CONDENSING THE STEINBERG MODULE

Gerhard Hiss
joint work with Thomas Breuer, Frank Lübeck and Klaus Lux

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Compute the modular character tables of the Atlas groups.

The modular character tables are completely known for the following sporadic groups (and their decorations):

M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL (10 groups)
An Atlas of Brauer Characters, Jansen, Lux, Parker, Wilson, '95

He, Ru, Suz, $O'N$, Co_3, Co_2, Fi_{22}, HN, Fi_{23} (9 groups)
various authors (1988 – 2017)

non-sporadic Atlas groups: complete up to $\mathrm{Sp}_8(2)$ (58 groups)
various authors (1941 – 2018)

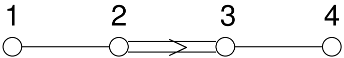
non-sporadic simple Atlas group of smallest order with an incomplete modular character table: $F_4(2)$ (state in 2018)

THE CHEVALLEY GROUP $F_4(2)$

$F_4(2)$: simple Chevalley group of order

$$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17 = 3\,311\,126\,603\,366\,400.$$

Automorphism group of simple Lie algebra of type F_4 over $\mathbb{F}_2$;
Dynkin diagram:



$\mathrm{Aut}(F_4(2)) = F_4(2).2$

Universal cover: $2.F_4(2)$

THE CHARACTER TABLES OF $F_4(2)$

Knowledge of characters of $F_4(2)$ and $2.F_4(2)$ in 1997

Characteristic	Authors	Remark
0	Atlas [1985]	complete
2	Steinberg [1963]	complete
	Veldkamp [1970]	
3	White [1992], H. [1997]	partial results
5, 7	White [1992], H. [1997]	complete
13, 17	White [1992]	complete

Completed for $F_4(2)$ and $2.F_4(2)$ in characteristic 3 by Breuer, Lübeck, Lux and H. in 2018 (published in 2019).

Some questions remain for $F_4(2).2$ and $2.F_4(2).2$.

Some character degrees

Degrees of the irreducible Brauer characters of the principal 3-block of $F_4(2)$:

1	833	1105	1105	1326	21658
20722	22372	22372	63700	77077	77077
183600	215747	215747	182274	270725	496146
496146	1061242	**1157377**	**1157377**	1248428	1248428
1551199	**6194188**				

Plain entries: Degrees already known in 1997.

Boldface entries: Degrees determined in 2018 by condensing the Steinberg module of $F_4(2)$.

CONDENSATION IN THEORY . . . [GREEN 1980]

Let k be a field and A a finite-dimensional k-algebra.

Let $\iota \in A$ be an idempotent, i.e., $0 \neq \iota = \iota^2$ (a projection).

Get an exact functor: $\mathrm{mod}\text{-}A \to \mathrm{mod}\text{-}\iota A\iota$, $V \mapsto V\iota$.

If $S \in \mathrm{mod}\text{-}A$ is simple, then $S\iota = 0$ or simple (so a composition series of a module V is mapped to a composition series of $V\iota$).

If $S\iota \neq 0$ for all simple $S \in \mathrm{mod}\text{-}A$, then this functor is an equivalence of categories.

(A and $\iota A\iota$ have the same representations.)

Let G be a finite group and k a field, take $A = kG$.

Let $K \leq G$ with $\mathrm{char}(k) \nmid |K|$ (condensation subgroup).

Put $\iota := 1/|K| \sum_{x \in K} x \in kG$ (condensation idempotent).

For V a kG-module, $V\iota$ is the set of K-fixed points in V.

Task: Given $g \in G$, determine action of $\iota g \iota$ on $V\iota$,
without explicit computation of action of g on V!

Theorem (This can be done, if . . .)

V is a permutation module (Thackray-Parker, 1981)
V is a tensor product (Lux-Wiegelmann, 1998)
V is an induced module (Müller-Rosenboom, 1999)

Now apply the MeatAxe to the matrices of $\iota g_1 \iota, \iota g_2 \iota, \ldots$

Generation

Generation Problem: How to choose $g_1, g_2, \ldots \in G$ such that $\iota kG\iota = \langle \iota g_1 \iota, \iota g_2 \iota, \ldots \rangle$?

Theorem (Felix Noeske, 2005)

Suppose $K \trianglelefteq N \leq G$.
X: set of double coset representatives of $N \backslash G / N$
Y: set of generators of N modulo K, i.e. $N = \langle Y, K \rangle$

Then $\quad \iota kG\iota = \langle \iota X \iota, \iota Y \iota \rangle \quad$ as k-algebras.

