RWTH Aachen

Mathemagic with a Deck of Cards

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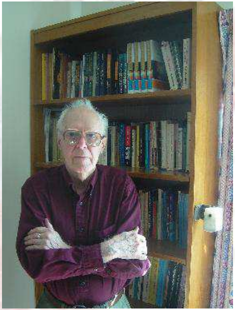
Mathemagic with a Deck of Cards

There seems to be no end to the mathemagical entertainment to be had with a simple deck of cards, drawing from ideas in algebra and discrete mathematics, and probability. We'll present a sampling, using ideas both recent and classical.

The material is very much in the spirit of influential and prolific writer Martin Gardner (1914-2010)— and it can be used to liven up mathematics classes and parties, as well as providing jumping-off points for undergraduate independent study and seminars.

Martin Gardner: The Best Friend Mathematics Ever Had



—standing by every word he ever wrote.

He authored over 100 books, on mathematics, puzzles, magic, physics, philosophy, and *Alice in Wonderland*.

Martin's associates included M.C. Escher, S. Dalí & I. Asimov.

www.martin-gardner.org — @WWMGT — @MGardner100th

Scientific American

Martin wrote 300 "Mathematical Games" columns for *Scientific American* between 1956 and 1986. They were later issued in 15 books, and on a single searchable CD-rom.

He introduced generations to hexaflexagons; the Soma cube; origami; Eleusis; Fermat's last theorem; rep-tiles; tangrams; pentominoes; polyominoes; the art of M. C. Escher; the $3n + 1$ problem; Conway's game of Life; the four-color map problem; RSA cryptography; fractals; and paradoxes from A to Z (all crows are black, infinity, Newcomb's, nontransitive dice, the unexpected hanging, Zeno's). And mathematical card magic.

Along with James Randi and Card Sagan, he was one of the founding fathers of modern skepticism, especially in relation to debunking pseudoscience, quacks and dubious medical claims.

Additional Certainties

Shuffle the deck well, and have cards selected by two spectators.

They remember their cards, share the results with each other, and tell you the sum of the chosen card values.

You soon announce what each individual card is!

Secret Number 1:

When several numbers are added up, each one can be determined from the sum.

Secret Number 2:

Totally free choices of cards are offered, but only from a controlled small part of the deck.

The possibilities are narrowed down by having half a dozen key cards at the top of the deck at the start, in any order, and keeping them there throughout some fair-looking shuffles.

The mathematics is about numbers (card values).

Secret Number 3:

> *You've memorized the suits of the top half dozen key cards.*

We use the Fibonacci numbers: start with 1, 2; add to get the next one. Repeat.

$$1 + 2 = 3,$$

$$2 + 3 = 5,$$

$$3 + 5 = 8,$$

$$5 + 8 = 13, \text{ and so on.}$$

The list of Fibonacci numbers continues forever:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

However, for magic with playing cards, we focus on the first six of these, namely the **Little Fibs**, i.e., 1, 2, 3, 5, 8, and 13, agreeing that 1 = Ace and 13 = King. (Fib = Lüge/Schwindelei)

Now, consider some particular cards with those values, for instance,

A♣, 2♡, 3♠, 5♢, 8♣, K♡ (CHaSeD order).

If any two are selected from these, they can be determined from the sum of their values, because of the *unaddition property*:

Any possible total can only arise in one way.

Zeckendorf–Fibonacci

Little-known amazing fact:

> *Every positive whole number can be written in exactly one way as a sum of* different *and* non-consecutive *Fibonacci numbers: 1, 2, 3, 5, 8, 13, 21, 34, ...*

This was apparently not noticed before 1939, when amateur mathematician Edouard Zeckendorf from Belgium spotted it. He published it in 1972, after he'd retired.

For instance, $6 = 5 + 1$ (not $3 + 2 + 1$ or $3 + 3$), and $10 = 8 + 2$, and $20 = 13 + 5 + 2$.

It's easy to break up any sum into the two Fibs it's made up from: just peel off the largest possible Fib, what's left is the other one. (This is a kind of greedy algorithm.)

Zeckendorf–Fibonacci

E.g., $27 = 21 + 6 = 21 + 5 + 1$. Note, $21 = 21$.

