2.4. *Let $i, j \in \{1, \ldots, n\}$ and let $a \in e_j\mathbf{A}e_i$.*

(a) *Suppose that $a \neq 0$. Then $[a] \in \mathcal{K}_j$ is maximal, if and only if the following holds: If $i' \in \{1, \ldots, n\}$ and $b \in e_j\mathbf{A}e_{i'}$ with $ab \neq 0$, then $i' = i$ and $b \in (e_i\mathbf{A}e_i)^*$.*

(b) *Suppose that $a\mathbf{A} \neq e_j\mathbf{A}$. Then $[a] \in \mathcal{K}_j$ is minimal, if and only if the following holds: If $i' \in \{1, \ldots, n\}$ and $a' \in e_j\mathbf{A}e_{i'}$, $b \in e_{i'}\mathbf{A}e_i$ with $a = a'b$, then $i' = i$ and $b \in (e_i\mathbf{A}e_i)^*$ (i.e. $[a] = [a']$), or $i' = j$ and $a' \in (e_j\mathbf{A}e_j)^*$.*

Proof. This follows directly from Lemma 2.2. □

3. The socle of $\mathbf{E}_\mathbf{E}$

Throughout this section we assume Hypothesis 1.1. This implies that $\mathbf{E}$ is a basic algebra, i.e., each simple $\mathbf{E}$-module is one-dimensional. Homomorphisms are written and composed from the left, i.e., $\varphi(y)$ denotes the image of $y \in Y$ under $\varphi \in \mathbf{E}$, and $\varphi\psi(y) = \varphi(\psi(y))$ for $y \in Y$. All $\mathbf{A}$-modules are assumed to be right modules and finitely generated, unless explicitly stated otherwise. Recall that F denotes the covariant Hom-functor $\mathrm{Hom}_\mathbf{A}(Y, -)$ from the category of finitely generated right $\mathbf{A}$-modules to the category of finitely generated right $\mathbf{E}$-modules.

For $1 \leq j \leq n$ we write $\varepsilon_j \in \mathbf{E}$ for the projection of Y onto Y_j. Then $\sum_{j=1}^{n} \varepsilon_j = \mathrm{id}_Y$ is a decomposition of id_Y into pairwise orthogonal primitive idempotents. We identify $\mathbf{E}_j := \mathrm{Hom}_\mathbf{A}(Y, Y_j)$ with $\varepsilon_j\mathbf{E}$ for $1 \leq j \leq n$, and $\mathrm{Hom}_\mathbf{A}(Y_i, Y_j)$ with $\varepsilon_j\mathbf{E}\varepsilon_i$ for $1 \leq i, j \leq n$.

3.1. **The general case.** The socle of $\mathbf{E}_\mathbf{E}$ equals the direct sum of the socles of the $\mathbf{E}_j$. We may therefore restrict our attention to the socles of the latter. These have been investigated in Section 2. We begin with a further reduction.

Lemma 3.1. *Fix $1 \leq i, l \leq n$. Let $\psi \in \varepsilon_l \mathbf{E} \varepsilon_i$ be such that $\langle \psi \rangle_k$ is a simple submodule of $\mathbf{E}_l$. Then there is a simple submodule S of Y_j for some $1 \leq j \leq n$ and an element $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$, such that $\varphi(Y_i) = S$ and $\langle \psi \rangle_k \cong \langle \varphi \rangle_k$ as $\mathbf{E}$-modules.*

Proof. Put $M := \psi(Y_i)$ and let S' denote a simple quotient of M. Thus S' is isomorphic to a head constituent of Y, and hence there is some $1 \leq j \leq n$ and a simple submodule S of Y_j isomorphic to S'. Let ι denote a non-zero map $M \to S$ and put $\varphi := \iota\psi$. Then $\varphi\mathbf{E} = \iota\psi\mathbf{E} = \iota\langle \psi \rangle_k = \langle \varphi \rangle_k$. Thus $\langle \varphi \rangle_k$ is a simple submodule of $\mathbf{E}_j$ isomorphic to $\langle \psi \rangle_k$. □

The above observation motivates the following definition, where we use the notation from Definition 2.3. Note that if $\varphi, \varphi' \in \varepsilon_j \mathbf{E} \varepsilon_i$ are equivalent in the sense of that definition, we have $\varphi(Y) = \varphi'(Y)$ by Lemma 2.2(b), and if $[\varphi] \leq [\varphi']$, we have $\varphi'(Y) \leq \varphi(Y)$.

