# CORRIGENDUM TO "THE WEIL-STEINBERG CHARACTER OF FINITE CLASSICAL GROUPS", REPRESENT. THEORY 13(2009), 427-459

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Abstract. This paper corrects the statement and the proof of Theorem 1.5 of the paper quoted in the title.

Theorem 1.5 of our paper [1] requires a correction. Below we provide a new statement of this theorem and correct the proof. A mistake in the original proof of Theorem 1.5 is due to missing the multiple 2 at a certain point of the proof (see [1, page 456, line 23]).

Let $\mathbf{G}$ be a simple algebraic group of type $C_n$ defined over a field of characteristic 2 and $G = Sp(2n, q)$, $q = 2^k$. If $\mu$ is a dominant weight of $\mathbf{G}$ then $\varphi_\mu$ denotes the irreducible representation of $\mathbf{G}$ with highest weight $\mu$, and $\Phi_\mu$ is the representation of $G$ afforded by the principal indecomposable module corresponding to $(\varphi_\mu)_G$ if $\mu$ is a $q$-restricted weight. In addition, $\omega := (\varphi_{(q-1)\lambda_n})_G$. Let $st$ be the 2-modular Steinberg representation of $G$. Recall that $st = (\varphi_{(q-1)(\lambda_1+\cdots+\lambda_n)})_G = \Phi_{st}$, where $\lambda_1, \dots, \lambda_n$ are the fundamental weights of $\mathbf{G}$. The standard Frobenius endomorphism $\mathbf{G} \to \mathbf{G}$ is denoted by $Fr_0$, and it acts on the representations and the weights of $\mathbf{G}$ (so $Fr_0(\mu) = 2\mu$).

Theorem 1.5 in [1] has to be corrected as follows:

**Theorem 1.5** Let $\lambda_1, \dots, \lambda_n$ be the fundamental weights of $\mathbf{G}$, and $\tau = (q-1)(\lambda_1 + \cdots + \lambda_{n-1})$. Then $\omega \otimes st = st \oplus \Phi_\tau$.

Recall that $\varepsilon_1, \dots, \varepsilon_n$ denote the weights of $\mathbf{G}$ introduced in [2, Planchee III]. The following lemma is a refinement of [1, Lemma 7.2(1)].

**Lemma** $\varphi_{2\lambda_n}$ is the only composition factor of $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$ occurring with multiplicity 1.

Proof. Let $M$ be the $\mathbf{G}$-module afforded by the representation $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$. Note that the weights of $\varphi_{\lambda_n}$ and hence of $M$ are known. In terms of $\varepsilon_j$ the weights of $\varphi_{\lambda_n}$ are $\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n$, so the weights of $M$ are $\sum_{i \in N} \pm 2\varepsilon_i$, where $N$ can be any subset of $\{1, \dots, n\}$ (possibly empty; in this case the weight in question is meant to be the zero weight). It follows that $2\lambda_i = 2\varepsilon_1 + \cdots + 2\varepsilon_i$ $(i = 1, \dots, n)$ occur as weights of $M$.

Let $\mathbf{H} = GL(2n, \overline{F}_2)$ and let $\varepsilon'_1, \dots, \varepsilon'_{2n}$ be the weights of the natural $\mathbf{H}$-module $V$. One can embed $\mathbf{G}$ into $\mathbf{H}$ so that a maximal torus $\mathbf{T}$ of $\mathbf{G}$ is contained in a maximal torus $\mathbf{T}'$ of $\mathbf{H}$, and $\varepsilon_i = \varepsilon'_i|_{\mathbf{T}}$, $\varepsilon'_{n+i}|_{\mathbf{T}} = -\varepsilon_i$ for $i = 1, \dots, n$. Let $V_i$ $(1 \le i \le 2n)$ be the $i$-th exterior power of $V$, and $V_0$ the trivial $\mathbf{H}$-module. Set $R = \oplus_{i=0}^{2n} V_i$. Then the weights of $R$ are 0 and $\varepsilon'_{j_1} + \cdots + \varepsilon'_{j_i}$, where $1 \le i \le 2n$ and $0 < j_1 < j_2 < \cdots < j_i \le 2n$. It follows that the weights of $R' := R_{\mathbf{G}}$ are 0 and $\pm\varepsilon_{j_1} \pm \cdots \pm \varepsilon_{j_i}$ where $1 \le i \le n$ and $0 < j_1 < j_2 < \cdots < j_i \le n$. Therefore, the weights of $Fr_0(R')$ and $M$ are the same.