$51 = 34 + 17 = 34 + 13 + 4 = 34 + 13 + 3 + 1$.

$$\begin{aligned} 8 &= 5 + 3, \\ 10 &= 8 + 2, \\ 14 &= 13 + 1, \\ 18 &= 13 + 5, \\ &\dots \end{aligned}$$

Do other lists of numbers work? Yes! The Lucas sequence 2, 1, 3, 4, 7, 11, 18, ..., is a kind of generalized Fibonacci sequence. If we omit the 2, it too has the desired "unaddition" property.

We don't even need generalized Fibonacci sequences.

The numbers 1, 2, 4, 6, 10 work. So do 1, 2, 5, 7, 13. And?

Can we generalize?

Twice as impressive?

Can you have four cards picked from the six Little Fibs, the total revealed, and name all four cards?

Hint: all six values add up to 32.

If four cards are chosen, and you're told the values sum to 22, then the two *not* chosen must add up to $32 - 22 = 10$, so *they* are the 8 and 2.

Hence the chosen cards are the Ace, 3, 5, and King.

Three Scoop Miracle

Hand out the deck for shuffling. A spectator is asked to call out her favourite ice-cream flavour; let's suppose she says, "Chocolate."

Take the cards back, and take off about a quarter of the deck. Mix them further until told when to stop.

Deal cards, one for each letter of "chocolate," before dropping the rest on top as a topping. This spelling/topping routine is repeated twice more—so three times total.

Emphasize how random the dealing was, since the cards were shuffled and you had no control over the named ice-cream flavour.

Have the spectator press down hard on the final top card, asking her to magically turn it into a specfic card, say the 4♠.

When that card is turned over it is found to be the desired card.

Low Down Triple Dealing

The key move here is a *reversed transfer* of a fixed number of cards in a packet—at least half—from top to bottom, done three times total.

The dealing out of k cards from a packet that runs $\{1, 2, \ldots, k-1, k, k+1, k+2, \ldots, n-1, n\}$ from the top down, and then dropping the rest on top as a unit, yields the rearranged packet

$$\{k+1, k+2, \ldots, n-1, n, k, k-1, \ldots, 2, 1\}.$$

True, but hardly inspiring, or revealing!

Low Down Triple Dealing

When $k \geq \frac{n}{2}$, doing this three times brings the original bottom card(s) to the top.

Given a flavour of length k and a number $n \leq 2k$, the packet of size n breaks symmetrically into three pieces T, M, B of sizes $n-k$, $2k-n$, $n-k$, such that the count-out-and-transfer operation (of k cards each time) is

$$T, M, B \rightarrow B, \overline{M}, \overline{T},$$

where the bar indicates complete reversal.

Low Down Triple Dealing Generalized?

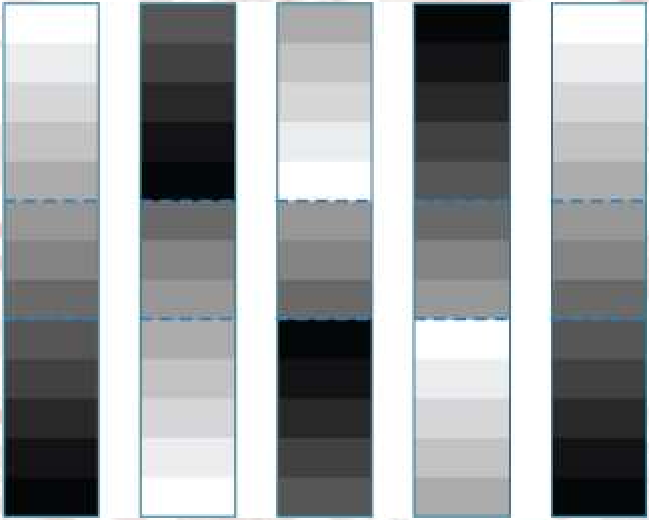
Using this approach, the Bottom to Top (with three moves) property can be proved. Actually . . .

The Bottom to Top Property is only 75% of the story. Here's the real scoop:

> **The Period 4 Principle** *If four reversed transfers of k cards are done to a packet of size n, where $k \geq \frac{n}{2}$, then every card in the packet is returned to its original position.*

Why does this work?