Definition 3.2. Fix some $1 \leq j \leq n$ and an $\mathbf{A}$-submodule $0 \neq S \leq Y_j$. Put

$$\mathcal{K}_{S,j} := \{[\varphi] \mid 0 \neq \varphi \in \bigcup_{i=1}^{n} \varepsilon_j \mathbf{E} \varepsilon_i, \varphi(Y) \leq S\}.$$

Suppose that $S \leq S' \leq Y_j$. Then $\mathcal{K}_{S,j} \subseteq \mathcal{K}_{S',j}$, and the partial order on $\mathcal{K}_{S',j}$ restricts to that of $\mathcal{K}_{S,j}$. Moreover, $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, if and only if it is maximal in $\mathcal{K}_{S',j}$. In particular, if $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, it is also maximal in $\mathcal{K}_{\varphi(Y),j}$ and in $\mathcal{K}_{Y_j,j}$.

If an $\mathbf{A}$-module can be embedded into two distinct direct summands Y_j and $Y_{j'}$ of Y, the configuration we consider is "independent" of the particular direct summand used to define it.

Lemma 3.3. *Assume that $S' \leq Y_{j'}$ for some $1 \leq j' \leq n$ and that $\iota : S \to S'$ is an isomorphism. Then*

$$\mathcal{K}_{S,j} \to \mathcal{K}_{S',j'}, \quad [\varphi] \mapsto [\iota\varphi],$$

is a bijection preserving maximal elements. If $[\varphi] \in \mathcal{K}_{S,j}$ spans a simple submodule of $\mathbf{E}_{\mathbf{E}}$, then so does $[\iota\varphi]$, and the two $\mathbf{E}$-modules are isomorphic.

Proof. This is obvious. □

We collect the main results of this section in the following corollary.

Corollary 3.4. (a) *Fix some $1 \leq j \leq n$ and a submodule $S \leq Y_j$. Then there is a bijection between the simple submodules of $F(S)$ and the maximal elements of $\mathcal{K}_{S,j}$.*

(b) *Suppose that Y_j is projective and that $S = \mathrm{soc}(Y_j) \cong \mathrm{hd}(Y_j)$. Then $\mathcal{K}_{S,j}$ has a unique maximal element $[\varphi]$ with $\varphi \in \mathrm{Hom}_{\mathbf{A}}(Y_j, S) \leq F(S)$. Moreover, the head and the socle of $\mathbf{E}_j$ are simple and isomorphic.*

Proof. (a) This is clear by the results of Section 2 and the remarks following Definiton 3.2.

(b) Let $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_j$ denote the epimorphism of Y_j onto S. Then $[\varphi]$ is the unique maximal element of $\mathcal{K}_{S,j}$ by our assumptions on Y_j. Let $\langle \psi \rangle_k \leq \mathbf{E}_j$ be a simple submodule. By Lemma 2.1, we may assume that $\psi \in \varepsilon_j \mathbf{E} \varepsilon_i$ for some $1 \leq i \leq n$. Let M be the image of ψ in Y_j. Note, that $M \leq Y_j$ has S as socle, so that there is a homomorphism $\tilde{\varphi} : Y_j \to S \leq M$.

By the projectivity of Y_j, there is a homomorphism $\eta : Y_j \to Y_i$, such that $\psi\eta = \tilde{\varphi}$. Since $\langle \psi \rangle_k$ is an $\mathbf{E}$-submodule we have $\langle \varphi \rangle_k = \langle \tilde{\varphi} \rangle_k = \langle \psi\eta \rangle_k = \langle \psi \rangle_k$. □

Next, we determine the isomorphism types corresponding to the maximal elements of $\mathcal{K}_{S,j}$.

Lemma 3.5. *Let $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$ and assume that $[\varphi] \in \mathcal{K}_{S,j}$ is maximal for some $0 \neq S \leq Y_j$. Then the simple socle constituent $\langle \varphi \rangle_k$ of $\mathbf{E}_j$ is isomorphic to the head constituent of $\mathbf{E}_i$.*

Proof. For any $1 \leq l \leq n$ we have

$$\mathrm{Hom}_{\mathbf{E}}\left(\varepsilon_l \mathbf{E}, \langle \varphi \rangle_k\right) \cong \langle \varphi \rangle_k \varepsilon_l \tag{1}$$

as k-vector spaces. Now $\langle \varphi \rangle \varepsilon_l$ is non-trivial, if and only if $l = i$. In this case, $\langle \varphi \rangle_k \varepsilon_i = \langle \varphi \rangle_k$. By Equation (1), the assertion is now obvious. □

If $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, then $\varphi(Y)$ need not be simple. An example for this is given in the following, which also illustrates Lemma 3.1.