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Furthermore, 0 and $\lambda_1,\dots,\lambda_n$ are the only dominant weights of $R'$. Indeed, $\varepsilon_1=\lambda_1$ and $\varepsilon_i=\lambda_i-\lambda_{i-1}$ for $i>1$. Suppose that a non-zero weight $\sum_{i\in N}\pm\varepsilon_i=\sum_{i\in N}\pm(\lambda_i-\lambda_{i-1})$ is dominant. Then the coefficient of $\lambda_i$ is non-negative for every $i$. It follows that $1\in N$ and if $i\in N, i>1$ then $i-1\in N$. So $\sum_{i\in N}\pm\varepsilon_i$ is dominant if and only if it is of shape $\sum_{i=1}^k\varepsilon_i$, $k=1,\dots,n$, as claimed. Therefore, if $\mu$ is the highest weight of an irreducible constituent $\varphi_\mu$ of $R'$ then $\mu=\lambda_i$ for some $i$ or 0.

Let $m'_i$ $(i=1,\dots,n)$ be the multiplicity of $\varphi_{2\lambda_i}$ in $Fr_0(R')$ and $m_i$ the multiplicity of $\varphi_{2\lambda_i}$ in $M$; in addition, let $m'_0, m_0$ be the multiplicity of the trivial $G$-module in composition series of $Fr_0(R'), M$, respectively. We show that $m_i=m'_i$. As the restriction $\varphi_{2\lambda_i}$ to $Sp(2n,2)$ is irreducible, and $(\varphi_{2\lambda_i})_{Sp(2n,2)}$ and $(\varphi_{2\lambda_j})_{Sp(2n,2)}$ are non-equivalent for $i\neq j$, it follows that $m'_i$, resp., $m_i$ is the multiplicity of the irreducible representation $(\varphi_{2\lambda_i})_{Sp(2n,2)}$ in $Sp(2n,2)$-composition series of $Fr_0(R')$, resp., $M$. Similarly, $m'_0$ and $m_0$ is the multiplicity of the trivial $Sp(2n,2)$-module in $Fr_0(R'), M$, respectively. By [1, Lemma 7.2], the composition factors of $\oplus_{i=0}^{2n}(V_i)_{Sp(2n,2)}$ coincide with the composition factors of $(\varphi_{\lambda_n}\otimes\varphi_{\lambda_n})_{Sp(2n,2)}$ with regarding their multiplicities. It follows that $m'_0=m_0$ and $m'_i=m_i$ for $i=1,\dots,n$.

Note that $(V_i)_{Sp(2n,\overline{F}_2)}\cong(V_{2n-i})_{Sp(2n,\overline{F}_2)}$ and $(V_i)_{Sp(2n,\overline{F}_2)}$ contains $\varphi_{\lambda_i}$ for $i=1,\dots,n$. In addition, $(V_0)_{Sp(2n,\overline{F}_2)}\cong(V_{2n})_{Sp(2n,\overline{F}_2)}$. It follows that $\varphi_{\lambda_n}$ is the only composition factor of $R'$ which may occur with multiplicity 1. By general theory, $\varphi_{2\lambda_n}$ does occur in $M$ with multiplicity 1.

