Representations of finite groups of Lie type

Lecture I: Harish-Chandra philosophy

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

Summer School
Finite Simple Groups and Algebraic Groups:
Representations, Geometries and Applications
Berlin, August 31 – September 10, 2009

MOTIVATION: CLASSIFICATION OF REPRESENTATIONS

Let G be a finite group and k an algebraically closed field.

1. There are only finitely many irreducible k-representations of G up to equivalence.
2. Classify all irreducible representations of G.

AIM

Classify all irreducible representations of all finite simple groups and related finite groups.

By the classification of the finite simple groups, most finite simple groups are finite groups of Lie type.

In the following, unless otherwise said, let G be a finite reductive group of characteristic p.

FINITE REDUCTIVE GROUPS: RECOLLECTION

Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbb{F}}_p$ and let F be a Frobenius map of $\mathbf{G}$.

Then $G := \mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$ is a finite group, called a finite reductive group.

A finite reductive group is a finite group of Lie type, but the latter term is usually regarded in a broader sense.

For example, $\mathrm{PSL}_n(q)$ is a finite group of Lie type, but not a finite reductive group unless n and $q-1$ are coprime (in which case $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q)$).

This can be seen from the order formula for finite reductive groups (cf. Jean Michel's talk).

LEVI SUBGROUPS: RECOLLECTION

Recall that there is a distinguished class of subgroups of G, the parabolic subgroups.

One way to describe them is through the concept of split BN-pairs of characteristic p.

A parabolic subgroup P has a Levi decomposition $P = LU$ with $L \cap U = \{1\}$, where $U = O_p(P) \lhd P$ is the unipotent radical of P, and L is a Levi subgroup of G.

Levi subgroups of G resemble G; in particular, they are again groups of Lie type.

Inductively, we may use the representations of the Levi subgroups to obtain information about the representations of G.

This is the idea behind Harish-Chandra philosophy.

BN-PAIRS

This axiom system was introduced by Jacques Tits.

DEFINITION

The subgroups B and N of the group G form a BN-pair, if:

1. $G = \langle B, N \rangle$;
2. $T := B \cap N$ *is normal in* N;
3. $W := N/T$ *is generated by a set* S *of involutions;*
4. *If* $\dot{s} \in N$ *maps to* $s \in S$ *(under* $N \to W$*), then* $\dot{s}B\dot{s} \neq B$;
5. *For each* $n \in N$ *and* $\dot{s}$ *as above,* $(B\dot{s}B)(BnB) \subseteq B\dot{s}nB \cup BnB$.

W is called the Weyl group of the *BN*-pair G. It is a Coxeter group with Coxeter generators S.

Any conjugate of B is a Borel subgroup of G. A parabolic subgroup is one containing a Borel subgroup.

THE BN-PAIR OF $\mathrm{GL}_n(k)$ AND OF $\mathrm{SO}_n(k)$

Let k be a field and $G = \mathrm{GL}_n(k)$. Then G has a BN-pair with:

- B: group of upper triangular matrices;
- N: group of monomial matrices;
- $T = B \cap N$: group of diagonal matrices;
- $W = N/T \cong S_n$: group of permutation matrices.

Suppose that n is odd and $\mathrm{char}(k) \neq 2$. Define the special orthogonal group by $\mathrm{SO}_n(k) := \{g \in \mathrm{SL}_n(k) \mid g^{tr} J g = J\}$, where $J = (\delta_{i,n-j+1})$.

If B, N are as above, then

$$B \cap \mathrm{SO}_n(k), N \cap \mathrm{SO}_n(k)$$

is a BN-pair of $\mathrm{SO}_n(k)$.

Split BN-pairs of characteristic p

Let G be a group with a BN-pair (B, N).

This is said to be a split BN-pair of characteristic p, if the following additional hypotheses are satisfied:

6. $B = UT$ with $U = O_p(B)$, and T a complement of U.
7. $\bigcap_{n \in N} nBn^{-1} = T$. (Recall $T = B \cap N$.)

Examples

1. *A semisimple algebraic group over $\bar{\mathbb{F}}_p$ and a finite group of Lie type of characteristic p have split BN-pairs of characteristic p.*

2. *If $G = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ or $\mathrm{GL}_n(q)$, q a power of p, then U is the group of upper triangular unipotent matrices.*

   In the latter case, U is a Sylow p-subgroup of G.