Proof without words



The inevitability of coincidences I (Birthdays)

With over 60 randomly selected people, the chances of a birthday match are overwhelming, though you *can* handpick 366 people with different birthdays!

With just 23 randomly selected people, the chances of a birthday match are a little over 50%.

("The Birthday Paradox")

The key to estimating such probabilities is to turn things around, and focus on the chances of there being no match, noting that

$$\textit{Prob}(\geq \textit{one match}) = \textit{1 - Prob(no match)}.$$

Poker with Ten Random Cards

If *k* cards are picked at random, then since there are four cards of each value, the chance of getting at least one matching value (i.e., *a pair or better*) is:

$$1 - \frac{52}{52} \times \frac{48}{51} \times \frac{44}{50} \times \ldots \times \frac{52-4k+4}{52-k+1}.$$

For 5 cards, this comes out to be about 50%. For 8 or 9 cards, it's about 89% or 95%, respectively.

For 10 cards, it's about 98%.

Only 2% of the time will you be unlucky!

The inevitability of coincidences II (Cards)

Given 10 random cards, *assuming that there is at least one matching pair*, the winning poker trick is based on your making sure that the "winning cards" are in the bottom four, with as little suspicion-arousing rearrangement as possible.

Those are positions seven to ten, from the top.

The "losing cards" are in positions three to six.

The top two cards must not impact which of you wins (i.e., they aren't "deal-breakers").

Bill Simon's contribution (1964)

> *It's easy to give the illusion of free choices while really controlling exactly how to split a packet of 8 cards into two piles of 4.*

In fact, you retain control of the division in one key sense: the bottom 4 cards end up in the second pile.

E.g., if you start with 4 red cards of top of 4 blacks, the piles maintain color separation, with the reds in the first pile and the blacks in the second pile.

For poker purposes, it's winning and losing cards that are controlled, not reds and blacks.

Classic Fitch Cheney trick ("Given Any Five Cards . . . ")

In 1951, one of the best card tricks ever invented was published. Martin Gardner mentioned it in his *Scientific American* column. Since the 1980s, it has found new popularity thanks to the efforts of Art Benjamin, Elwyn Berlekamp, Paul Zorn, and others.

> *Five random playing cards from a regular deck are given to a volunteer. He hands one back, and places the remaining four in a face-up row.*
>
> *You, having not seen anything so far, look at the row of cards, and promptly name the hidden fifth card.*

How can this be done, without any verbal or physical cues?

You and "the volunteer" have agreed in advance on a strictly mathematical system of communication . . .

Fitch Cheney's Five-Card Trick

This superb effect is due to William Fitch Cheney Jnr (1904-1974), who in 1927 received MIT's first PhD in Mathematics.

The volunteer chooses which card to hand back, and in what order to place the other four. Three main ideas make this possible.

1. There must be (at least) two cards of the same suit. We may as well assume it's two Clubs. One Club is handed back, and by placing the remaining four cards in some particular order, the volunteer effectively tells you the identity of the Club handed back.

2. The volunteer places the retained Club in one key position (e.g., the first) to communicate the suit to you.

The volunteer uses the other three positions for the placement of the remaining cards, which can be arranged in $3! = 6$ ways.

3. This is where it gets tricker . . .

Fitch Cheney's Five-Card Trick

If you both agree in advance on a one-to-one correspondence between the six possible permutations and $1, 2, \ldots, 6$, then the volunteer can communicate one of six things.

What *can* one say about these other three cards? Not much—for instance, some or all of them could be Clubs too, or there could be other suit matches!

However, one thing *is* certain: they are all distinct, so with respect to some total ordering of the entire deck, one of them is LOW, one is MEDIUM, and one is HIGH.

Asssume suits are in CHaSeD order.

This permits for an unambiguous and easily remembered way to communicate a number between 1 and 6.

Fitch Cheney's Five-Card Trick

But surely 6 isn't enough?

The hidden card could in general be any one of 12 Clubs!

This brings us to the third main idea:

3. The volunteer must be careful as to which card he hands back.

Considering the 13 card values, 1 (Ace), $2, 3, \ldots, 10$, J, Q, K, as being arranged clockwise on a circle, we see that the two suit match cards are at most 6 values apart, i.e., counting clockwise, one of them lies at most 6 vertices past the other.