**Proof of Theorem 1.5**. Let $\nu=a_1\lambda_1+\cdots+a_n\lambda_n$, where $0\leq a_1,\dots,a_n\leq q-1$, and $\nu'=a_1\lambda_1+\cdots+a_{n-1}\lambda_{n-1}$. We show that $\Phi_\nu$ is a direct summand of $\omega\cdot st$ if and only if $\nu=(q-1)(\lambda_1+\cdots+\lambda_n)$ or $\tau$. By [1, Lemma 7.4], it suffices to show that $st$ is an irreducible constituent of $(\varphi_\nu\otimes\varphi_{(q-1)\lambda_n})_G$ if and only if $\nu'=\tau$ and $a_n=0$ or $q-1$.

It can be deduced from Steinberg [3, Corollary to Theorem 41 and Theorem 43] that $\varphi_\nu=\varphi_{\nu'}\otimes\varphi_{a_n\lambda_n}$. In particular, $\varphi_{\nu'}\otimes\varphi_{(q-1)\lambda_n}=\varphi_{\nu'+(q-1)\lambda_n}$.

If $a_n=0$ then $\nu=\nu'$ so the representation $\varphi_\nu\otimes\varphi_{(q-1)\lambda_n}=\varphi_{\nu'+(q-1)\lambda_n}$ is irreducible. As $\nu'+(q-1)\lambda_n$ is a $q$-restricted dominant weight, $(\varphi_{\nu'+(q-1)\lambda_n})_G$ is irreducible, so it is not equal to $st$ unless $\nu'=\tau$. If $\nu=\nu'=\tau$ then $(\varphi_{\nu'+(q-1)\lambda_n})_G=st$, so $st$ is a direct summand of $(\varphi_\nu\otimes\varphi_{(q-1)\lambda_n})_G$ (when $a_n=0$).

Suppose that $a_n>0$. It follows from [1, Corollary 1.3] that every principal indecomposable module $\Phi_\nu$ occurs as a direct summand of $(\varphi_{(q-1)\lambda_n})_G\otimes st$ with multiplicity at most 1; by [1, Lemma 7.4], this implies that $st$ occurs as an irreducible constituent of $(\varphi_\nu\otimes\varphi_{(q-1)\lambda_n})_G$ with multiplicity at most 1. Therefore, the constituents occurring with multiplicity greater than 1 can be ignored.

We have

$$\varphi_\nu\otimes\varphi_{\lambda_{(q-1)\lambda_n}}=\varphi_{\nu'}\otimes\varphi_{a_n\lambda_n}\otimes\varphi_{(q-1)\lambda_n}.$$

We show that a composition factor of $(\varphi_{a_n\lambda_n}\otimes\varphi_{(q-1)\lambda_n})_G$ have multiplicity greater than 1, unless $a_n=q-1$, and it is of form $(\varphi_{(q-1)\lambda_n})_G$.

Let $a_n=\sum_{i=0}^{k-1}2^ib_i$ be the 2-adic expansion of $a_n$ (so $0\leq b_i\leq 1$). Then

$$\varphi_{a_n\lambda_n}\otimes\varphi_{(q-1)\lambda_n}=(\varphi_{b_0\lambda_n}\otimes\varphi_{\lambda_n})\otimes Fr_0(\varphi_{b_1\lambda_n}\otimes\varphi_{\lambda_n})\otimes\cdots\otimes Fr_0^{k-1}(\varphi_{b_{k-1}\lambda_n}\otimes\varphi_{\lambda_n}).$$