The Steinberg module, I [Steinberg, 1956]

G finite group with split BN-pair of characteristic p, k a field

$B = UT$ Borel subgroup of G, $U \trianglelefteq B$, $U \cap T = 1$, $U \in \mathrm{Syl}_p(G)$

$T \trianglelefteq N$, $W = N/T$ Weyl group of G

$$e := [B] \sum_{w \in W} (-1)^{\ell(w)} \dot{w} \in kG$$

$\dot{w} \in N$ inverse image of $w \in W$, $[B] = \sum_{x \in B} x \in kG$

St $:= ekG$: Steinberg module of kG

k-basis of St: $\{eu \mid u \in U\}$, dimension of St: $|U|$

THE STEINBERG MODULE, II [STEINBERG, 1956]

THEOREM (ROBERT STEINBERG, 1956)

Let G, B, U and k be as above. Then

(i) $\mathrm{St}_U \cong kU$, *the right regular kU-module;*

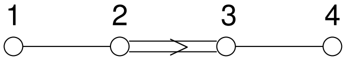
(ii) *W.r.t. the basis $\{eu \mid u \in U\}$, the matrix of $g \in G$ on St has entries 1, -1, or 0, with at most $|W|$ non-zero entries in each row;*

(iii) St *is irreducible if and only if $\mathrm{char}(k) \nmid [G : B]$.*

Steinberg gives explicit formulae for the matrices in (ii).

Our case of interest

Now let $G = F_4(2)$; dimension of St: $2^{24} = 16\,777\,216$

Dynkin diagram: 

Fix $i \in \{1, 2, 3, 4\}$.

$s_i \in W \leq G$: corresp. fundamental reflection; $W = \langle s_1, \ldots, s_4 \rangle$

$U_i \leq U$: corresponding root subgroup; $|U_i| = 2$

Fix $u \in U$, write $u = u_i u_i'$ with $u_i \in U_i$ and $u_i' \in U^{s_i} \cap U$,
i.e. $s_i u_i' s_i = u_i'' \in U$.

Then

$$(eu)s_i = \begin{cases} eu_iu_i'' - eu_i'' & \text{if } u_i \neq 1, \\ -eu_i'' & \text{if } u_i = 1. \end{cases}$$

THE CONDENSATION SUBGROUP

Let $k = \mathbb{F}_3$.

Knew a priori from our 1997 results:
composition series of St (over k) will solve our problem.

P: parabolic subgroup of G of type C_3

$U_P \leq U$: unipotent radial of P $(= O_2(P))$; $|U_P| = 2^{15}$

(Condensation with unipotent radicals is Harish-Chandra restriction. This is well understood on a theoretical level.)

Condensation subgroup: $K := Z(U_P)$; $|K| = 2^7$

Notice: $K \trianglelefteq P$.

Dimension of condensed Steinberg module: $2^{17} = 131\,072$

Feasible for Richard Parker's MeatAxe64

How to condense?

Notice: The condensation subgroup K is contained in U.

$\mathcal{R}(U/K)$: coset representatives for U/K

$$\text{St has } k\text{-basis } \{eu \mid u \in U\} \tag{1}$$

$$\text{St}\iota \leq \text{St has } k\text{-basis } \{eu\iota \mid u \in \mathcal{R}(U/K)\} \tag{2}$$

Let $a \in kG$.

$(\gamma_{u,u'})_{u,u' \in U}$: matrix of a on St with respect to basis (1)

$(\kappa_{u,u'})_{u,u' \in \mathcal{R}(U/K)}$: matrix of $\iota a \iota$ on Stι with respect to basis (2)

Then

$$\kappa_{u,u'} = \frac{1}{|K|} \sum_{v,v' \in K} \gamma_{uv,u'v'}.$$

Which elments to condense?

Want to apply Noeske´s criterion: Need

Double coset representatives for P:

$$b_1 := 1,$$

$$b_2 := s_1,$$

$$b_3 := s_1 s_2 s_3 s_2 s_1,$$

$$b_4 := s_1 s_2 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1,$$

$$b_5 := s_1 s_2 s_3 s_2 s_4 s_3 s_2 s_1$$

Generators of P modulo K:

$$u_1, \ldots, u_4, s_2, s_3, s_4 \text{ with } 1 \neq u_i \in U_i$$

Comments

Most time consuming: Computing the 11 condensed matrices; each is about 2.5 GB big.

Richard Parker chopped the condensed Steinberg module of dimension 131 072 into smaller modules, of dimensions around 40 000, using his MeatAxe64.

Thank's again, Richard!

The remaining modules were chopped with Ringe´s C-MeatAxe.

The entire chopping process took only a few hours of CPU time.

Thank you for your attention!