Example 3.6. (Naehrig, [10]) Suppose that $\mathrm{char}(k) = 2$ and that $G = S_7$ is the symmetric group on seven letters. Let Y denote the component of $\mathrm{Ind}_P^G(k)$ in the non-principal 2-block of kG.

Then Y has three indecomposable direct summands. These are uniserial and of dimensions 6, 20, and 28, respectively. In what follows, we will denote modules by their dimension, e.g., **6** denotes a module of dimension 6. The ascending composition series are given as follows.

$$\begin{aligned} Y_1 &: \quad \mathbf{6} \\ Y_2 &: \quad \mathbf{6}, \mathbf{8}, \mathbf{6} \\ Y_3 &: \quad \mathbf{8}, \mathbf{6}, \mathbf{6}, \mathbf{8}, \end{aligned}$$

where **6** and **8** are simple. Then

$$\mathcal{K}_{\mathbf{6},1} = \{[\varphi_1] : Y_1 \to \mathbf{6}, [\varphi_2] : Y_2 \to \mathbf{6}\}$$

and

$$\mathcal{K}_{\mathbf{8},3} = \{[\psi] : Y_3 \to \mathbf{8}\}.$$

We immediately see that $[\varphi_2]$ and $[\psi]$ are the maximal elements of $\mathcal{K}_{\mathbf{6},1}$ and $\mathcal{K}_{\mathbf{8},3}$, respectively. By Lemmas 3.3 and 3.5, there are two isomorphism types of socle constituents of the corresponding **E**-PIMs $F(Y_1)$, $F(Y_2)$ and $F(Y_3)$. Our analysis shows that $\mathrm{soc}(F(Y_1)) = \langle\varphi_2\rangle_k$ and $\mathrm{soc}(F(Y_3)) = \langle\psi\rangle_k$ are simple, while $\mathrm{soc}(F(Y_2)) \cong \langle\varphi_2, \psi\rangle_k$. In fact, the socle constituent of $F(Y_2)$ isomorphic to $\langle\psi\rangle_k$ is spanned by a homomorphism $\eta : Y_3 \to Y_2$ with image **14**.

3.2. **Proof of Theorem 1.2.** Let S be a simple quotient of Y. We may assume that $S \leq Y_j$ for some $1 \leq j \leq n$. By assumption and Corollary 3.4(b), $\mathcal{K}_{S,j}$ contains a unique maximal element. By Part (a) of this corollary, the socle of $F(S)$ is simple. Suppose that $S' \leq Y_{j'}$ is simple and $S \not\cong S'$. Then the maximal elements in $\mathcal{K}_{S,j}$ and $\mathcal{K}_{S',j'}$ arise from different direct summands of Y. Lemma 3.5 implies that $\mathrm{soc}(F(S)) \not\cong \mathrm{soc}(F(S'))$. Thus the given map is injective. By Lemma 3.1 it is also surjective.

4. Blocks with a cyclic defect group

In this second part of our paper we apply the previous results to the case of p-blocks of finite groups with cyclic defect groups. In particular, we are able to explain the computational results of [10] in the case of groups with cyclic Sylow p-subgroups. There, the third author found that in all computed examples, the PIMs of the modular Hecke algebra had simple socles. Moreover, the number of PIMs with the same socle (up to isomorphism) was at most two.

Throughout this section, let G be a finite group and let P be a Sylow p-subgroup of G. The main difficulty is to describe the indecomposable direct summands of $\mathrm{Ind}_P^G(k)$ in cyclic blocks of kG. Using Green correspondence, one can reduce this problem to the centralizer of a subgroup of order p. This is p-nilpotent, if P is cyclic.

4.1. **Preliminaries.** Let **B** be a block of kG with a cyclic defect group $Q \leq P$, and let $P_1 \leq Q$ have order p. Put $N := N_G(P_1)$, $C := C_G(P_1)$. By **b** we denote the Brauer correspondent of **B** in kN. The following lemma restricts the structure of C and of N in a special case.