If $b_i=0$ then $\varphi_{b_i\lambda_n}\otimes\varphi_{\lambda_n}=\varphi_{\lambda_n}$, otherwise $b_i=1$ and then, by the lemma above, the only composition factor of $(\varphi_{b_i\lambda_n}\otimes\varphi_{\lambda_n})_G$ occurring with multiplicity 1 is $(\varphi_{2\lambda_j})_G$ for $1\leq j\leq n$. Therefore, the composition factors of $\varphi_{\lambda_n}\otimes\varphi_{\lambda_n}$ distinct from $\varphi_{2\lambda_n}$ can be ignored. This means that we have to decide whether $st$ occurs as a composition factor of the representation obtained from

$$(\varphi_{\nu'}\otimes\varphi_{b_0\lambda_n}\otimes\varphi_{\lambda_n}\otimes\cdots\otimes Fr_0^{k-1}(\varphi_{b_{k-1}\lambda_n}\otimes\varphi_{\lambda_n}))_G$$

by omitting $\varphi_{b_i\lambda_n}$ whenever $b_i = 0$, and replacing $\varphi_{b_i\lambda_n} \otimes \varphi_{\lambda_n}$ by $\varphi_{2\lambda_n}$ whenever $b_i = 1$. Let $B = \{i \in \{0,\dots,k-1\} : b_i = 1\}$.

Then we can write the resulting expression as

$$(\otimes_{i\in B} Fr_0^i(\varphi_{2\lambda_n}) \otimes_{i\notin B} Fr_0^i(\varphi_{\lambda_n}))_G = (\otimes_{i\in B} Fr_0^{i+1}(\varphi_{\lambda_n}) \otimes_{i\notin B} Fr_0^i(\varphi_{\lambda_n}))_G.$$

We first consider the set $B' := \{i \in B : i+1 \notin B\}$ (if $i = k-1$ then $i+1$ is regarded to be 0). Suppose that $B'$ is non-empty (this means that $a_n \neq 0$ and $a_n \neq 2^k - 1$). For $i \in B'$ the lemma above applied to the term $(Fr_0^{i+1}(\varphi_{\lambda_n}) \otimes Fr_0^{i+1}(\varphi_{\lambda_n}))_G = (Fr_0^{i+1}(\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}))_G$ tells us that, by the above reason, we can replace $\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}$ by $\varphi_{2\lambda_n}$ so $(Fr_0^{i+1}(\varphi_{\lambda_n} \otimes \varphi_{\lambda_n}))_G$ is replaced by $(Fr_0^{i+2}(\varphi_{\lambda_n}))_G$ ($i \in B'$). One can continue the analysis by repeating this reasoning, but it is more efficient to observe that the process is parallel to the addition of the residues of integers modulo $2^k - 1$. In order to justify this claim, we first compute $(\varphi_{a_n\lambda_n} \otimes Fr_0^i(\varphi_{\lambda_n})_G$ for every $i = 0,\dots,k-1)$, and then in general.

Note that

$$(\varphi_{a_n\lambda_n} \otimes \varphi_{2^i\lambda_n})_G = (Fr_0^i(\varphi_{b_i\lambda_n} \otimes \varphi_{\lambda_n}) \otimes_{j\neq i} \varphi_{2^j b_j\lambda_n})_G.$$

If $b_i = 0$ then $(\varphi_{b_i\lambda_n})_G = 1_G$ so $Fr_0^i(\varphi_{b_i\lambda_n} \otimes \varphi_{\lambda_n})_G = (Fr_0^i(\varphi_{\lambda_n}))_G$. Suppose that $b_i = 1$. In view of the lemma above, the composition factors of $\varphi_{b_i\lambda_n} \otimes \varphi_{\lambda_n}$ other than $\varphi_{2\lambda_n}$ occur with multiplicity greater than 1, so they are immaterial for our purpose. Therefore, we are left with $\varphi_{2\lambda_n}$, so it suffices to compute $(Fr_0^i(\varphi_{2\lambda_n}) \otimes_{j\neq i} \varphi_{2^j b_j\lambda_n})_G$. This is equal to $(Fr_0^{i+1}(\varphi_{\lambda_n}) \otimes_{j\neq i} \varphi_{2^j b_j\lambda_n})_G$. If $i+1 = k$ then $(Fr_0^k(\varphi_{\lambda_n}))_G = (\varphi_{2^k\lambda_n})_G \cong (\varphi_{\lambda_n})_G$, and the replacement of $(\varphi_{2^k\lambda_n})_G$ by $(\varphi_{\lambda_n})_G$ is parallel to taking the residue modulo $2^k - 1$. Next, if $b_{i+1} = 0$ then we stop, otherwise we repeat the same trick and obtain $(Fr_0^{i+2}(\varphi_{\lambda_n}) \otimes_{j\neq i,i+1} \varphi_{2^j b_j\lambda_n})_G$. The output of the procedure will be $\varphi_{a_n(i)}$, where $a_n(i) = (a_n + 2^i)(\bmod 2^k - 1)$.

