DARBOUX INTEGRATING FACTORS: INVERSE PROBLEMS

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Abstract. We discuss planar polynomial vector fields with prescribed Darboux integrating factors, in a nondegenerate affine geometric setting. We establish a reduction principle which transfers the problem to polynomial solutions of certain meromorphic linear systems, and show that the space of vector fields with a given integrating factor, modulo a subspace of explicitly known “standard” vector fields, has finite dimension. For several classes of examples we determine this space explicitly.

1. Introduction

Given a planar polynomial vector field, there is the classical problem (going back to Darboux and Poincaré) to determine an integrating factor of Darboux type, or to verify that no such factor exists. The papers and survey papers by Schlomiuk [14, 15, 16] give a good introduction to this field of research. A preliminary question is to determine the invariant algebraic curves of the vector field, or to ensure that no such curves exist. Several results, mostly for settings with certain additional conditions, are known; we mention Cerveau and Lins Neto [4], Carnicer [3], Camacho and Sad [2], Zoladek [20], and Esteves and Kleiman [10]. An elementary approach, which also discusses integrating factors, is given in [19]. However, it seems that a general solution of this problem is still not within reach. Considering and solving the corresponding inverse problems seems to be essential in order to obtain a proper understanding of the situation. Considering and solving the corresponding inverse problems seems to be essential in order to obtain structural insight and a proper understanding of the situation. These inverse problems are not trivial, but they are accessible.

The solution of the inverse problem for curves in the projective plane, in the case that all irreducible curves are smooth and all intersections transversal, has been considered by several authors. An elementary exposition is given in [7]. The nondegenerate affine setting with smooth irreducible curves and only transversal intersections was resolved, some years ago, using mainly tools from elementary commutative algebra; see [5] and [18]. A complete solution for the general problem in the affine plane - modulo standard tasks of algorithmic algebra - is presented in [6]. In the cited papers the strategy was to start with a linear space of vector fields that are known to admit the given curves. Then one proceeded to show that the list is already complete, or at least that the quotient space of all vector fields admitting the curves modulo the known ones

1991 *Mathematics Subject Classification.* Primary 34C05, 34A34, 34C14.

Key words and phrases. polynomial differential system, invariant algebraic curve, integrating factor.

J.L. and Ch.P. are partially supported by a MEC/FEDER grant MTM 2005-06098-C02-01 and by a CIRIT grant number 2001SGR 00550. Ch.P. is additionally partially supported by a MEC/FEDER grant MTM2006-00478. C.Ch. and S.W. acknowledge the hospitality and support of the CRM and Mathematics Department at Universitat Autònoma de Barcelona during visits when this manuscript was prepared.

has finite dimension. The remaining finite-dimensional problem is then amenable to methods of algorithmic algebra.

The inverse problem for integrating factors, which we will discuss in the present paper, seems harder to tackle. In [12] and [5] the case of inverse polynomial integrating factors was settled if the above-mentioned affine nondegeneracy conditions hold for the underlying curves. Moreover, in the nondegenerate projective setting the case of arbitrary exponents was resolved. Similar to the strategy for curves, the basic approach was to obtain a sufficiently large inventory of vector fields that admit a given integrating factor, and then to prove that no further ones exist.

The main theme of the present paper is a general characterization of vector fields with prescribed integrating factors in the affine nondegenerate geometric scenario. (In other words, the underlying projective curves may have degenerate singular points in the projective plane as long as they are restricted to one line.) The main result is that, modulo a subspace of 'known' vector fields admitting a prescribed integrating factor, only a finite dimensional problem remains. To obtain this result, we will employ a quite simple reduction principle, which leads to a system of meromorphic linear differential equations for functions of one variable, and more precisely to polynomial solutions of this system. Since the solution space of this meromorphic system is finite dimensional, finite dimensionality of our quotient space follows. Moreover, one obtains a computational access to the determination of the quotient space. For several classes of examples we will prove that the dimension of the quotient space is zero, but we will also exhibit cases when the dimension is positive.

The remaining open problem is the degenerate affine setting; including irreducible curves with singularities as well as non-transversal intersections. In the final section of the paper we show that this problem can in part be addressed by extending the space of vector fields from polynomials to the larger class whose entries are polynomial in one variable and rational in the other. Variants of the arguments from previous sections show finite dimension of the corresponding quotient space modulo "known" vector fields, and examples illustrate that explicit results can be obtained by computation. But transferring results back to the case of polynomial vector fields is a nontrivial problem.