Harish-Chandra induction

View G as a finite group with a split BN-pair of characteristic p.

Let $\mathfrak{k}$ be a commutative ring (with 1).

Let L be a Levi subgroup of G, and M a $\mathfrak{k}L$-module, free and finitely generated as $\mathfrak{k}$-module.

Let P be a parabolic subgroup with Levi complement L. Write $\widetilde{M}$ for the inflation of M to P.

Put

$$R^G_{L\subset P}(M) := \mathfrak{k}G \otimes_{\mathfrak{k}P} \widetilde{M},$$

the $\mathfrak{k}G$-module obtained from inducing $\widetilde{M}$ from P to G.

$R^G_{L\subset P}(M)$ is called a Harish-Chandra induced module.

INDEPENDENCE

THEOREM

If p is invertible in $\mathfrak{k}$, then $R^G_{L\subset P}(M)$ is independent of the choice of P with Levi complement L.

- Lusztig, 1970s (?): $\mathfrak{k}$ a field of characteristic 0
- Dipper-Du, 1993: $\mathfrak{k}$ a field of characteristic $\neq p$
- Howlett-Lehrer, 1994: p invertible in $\mathfrak{k}$

To prove the theorem following Howlett and Lehrer, first note:

$$R^G_{L\subset P}(M) \cong \mathfrak{k}Ge_U \otimes_{\mathfrak{k}L} M,$$

with $e_U = \frac{1}{|U|}\sum_{u\in U} u \in \mathfrak{k}G$.

The permutation module $\mathfrak{k}Ge_U = \mathfrak{k}[G/U]$ is a $\mathfrak{k}G$-bimodule-$\mathfrak{k}L$.

On the Howlett-Lehrer proof

Let $P' = LU'$ be another parabolic subgroup of G with Levi complement L.

Proposition (Howlett-Lehrer, 1994)

There is a $\mathfrak{k}G$-bimodule-$\mathfrak{k}L$ isomorphism $\mathfrak{k}Ge_U \to \mathfrak{k}Ge_{U'}$.

To prove this, we may assume that $B \subseteq P$, i.e. P is a standard parabolic subgroup.

Furthermore, there is $w \in W$ such that $V := {}^wU'$ is standard.

Proposition (Howlett-Lehrer, 1994)

The map $\mathfrak{k}Ge_V \to \mathfrak{k}Ge_U$, $x \mapsto x[e_V we_U]$ is a $\mathfrak{k}G$-isomorphism, which yields the desired bimodule isomorphism $\mathfrak{k}Ge_U \to \mathfrak{k}Ge'_U$.

AN EXAMPLE: $\mathrm{GL}_3(q)$

Let $G = \mathrm{GL}_3(q)$, where q is a power of p,

$$L = \left\{ \left[\begin{array}{cc|c} \star & \star & 0 \\ \star & \star & 0 \\ \hline 0 & 0 & \star \end{array} \right] \right\}, P = \left\{ \left[\begin{array}{cc|c} \star & \star & \star \\ \star & \star & \star \\ \hline 0 & 0 & \star \end{array} \right] \right\}, P' = \left\{ \left[\begin{array}{cc|c} \star & \star & 0 \\ \star & \star & 0 \\ \hline \star & \star & \star \end{array} \right] \right\}.$$

If $w = (1,2,3) \in S_3 = W$, then ${}^wP' = \left\{ \left[\begin{array}{c|cc} \star & \star & \star \\ \hline 0 & \star & \star \\ 0 & \star & \star \end{array} \right] \right\}$.

Thus $\mathfrak{k}Ge_U \cong \mathfrak{k}Ge_V$ with

$$U = \left\{ \left[\begin{array}{cc|c} 1 & 0 & \star \\ 0 & 1 & \star \\ \hline 0 & 0 & 1 \end{array} \right] \right\} \quad \text{and} \quad V = \left\{ \left[\begin{array}{c|cc} 1 & \star & \star \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \right\}.$$

Notice that U and V are **not** conjugate in G.

Centraliser algebras

From now on we suppress the P from the notation for Harish-Chandra induction, i.e. we write R_L^G for $R_{L\subset P}^G$.