Lemma 4.1. *Suppose that $Q = P$ is a Sylow p-subgroup of G. Then C is p-nilpotent and N/C is cyclic of order dividing $p-1$. In particular, N is p-solvable.*

Proof. Since $C \trianglelefteq N$ and $|N/C| \mid p-1$, it suffices to prove the first assertion.

Now p does not divide $|C' \cap Z(C)|$ by a well known transfer argument (see e.g., [8, Theorem (5.6)]). Hence P_1 is not contained in C'. Thus C' is a p'-group, since P_1 is the unique subgroup of C of order p. The result follows. □

Lemma 4.2. *Suppose that* $\mathbf{B}$ *has defect* 1 *or is the principal block. Then every non-projective indecomposable direct summand* U *of* $\mathrm{Ind}_P^G(k)$ *in* $\mathbf{B}$ *has maximal vertex and a simple Green-correspondent.*

Proof. Suppose first that $\mathbf{B}$ has defect 1. The simple $\mathbf{b}$-modules have vertex Q and thus have trivial source. Hence the non-projective trivial source $\mathbf{b}$-modules are exactly the simple $\mathbf{b}$-modules.

Now assume that $\mathbf{B}$ is the principal block. Then C is p-nilpotent by Lemma 4.1. The restriction of $\mathrm{Ind}_P^C(k)$ to the principal block of kC is a direct sum of trivial modules. Thus the restriction of $\mathrm{Ind}_P^N(k)$ to the principal block of kN is semisimple. Since U is a direct summand of $\mathrm{Ind}_N^G(\mathrm{Ind}_P^N(k))$, the result follows. □

4.2. **The proof of Theorem 1.3.** Suppose first that G is p-solvable. Then the Brauer tree of $\mathbf{B}$ is a star with its exceptional node at its centre (see [5, Lemma X.4.1]). This implies that any $\mathbf{B}$-module is uniserial, and we are done.

No suppose that the defect of $\mathbf{B}$ is 1 or that $\mathbf{B}$ is the principal block. By Lemma 4.2, every non-projective indecomposable direct summand U of Y contained in $\mathbf{B}$ has a simple Green correspondent in $\mathbf{b}$. By [1, Lemma 22.3], this implies that U is uniserial, thus proving the first claim of Theorem 1.3.

Let us now turn to the proof of the remaining assertions of the theorem. By replacing $\mathbf{E}$ by its basic algebra, we may assume that Y satisfies Hypothesis 1.1(a). The second part of this hypothesis is clearly satisfied by $\mathrm{Ind}_P^G(k)$ and hence by Y. In Cases (b) and (c), the indecomposable direct summands of Y are either projective or of maximal vertex with a simple Green correspondent (see Lemma 4.2). In particular, Hypothesis (a) of Theorem 1.2 is satisfied. Fix j, $1 \leq j \leq n$. If Y_j is projective, then $\mathbf{E}_j$ has a simple socle by Corollary 3.4(b). Suppose that Y_j is not projective. Then Y_j is uniserial, and the composition factors of Y_j, from top to bottom, arise from a cyclic walk around a vertex of the Brauer tree of the block containing Y_j (see [1, Theorem 22.1, Lemma 22.3]).

Let $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$ for some $1 \leq i \leq n$ be such that $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, where $S := \varphi(Y) \leq Y_j$. We claim that S is simple. If not, let S_0 and S_1 denote the socle and the head of S, respectively. Suppose first that Y_i is projective. Let Y_l denote the indecomposable direct summand of Y which is in the Ω^2-orbit of Y_j and which has socle S_1 (such a direct summand exists since $\Omega^2_{kP}(k) \cong k$ and since the Heller operator "commutes" with induction). Then there is an embedding $\psi : Y_l \to Y_i$ such that $\psi(Y_l) \not\leq \ker(\varphi)$. Indeed, the multiplicity of S_0 as a composition factor of Y_i is equal to the multiplicity of S_0 as a composition factor of Y_l. Thus $\varphi\psi \neq 0$, contradicting the maximality of $[\varphi]$. Finally, suppose that Y_i is not projective. Then there is an $1 \leq l \leq n$ and a homomorphism $\psi : Y_l \to Y_i$ with $\psi(Y_l) = \mathrm{rad}(Y_i)$. Again, $\varphi\psi \neq 0$, a contradiction.