We will further investigate the degenerate affine setting in forthcoming work. In addition to continuing the approach from Section 7 of this paper we will also employ sigma processes to remove degenerate singular points from the affine plane.

2. Basics and known results

We consider a complex polynomial vector field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \tag{1}$$

on $\mathbb{C}^2$, sometimes also written as

$$X = \begin{pmatrix} P \\ Q \end{pmatrix},$$

and irreducible pairwise relatively prime polynomials $f_1, \ldots, f_r$, with $f := f_1 \cdots f_r$. The degree of f will be denoted by $\delta(f)$; the degree of X is defined here as $\max\{\delta(P), \delta(Q)\}$. We will also consider degrees δ_x, δ_y with respect to the individual variables.

The question we will address in the present paper is the following: Given nonzero complex constants $d_1, \ldots, d_r$, under which conditions does X admit the *Darboux integrating factor* $f_1^{-d_1} \cdots f_r^{-d_r}$?

It is well known (see, e.g. [7] and [5]) that the complex zero set of f (which is generally a reducible curve in $\mathbb{C}^2$) is then invariant for the vector field; equivalently there are polynomials $L_1, \ldots, L_r$ such that

$$Xf_i = L_i \cdot f_i, \quad 1 \le i \le r.$$

We will briefly say that in this case *the vector field X admits f*, or *admits the curve given by $f = 0$*. The vector field X admits the Darboux integrating factor above if and only if

$$d_1 \cdot L_1 + \cdots d_r \cdot L_r = \operatorname{div} X.$$

The respective zero sets of f and the f_i in $\mathbb{C}^2$ will be denoted by C and C_i. As usual, we call a point z with $f(z) = f_x(z) = f_y(z) = 0$ a singular point of C, and analogously for the C_i. The Hamiltonian vector field of f is defined by

$$X_f = -f_y \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial y}.$$

For given f, the vector fields admitting f form a linear space $\mathcal{V}_f$. We are interested in the structure of the subspace of $\mathcal{V}_f$ whose elements admit the Darboux integrating factor $f_1^{d_1} \cdots f_r^{d_r}$. First, let us collect some facts and properties that are known from previous work. (See [18], [7], [12], [5]). Clearly, vector fields of the type

$$X = \sum_i a_i \frac{f}{f_i} \cdot X_{f_i} + f \cdot \widetilde{X} \tag{2}$$

with polynomials a_i and a polynomial vector field $\widetilde{X}$, admit f. These vector fields form a subspace of $\mathcal{V}_f$ which will be denoted $\mathcal{V}_f^1$. Following [12], but reorganizing a little, we introduce two generic nondegeneracy conditions:

(ND1) Each C_i is nonsingular.
(ND2) All singular points of C have multiplicity one (thus when two irreducible components intersect, they intersect transversally, and no more than two irreducible components intersect at one point).

Note that - in contrast to [5] - there is no condition with regard to the behavior at infinity. The following result is from [7], with an improvement in [5], Theorem 3.4 (in conjunction with Theorem 3.6); see also [12] and [6].

Theorem 1. *If the conditions (ND1) and (ND2) hold then $\mathcal{V}_f = \mathcal{V}_f^1$.*

This is our starting point for turning to integrating factors. As above, fix complex constants $d_1, \ldots, d_r$, all of them nonzero. The vector fields with Darboux integrating factor

$$\left(f_1^{d_1} \cdots f_r^{d_r}\right)^{-1} \tag{3}$$

form a linear space $\mathcal{F}_f = \mathcal{F}_f(d_1, \ldots, d_r)$, which is a subspace of $\mathcal{V}_f$. Let us first exhibit some of its elements; cf. [18] and [5]. Given an arbitrary polynomial g, define

$$Z_g = Z_g^{(d_1,\ldots,d_r)}$$

to be the Hamiltonian vector field of

$$g / \left(f_1^{d_1-1} \cdots f_r^{d_r-1}\right).$$

Lemma 2. *The following statements hold.*

(a) *The vector field*

$$f_1^{d_1}\cdots f_r^{d_r}\cdot Z_g = f\cdot X_g - \sum_{i=1}^{r}(d_i-1)g\frac{f}{f_i}\cdot X_{f_i} \tag{4}$$

is polynomial and admits the integrating factor $(f_1^{d_1}\cdots f_r^{d_r})^{-1}$. *The vector fields of this type form a subspace* $\mathcal{F}_f^0=\mathcal{F}_f^0(d_1,\ldots,d_r)$ *of* $\mathcal{F}_f(d_1,\ldots,d_r)$.