With L and M as before, we write

$$\mathcal{H}(L, M) := \operatorname{End}_{\mathfrak{k}G}(R_L^G(M)).$$

for the endomorphism ring of $R_L^G(M)$.

$\mathcal{H}(L, M)$ is also called the centraliser algebra or Hecke algebra of $R_L^G(M)$.

$\mathcal{H}(L, M)$ is used to analyse the submodules and quotients of $R_L^G(M)$ via Fitting correspondence.

THE FITTING CORRESPONDENCE

Let A be a ring, X an A-module and $E := \mathrm{End}_A(X)$.

PROPOSITION (FITTING CORRESPONDENCE)

Suppose that $X = X_1 \oplus \cdots \oplus X_n$ is a direct decomposition of X into A-submodules X_i.
Put $E_i := \mathrm{Hom}_A(X, X_i)$, $1 \leq i \leq n$, viewed as a subset of E.
Then the following hold:

1. *The E_i are (right) ideals of E and $E = E_1 \oplus \cdots \oplus E_n$.*
2. *$E_i \cong E_j$ as E-modules if and only if $X_i \cong X_j$ as A-modules.*
3. *E_i is indecomposable as an E-module if and only if X_i is indecomposable as an A-module.*

This is an important link between the structures of X and of E.

Coxeter groups: Recollection

Recall that the Weyl group of G is a Coxeter group.

Let $M = (m_{ij})_{1 \leq i,j \leq r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$.

The group

$$W := W(M) := \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}},$$

is called the Coxeter group of M, the elements $s_1, \ldots, s_r$ are the Coxeter generators of W.

The relations $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ are called the braid relations.

In view of $s_i^2 = 1$, they can be written as $s_i s_j s_i \cdots = s_j s_i s_j \cdots$

THE IWAHORI-HECKE ALGEBRA

Let W be a Coxeter group with Coxeter matrix $M = (m_{ij})$.

Let $\mathfrak{k}$ be a commutative ring and $\mathbf{v} = (v_1, \ldots, v_r) \in \mathfrak{k}^r$ with $v_i = v_j$, whenever s_i and s_j are conjugate in W.

The algebra

$$\mathcal{H}_{\mathfrak{k},\mathbf{v}}(W) := \left\langle T_{s_1}, \ldots, T_{s_r} \mid T_{s_i}^2 = v_i 1 + (v_i - 1) T_{s_i},\ \text{braid rel's} \right\rangle_{\mathfrak{k}\text{-alg.}}$$

is the Iwahori-Hecke algebra of W over $\mathfrak{k}$ with parameter $\mathbf{v}$.

Braid rel's: $T_{s_i} T_{s_j} T_{s_i} \cdots = T_{s_j} T_{s_i} T_{s_j} \cdots$ (m_{ij} factors on each side)

FACT

$\mathcal{H}_{\mathfrak{k},\mathbf{v}}(W)$ is a free $\mathfrak{k}$-algebra with $\mathfrak{k}$-basis T_w, $w \in W$.

Note that $\mathcal{H}_{\mathfrak{k},\mathbf{1}}(W) \cong \mathfrak{k}W$, so that $\mathcal{H}_{\mathfrak{k},\mathbf{v}}(W)$ is a deformation of the group algebra $\mathfrak{k}W$.

THE THEOREM OF IWAHORI AND MATSUMOTO

Let $\mathfrak{k}[B/G]$ denote the permutation module on B/G.

This is a special case of a Harish-Chandra induced module.

Put $E := \mathrm{End}_{\mathfrak{k}G}(\mathfrak{k}[B/G])$.

THEOREM (IWAHORI/MATSUMOTO)

E is the Iwahori-Hecke algebra of W over $\mathfrak{k}$ with parameter $(q_i = [B \colon {}^{s_i}B \cap B])_{1 \leq i \leq r}$.

PROOF OF THE IWAHORI/MATSUMOTO RESULT, I

The set B/G is a $\mathfrak{k}$-basis of $\mathfrak{k}[B/G]$.

Use this basis to obtain a matrix representation of G over $\mathfrak{k}$.

The Schur basis of E is indexed by the orbits of G on $B/G \times B/G$.