We have thus proved that S is simple, which implies $S = \mathrm{soc}(Y_j)$. It now follows from Corollary 3.4 and the fact that there is at most one non-projective direct summand of Y with a given head, that $\mathcal{K}_{S,j}$ has a unique maximal element, and thus $\mathrm{soc}(\mathbf{E}_j)$ is simple. The last assertion follows from the fact that there are at most two non-isomorphic indecomposable direct summands of Y with isomorphic socles.

4.3. **The components of $\mathrm{Ind}_P^G(k)$ in groups with cyclic Sylow subgroups.** We return to the assumption and notation of Subsection 4.1. Assume in addition that a Sylow p-subgroup P of G is cyclic. Our aim is to describe the indecomposable direct summands of $\mathrm{Ind}_P^G(k)$ in terms of their Green correspondents.

4.3.1. *The p-nilpotent case.* We first investigate the situation in C. Thus assume in addition that G is p-nilpotent, i.e., $G = PM$ with $M = O_{p'}(G)$ being a normal p-complement. In this case $\mathrm{Ind}_P^G(k) \cong kM$ as kG-modules, where P acts on kM by conjugation.

Every p-block of kG has a unique simple module, and since P is cyclic, every indecomposable module in such a block is uniserial and uniquely determined by its composition factor V and its composition length ℓ. We write $J(\ell, V)$ for such an indecomposable kG-module, using a similar convention for subgroups of G.

Lemma 4.3. *Let V be a simple G-invariant kM-module and let ε denote the centrally primitive idempotent of kM corresponding to V. Then ε is G-invariant and εkM is a kG-module with P acting by conjugation. By V we also denote the unique extension of V to G. Then*

εkG is the block of kG containing V. Suppose that

$$\mathrm{Res}^G_P(V) = J(\ell_1, k) \oplus J(\ell_2, k) \oplus \ldots \oplus J(\ell_r, k)$$

with positive integers ℓ_i, $1 \leq i \leq r$. Then

$$\varepsilon kM \cong J(\ell_1, V) \oplus J(\ell_2, V) \oplus \ldots \oplus J(\ell_r, V)$$

as a kG-module.

Proof. It follows from [4, Theorem 13.13] that $\mathrm{End}_{kG}(\varepsilon kM) \cong \mathrm{End}_{kP}(V)$. The isomorphism induces a bijection $\pi \mapsto \pi'$ between the centrally primitive idempotents of $\mathrm{End}_{kG}(\varepsilon kM)$ and those of $\mathrm{End}_{kP}(V)$, such that $\pi\mathrm{End}_{kG}(\varepsilon kM)\pi \cong \pi'\mathrm{End}_{kP}(V)\pi'$. The dimension of $\pi\mathrm{End}_{kG}(\varepsilon kM)\pi$ equals the composition length of the direct summand $\pi\varepsilon M$ of εM. This implies the result. □

If ε is not P-invariant, we consider the stabilizer Q of ε in P and apply the lemma to QM. Then induction yields a Morita equivalence between the block εkQM and the kG-block covering εkM (see [2, Theorem 6.4.1]).

In the situation of Lemma 4.3, the module $\mathrm{Res}^G_P(V)$ is an endo-permutation kP-module by [4, Theorem 13.13]. Thus every indecomposable direct summand of $\mathrm{Res}^G_P(V)$ is an endo-permutation module as well. The indecomposable endo-permutation modules of a cyclic p-group are classified (see [13, Exercise (28.3)]).

Since the dimension of V is prime to p, the vertex of V equals P. Let S denote a source of V. Thus $S \mid \mathrm{Res}^G_P(V)$ and $V \mid \mathrm{Ind}^G_P(S)$. In particular, S is an indecomposable endo-permutation kP-module with vertex P. Moreover,

$$\mathrm{Res}^G_P(V) \mid \mathrm{Res}^G_P(\mathrm{Ind}^G_P(S)).$$

By Mackey's theorem, the indecomposable summands of $\mathrm{Res}^G_P(V)$ are of the form $\mathrm{Ind}^P_{P^g \cap P}(T)$, where $g \in G$ and T is an indecomposable summand of $\mathrm{Res}^P_{P^g \cap P}(S)$.