In general, applying this to $\varphi_{a_n\lambda_n} \otimes \varphi_{(q-1)\lambda_n} = (\varphi_{a_n\lambda_n} \otimes_{i=0}^{k-1} Fr_0^i(\varphi_{\lambda_n}))_G$, we obtain $(\varphi_{a_n'\lambda_n})_G$, where $a_n' = a_n + 1 + 2 + \cdots + 2^{k-1}(\bmod 2^k - 1) = a_n$.

This is also true if $B'$ is empty but $a_n = q - 1$. Indeed, the reasoning above for $i = 0,\dots,k-1$ remains valid, and we obtain $a_n(i) = (a_n + 2^i)(\bmod 2^k - 1) = 2^i$. Then again $a_n' = a_n = q - 1$.

Thus, we conclude that it suffices to decide whether $st$ is an irreducible constituent of $(\varphi_{\nu'} \otimes \varphi_{a_n\lambda_n})_G$. As mentioned above, $\varphi_{\nu'} \otimes \varphi_{a_n\lambda_n} = \varphi_{\nu'+a_n\lambda_n}$. Since $\nu' + a_n\lambda_n$ is a $q$-restricted dominant weight, $(\varphi_{\nu'+a_n\lambda_n})_G$ is irreducible. So $st$ is an irreducible constituent of $(\varphi_{\nu'+a_n\lambda_n})_G$ if and only if $(\varphi_{\nu'+a_n\lambda_n})_G = st$, equivalently, $\nu' + a_n\lambda_n = (q-1)(\lambda_1 + \cdots + \lambda_n)$. This implies $\nu' = (q-1)(\lambda_1 + \cdots + \lambda_{n-1}) = \tau$ and $a_n = q - 1$.

Therefore, $\omega \otimes st = \Phi_\tau \oplus st$, as required.

The comments to Theorem 1.5 in [1, page 430, line -12] concerning the decomposition numbers of $\omega \cdot St$ cannot not be applied to the new version of the theorem. In fact, we have:

**Corollary** $\dim \Phi_\tau = |G|_2 \cdot (\dim \varphi_{(q-1)\lambda_n} - 1) = |G|_2 \cdot (q^n - 1)$, and the ordinary character corresponding to $\Phi_\tau$ is multiplicity free.

Proof. $(\varphi_{(q-1)\lambda_n})_G \otimes st = \Phi_\tau \oplus st$, so $\dim \Phi_\tau = \dim st \cdot (\dim \varphi_{(q-1)\lambda_n} - 1)$. As $\dim st = |G|_2$ and $\dim \varphi_{(q-1)\lambda_n} = 2^{nk} = q^n$, the first claim follows. The second one follows from [1, Corollary 1.3].

Example: Let $G = Sp(2,q)$, $q$ even. Then $\tau = 0$; by the above corollary, $\dim \Phi_0 = q(q-1)$.

Example: Let $G = Sp(4,4)$, so $n = 2$. Then $\tau = 3\lambda_1$ and $|G|_2 = 2^8 = 256$. So $\dim \varphi_{3\lambda_2} = 4^2 = 16$. By the corollary above, $\dim \Phi_\tau = 256 \cdot 15 = 3840$.

## References

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