If $\mathcal{O}$ is such an orbit, the corresponding basis element $T_{\mathcal{O}}$ is defined by

$$[T_{\mathcal{O}}]_{i,j} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{O} \\ 0, & \text{if } (i,j) \notin \mathcal{O} \end{cases}$$

The orbits of G on $B/G \times B/G$ are in bijection with $B\backslash G/B$: $BxB \mapsto$ orbit of (xB, B).

By the Bruhat decomposition, $B\backslash G/B$ is in bijection with W.

Thus E has $\mathfrak{k}$-basis $T_w := T_{\text{orbit of } (wB,B)}$, $w \in W$.

PROOF OF THE IWAHORI/MATSUMOTO RESULT, II

Write $T_x T_y = \sum_{z \in W} a_{xyz} T_z$. Then a_{xyz} is the entry at the position (zB, B) in $T_x T_y$.

It is not difficult to check that

$$a_{xyz} = |zBx^{-1}B \cap ByB|/|B|.$$

Now let $x = y = s \in S$. Then $zBsB \subseteq BzsB \cup BzB$.

Thus $a_{ssz} \neq 0$ only if $z = s$ or $zs = s$, i.e. $z = 1$.

Suppose first that $z = 1$. Then $a_{ss1} = |BsB|/|B| = [B \colon {}^sB \cap B]$.

If $z = s$, we have $a_{sss} = |sBsB \cap BsB|/|B| = q_s - 1$, since $sBsB \subset B \cup BsB$ and $B \cap BsB = \emptyset$.

HARISH-CHANDRA CLASSIFICATION

From now on let k be an algebraically closed field with $\mathrm{char}(k) \neq p$.

A simple kG-module V is called cuspidal, if V is **not** a **submodule** of $R_L^G(M)$ for some **proper** Levi subgroup L of G.

Harish-Chandra philosophy (HC-induction, cuspidality) yields the following classification.

THEOREM (HARISH-CHANDRA (1968), LUSZTIG ('70S) (CHAR(k) $= 0$), GECK-H.-MALLE (1996) (CHAR(k) > 0))

$$\left\{ V \mid V \text{ simple } kG\text{-module} \right\} / \text{isomorphism}$$

$$\updownarrow$$

$$\left\{ (L, M, \theta) \;\middle|\; \begin{array}{c} L \text{ Levi subgroup of } G \\ M \text{ simple, cuspidal } kL\text{-module} \\ \theta \text{ simple } \mathcal{H}(L, M)\text{-module} \end{array} \right\} / \text{conjugacy}$$

MAIN STEPS IN HARISH-CHANDRA CLASSIFICATION, I

Let V be a simple kG-module.

Let L be a Levi subgroup of minimal order such that $V \leq R_L^G(M)$ for some kL-module M of minimal dimension.

Then M is simple since R_L^G is exact.

Moreover, M is cuspidal since Harish-Chandra induction is transitive and exact.

The pair (L, M) is uniquely determined from V up to conjugation in G (Mackey type formula and invariance).

MAIN STEPS IN HARISH-CHANDRA CLASSIFICATION, II

$R_L^G(M)$ is a direct sum of indecomposable kG-modules with simple socles.

These components are determined by their socles up to isomorphism.

Thus $V \leq R_L^G(M)$ determines an isomorphism type of components of $R_L^G(M)$.

By Fitting correspondence, the simple modules of $\mathcal{H}(L, M)$ are in bijection to the isomorphism types of components of $R_L^G(M)$.

Harish-Chandra series

Definition

Two simple kG-modules V and V' are said to lie in the same Harish-Chandra series, if V and V' determine the same cuspidal pair (L, M).

*In other words, if V **and** V' are submodules of $R_L^G(M)$ for some cuspidal KL-module M of some Levi subgroup L.*

Let $\mathcal{E}(L, M)$ denote the Harish-Chandra series determined by the cuspidal pair (L, M).

Remarks: The set of simple kG-modules (up to isomorphism) is partitioned into Harish-Chandra series.

The elements of $\mathcal{E}(L, M)$ are in bijection with the simple modules of $\mathcal{H}(L, M)$.