4.3.2. *The situation in N.* Now assume that G has a normal subgroup C such that G/C is cyclic of order dividing $p-1$ and that $C = PM$ is p-nilpotent.

Lemma 4.4. *Let V be simple kC-module, and let $W = J(\ell, V)$ be a uniserial kC-module of composition length $\ell \leq p^n$ (for the notation see Subsection 4.3.1). Write H for the inertia group of V in G, and put $e := |H : C|$.*

Since $|G/C|$ is prime to p, $\mathrm{Ind}^G_C(V)$ is semisimple,

$$\mathrm{Ind}^G_C(V) = V_1 \oplus \ldots \oplus V_e$$

with pairwise non-isomorphic simple kG-modules V_i. Moreover,

$$\mathrm{Ind}_C^G(W) \cong V_{1,\ell} \oplus \ldots \oplus V_{e,\ell},$$

where $V_{i,\ell}$ denotes the indecomposable kG-module with head isomorphic to V_i and composition length ℓ.

Proof. This is just an application of Clifford theory. □

4.3.3. *The general case.* From the preceding considerations we obtain information about the indecomposable direct summands of $\mathrm{Ind}_P^G(k)$ in a block of kG. Let $\mathbf{B}$ be block of kG with defect group $P_1 \leq Q \leq P$, and let $\mathbf{b}$ denote the Brauer correspondent of $\mathbf{B}$ in N. We also choose a block $\mathbf{c}$ of kC covered by $\mathbf{b}$. Then Q is a defect group of $\mathbf{c}$. An indecomposable direct summand of $\mathrm{Ind}_P^G(k)$ lying in $\mathbf{B}$ will be called a $\mathbf{B}$-component of $\mathrm{Ind}_P^G(k)$; an analogous notation is used for the blocks $\mathbf{b}$ and $\mathbf{c}$. The indecomposable $\mathbf{b}$-modules and $\mathbf{c}$-modules are uniserial; and we write $\ell(U)$ for the composition length of a uniserial module U. The Green correspondence between the indecomposable modules of $\mathbf{B}$ and those of $\mathbf{b}$ is denoted by f.

Proposition 4.5. *Let V be the simple $\mathbf{c}$-module and let S be the source of V. Then S is an indecomposable endo-permutation kQ-module with vertex Q and trivial P_1-action, and the following statements hold.*

(a) *If U is a non-projective $\mathbf{B}$-component of $\mathrm{Ind}_P^G(k)$ with vertex $R \leq Q$, then $\ell(f(U)) = |Q\colon R|\ell(T)$, where T is an indecomposable direct summand of $\mathrm{Res}_R^Q(S)$.*

(b) *If T is an indecomposable direct summand of $\mathrm{Res}_Q^C(V)$, then there is a a non-projective $\mathbf{B}$-component U of $\mathrm{Ind}_P^G(k)$ with $\ell(f(U)) = \ell(T)$.*

Proof. Clearly, V and hence S have vertex Q, since Q is a defect group of $\mathbf{c}$. Also, P_1 acts trivially on V, hence also on S.

We have $\mathrm{Ind}_P^G(k) \cong \mathrm{Ind}_N^G\left(\mathrm{Ind}_P^N(k)\right)$, and f sets up a vertex preserving one-to-one correspondence between the non-projective $\mathbf{B}$-components of $\mathrm{Ind}_P^G(k)$ and the non-projective $\mathbf{b}$-components of $\mathrm{Ind}_P^N(k)$.

Since $\mathrm{Ind}_P^N(k) \cong \mathrm{Ind}_C^N\left(\mathrm{Ind}_P^C(k)\right)$, it suffices to investigate the composition lengths of the $\mathbf{c}$-components of $\mathrm{Ind}_P^C(k)$ by Lemma 4.4. Write $M := O_{p'}(C)$. Then $C = PM$ by Lemma 4.1 and $\mathrm{Ind}_P^C(k) \cong kM$ as kC-module.

By the remark following Lemma 4.3, the block $\mathbf{c}$ covers $|P\colon Q|$ conjugate blocks $\mathbf{c}_i$ of kQM with defect group Q, Morita equivalent to $\mathbf{c}$. Let $\varepsilon_i \in kM$ denote the block idempotent of $\mathbf{c}_i$, with Q stabilising ε_1. Then $\varepsilon = \sum_{i=1}^{|P:Q|} \varepsilon_i$ is the block idempotent of $\mathbf{c}$, and $\mathrm{Ind}_{QM}^{PM}(\varepsilon_1 kM) \cong \varepsilon kM$, i.e. the Morita equivalence between $\mathbf{c}_1$ and $\mathbf{c}$ sends $\varepsilon_1 kM$ to εkM.