The volunteer gives this "higher" valued Club back to the spectator to hide. the volunteer will then use the "lower" Club and the other three cards to communicate the identity of the hidden card to you.

Fitch Cheney's Five-Card Trick

For example, if the volunteer has the 2♣ and 8♣, he hands back the 8♣. However, if he has the 2♣ and J♣, he hands back the 2♣.

In general, he saves one card of a particular suit and communicates another of the same suit, whose numerical value is k higher than the one saved, for some integer k between 1 and 6 inclusive.

Put this total CHaSeD linear ordering on the whole deck:

A♣, 2♣, . . . , K♣,
A♡, 2♡, . . . , K♡,
A♠, 2♠, . . . , K♠,
A♢, 2♢, . . . , K♢.

Fitch Cheney's Five-Card Trick

Mentally, he labels the three cards L (low), M (medium), and H (high) with respect to this ordering.

The 6 permutations of L,M,H are always ordered by rank, i.e., 1 = LMH, 2 = LHM, 3 = MLH, 4 = MHL, 5 = HLM and 6 = HML.

Finally, he orders the three cards in the pile from left to right according to this scheme to communicate the desired integer.

For example, if he is playing the J♣ and trying to communicate the 2♣ to you, then $k = 4$, and he plays the other three cards in the order MHL.

You know that the hidden card is a Club, decodes the MHL as 4, and mentally counts 4 past the visible J♣ (mod 13) to get the 2♣.

Fitch Four Glory

It's customary to extend this five-card trick to work for (much) larger decks—just try doing it for a regular deck with one joker added in—but we suggest a different direction to explore.

(Fitch Four) *Any four cards are handed to the volunteer. He glances at them briefly, and hands one back. It's placed face down to one side.*

He now places the remaining three cards left to right in a row on the table, some face up, some face down.

You, having not seen anything so far, look at the cards on display, and promptly name the hidden fourth card—even in the case where all three cards are face down!

Inspiring a Colm . . . and a column (at MAA.org)

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18 Jun 00

Dear Colm Mulcahy:

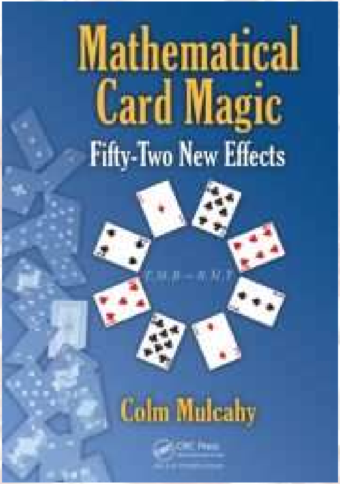
Many thanks for your letter and all the ~~greta~~ great enclosures. You've done a fine job of selecting good mathematical card tricks and explaining them so clearly. Maybe you should write a book about them? I think the MAA would be interested.

I knew Fitch Cheney slightly. He died many many years ago. I agree that his five-card trick is a gem.

I learned a lot from your articles on wavelets, and did

Extract from letter from Martin Gardner

Inspiring a book



Mathematical Card Magic:
Fifty-Two New Effects

AK Peters/CRC Press, 2013.

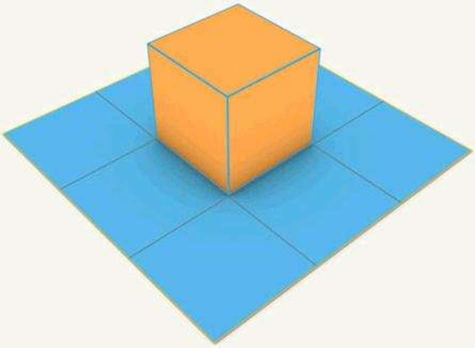
Hardback, full color, 380 pages.

13 main chapters, each with 4 effects.

Largely original material.

Endorsed by Max Maven, Ron Graham, Art Benjamin & Lennart Green.

Let's wrap it up
$= \frac{1}{2}$(komm zur Sache) $+ \frac{1}{2}$(wickeln Sie es)



Wrap the gold cube completely with the blue paper!

All cutting and folding must be along existing grid lines.

The paper must remain in one piece.