PROBLEMS IN HARISH-CHANDRA PHILOSOPHY

The above classification theorem leads to the three tasks:

1. Determine the cuspidal pairs (L, M).
2. For each of these, “compute” $\mathcal{H}(L, M)$.
3. Classify the simple $\mathcal{H}(L, M)$-modules.

State of the art in case $\mathrm{char}(k) = 0$ (Lusztig):

- Cuspidal simple kG-modules arise from étale cohomology groups of Deligne-Lusztig varieties.
- $\mathcal{H}(L, M)$ is an Iwahori-Hecke algebra (Lusztig, Howlett-Lehrer) corresponding to a Coxeter group, namely $W_G(L, M)$ (see below).
- $\mathcal{H}(L, M) \cong kW_G(L, M)$ (Tits deformation theorem).

THE RELATIVE WEYL GROUP

Let L be a Levi subgroup of G. The group $W_G(L) := (N_G(L) \cap N)L/L$ is the relative Weyl group of L.

Here, N is the N from the BN-pair of G.

It is introduced to avoid trivialities: If $G = \mathrm{GL}_n(2)$, and $L = T$ is the torus of diagonal matrices, then $L = \{1\}$ and $N_G(L) = G$.

Alternative definition: $W_G(L) = N_G(\mathbf{L})/L = N_{\mathbf{G}}(\mathbf{L})^F/\mathbf{L}^F$.

$W_G(L)$ is naturally isomorphic to a subgroup of W.

If M is a kL-module, $W_G(L, M) := \{w \in W_G(L) \mid {}^wM \cong M\}$.

EXAMPLE: $\mathrm{SL}_2(q)$

Let $G = \mathrm{SL}_2(q)$ and $\mathrm{char}(k) = 0$.

The group T of diagonal matrices is the only proper Levi subgroup; it is a cyclic group of order $q - 1$.

We have $W = W_G(T) = \langle T, s\rangle / T$ with $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Let M be a simple kT-module. Then $\dim M = 1$ and M is cuspidal, and $\dim R_T^G(M) = q + 1$ (since $[G : B] = q + 1$).

Case 1: $W_G(T, M) = \{1\}$. Then $\mathcal{H}(T, M) \cong k$ and $R_T^G(M)$ is simple.

Case 2: $W_G(T, M) = W_G(T)$. Then $\mathcal{H}(T, M) \cong kW_G(T)$, and $R_T^G(M)$ is the sum of two simple kG-modules.

STATE OF THE ART IN CASE CHAR$(k) \neq 0$

Suppose that $\mathrm{char}(k) = \ell > 0$.

- $\mathcal{H}(L, M)$ is a "twisted" "Iwahori-Hecke algebra" corresponding to an "extended" Coxeter group (Howlett-Lehrer (1980), Geck-H.-Malle (1996)), namely $W_G(L, M)$;

  parameters of $\mathcal{H}(L, M)$ not known in general.
- $G = \mathrm{GL}_n(q)$; everything known (Dipper-James, 1980s)
- G classical group, ℓ "linear"; everything known (Gruber-H., 1997).
- In general, classification of cuspidal pairs open.

EXAMPLE: $SO_{2m+1}(q)$ (GECK-H.-MALLE (1996))

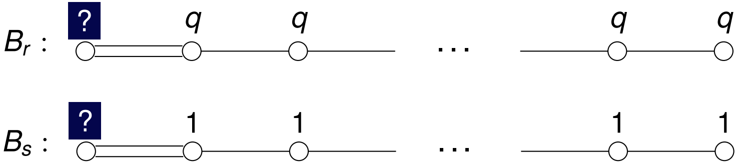
Let $G = SO_{2m+1}(q)$, assume that $\ell > m$, and put $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_\ell^*$.

Any Levi subgroup L of G containing a cuspidal unipotent (see later) module M is of the form

$$L = SO_{2m'+1}(q) \times GL_1(q)^r \times GL_e(q)^s.$$

In this case $W_G(L, M) \cong W(B_r) \times W(B_s)$, where $W(B_j)$ denotes a Weyl group of type B_j.

Moreover, $\mathcal{H}(L, M) \cong \mathcal{H}_{k,\mathbf{q}}(B_r) \otimes \mathcal{H}_{k,\mathbf{q}}(B_s)$, with $\mathbf{q}$ as follows:



Thank you for your listening!