We may thus assume that $Q = P$. Lemma 4.3 and the subsequent remarks imply the results. □

4.4. **Some examples.** In order to describe the modular Hecke algebra $\mathbf{E} = \mathrm{End}_{kG}(\mathrm{Ind}_P^G(k))$ in case P is cyclic, we have to determine the $\mathbf{B}$-components of $\mathrm{Ind}_P^G(k)$ for the blocks $\mathbf{B}$ of kG. By Lemma 4.4 and Proposition 4.5, this can be done locally, i.e. inside the p-nilpotent group $C = C_G(P)$. Mazza has shown in [9], that all indecomposable endo-permutation kP-modules occur as sources of simple modules in p-nilpotent groups with Sylow subgroup P. But by Lemma 4.3, apart from the sources, we have to take into account all indecomposable summands of the restrictions of the simple kC-modules to P.

Let us use Mazza's construction to consider two specific examples. Let P denote the cyclic group of order 7^2. Then P acts on a group M of order $13^3 \cdot 97^3$, the direct product of two extraspecial groups. By [9, 4.1, Theorem 5.3], the semidirect product PM has a simple module V of dimension $13 \cdot 97$ such that $\mathrm{Res}_P^{PM}(V) = (J(6,k) \oplus J(7,k)) \otimes (J(48,k) \oplus J(49,k)) \cong J(42,k) \oplus J(43,k) \oplus J(49,k)^{24}$. (The indecomposable direct summands of these tensor products can be computed with [12].)

Next, let P denote the cyclic group of order 17^2, and let $q = 577 = 2 \cdot 17^2 - 1$. By Mazza's construction, we get an action of P on the extraspecial q-group M_1 of order q^3 and exponent q in such a way that the semidirect product PM_1 has a representation V_1 of dimension 577 and $\mathrm{Res}_P^{PM_1}(V_1) = J(17^2 - 1, k) \oplus J(17^2, k)$. Now let M_2 denote the extraspecial group 2_-^{8+1} of minus type and order 512. Its automorphism group is an extension of an elementary abelian group of order 256 by the orthogonal group $O^-(8,2)$ (see [14, Theorem 1]). Let Q denote a Sylow 17-subgroup of $\mathrm{Aut}(M_2)$. A computation with GAP (see [6]) shows that Q has exactly two fixed points in its action on M_2 by conjugation. There is a simple kM_2-module V_2 of dimension 16, unique up to isomorphism. Letting P act on M_2 via the projection $P \to Q$, we find that V_2 extends to a kPM_2-module, also denoted by V_2. Since P has exactly one fixed point on $V_2^* \otimes_k V_2$, we have $\mathrm{Res}_P^{PM_2}(V_2) = J(16, k)$. Combining, we obtain an action of P on $M := M_1 \times M_2$, and a simple kPM-module $V = V_1 \otimes_k V_2$ such that $\mathrm{Res}_P^{PM}(V) = J(273, k) \oplus J(17^2, k)^{31}$. Thus the corresponding 17-block of PM contains a unique non-projective direct summand of $\mathrm{Ind}_P^{PM}(k)$, namely $J(273, V)$.

It is at least feasible, though not very likely, that there is a non-17-solvable group G such that $PM = C_G(P_1)/P_1$, where P_1 is the subgroup of order 17 of a cyclic defect group $\hat{P}$ of order 17^3 of a block $\mathbf{B}$ of G, whose Brauer tree is a straight line with 4 edges, say. Then, by the results of this section, the non-projective $\mathbf{B}$-component Y of $\mathrm{Ind}_{\hat{P}}^G(k)$

would consist of a direct sum of four indecomposable modules whose Brauer correspondents all have length 273. In this case, the heads of these modules would not be simple. Moreover, by the results of our first section, the Hom-functor F corresponding to $\mathbf{E} = \mathrm{End}_{kG}(Y)$ would not have the property that $F(S)$ has a simple socle for all simple $\mathbf{A}$-modules S. Such a hypothetical configuration could presumably only be ruled out with the help of the classification of the finite simple groups.

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