(b) *Given* $e_1,\ldots,e_r$ *such that every* d_i-e_i *is a nonnegative integer, one has*

$$Z_g^{(e_1,\ldots,e_r)} = Z_{g^*}^{(d_1,\ldots,d_r)},\ \text{with } g^* := f_1^{(d_1-e_1)}\cdots f_r^{(d_r-e_r)}\cdot g$$

and therefore

$$f_1^{(d_1-e_1)}\cdots f_r^{(d_r-e_r)}\cdot\left(f_1^{e_1}\cdots f_r^{e_r}\cdot Z_g^{(e_1,\ldots,e_r)}\right) = f_1^{d_1}\cdots f_r^{d_r}\cdot Z_{g^*}^{(d_1,\ldots,d_r)},$$

whence

$$f_1^{(d_1-e_1)}\cdots f_r^{(d_r-e_r)}\mathcal{F}_f^0(e_1,\ldots,e_r)\subseteq \mathcal{F}_f^0(d_1,\ldots,d_r).$$

Proof. Obvious. □

There exist vector fields in $\mathcal{F}_f(1,\ldots,1)$ that are not in $\mathcal{F}_f^0(1,\ldots,1)$: For all constants α_i and every vector field $\widehat{X}$ with divergence zero, the vector field

$$X = \sum_i \alpha_i\frac{f}{f_i}\cdot X_{f_i} + f\cdot\widehat{X} \tag{5}$$

clearly admits the integrating factor f^{-1}. More generally, if $d_1,\ldots,d_r$ are all nonnegative integers then

$$X = f_1^{d_1}\cdots f_r^{d_r}\left(\sum_{i=1}^{r}\frac{\alpha_i}{f_i}\cdot X_{f_i} + Z_h^{(d_1,\ldots,d_r)}\right), \tag{6}$$

with constants α_i and some polynomial h, admits the integrating factor $(f_1^{d_1}\cdots f_r^{d_r})^{-1}$.

The following result was shown in [5]; see also Section 3 below.

Theorem 3. *Suppose that conditions (ND1) and (ND2) hold, and that* $d_1,\ldots,d_r$ *are positive integers. Then a vector field* X *admits the integrating factor* $(f_1^{d_1}\cdots f_r^{d_r})^{-1}$ *if and only if* X *is of the form* (6).

The case of Darboux integrating factors that are not inverse polynomials has been resolved completely in [5] when additional nondegeneracy conditions at infinity hold:

Proposition 4. *Suppose that conditions (ND1) and (ND2) hold, and moreover that the homogeneous highest degree terms of* $f_1,\ldots,f_r$ *have no multiple prime factors and are pairwise relatively prime. Then a vector field* X *admits the integrating factor* (3), *with some* d_ℓ *not a nonnegative integer, if and only if* X *has the form* (4), *thus* $X=f_1^{d_1}\cdots f_r^{d_r}\cdot Z_g$ *for some polynomial* g.

One also knows some results about degenerate settings. The following is a special case of Prop. 3.4 of [18].

Proposition 5. *Let $f_1, \ldots, f_r$ be irreducible homogeneous (thus linear) polynomials. Then a vector field X admits the integrating factor (3) if and only if*

$$X = b \cdot \begin{pmatrix} x \\ y \end{pmatrix} + f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g$$

with a homogeneous polynomial b of degree $\sum d_i - 2$, and some polynomial g.

The methods employed in the proofs of Theorem 3 and Propositions 4 and 5 do not seem to be applicable for more general settings. Our goal for the present paper is to explore the scenario of Proposition 4 when no additional conditions at infinity are imposed. This requires different techniques. Achieving this goal may be a critical step since in the affine plane the nondegeneracy conditions (ND1) and (ND2) can be enforced by a series of sigma processes, due to a theorem by Bendixson and Seidenberg.

3. A reduction principle

We start with an elementary observation from [5], which also was at the basis for the proofs of Theorem 3 and Proposition 4.

Lemma 6. *Let X be of the form (2), namely*

$$X = \sum_{i=1}^{r} a_i \frac{f}{f_i} \cdot X_{f_i} + f \cdot \widetilde{X} \in \mathcal{V}_f^1$$

and assume that X admits the integrating factor (3), with some $d_\ell \neq 1$. Then

$$X + \frac{1}{d_\ell - 1} \cdot f_1^{d_1} \cdots f_r^{d_r} \cdot Z_{a_\ell} = f_\ell \cdot X^*,$$

and X^* admits the integrating factor

$$\left(f_1^{d_1} \cdots f_\ell^{(d_\ell - 1)} \cdots f_r^{d_r} \right)^{-1}.$$

Therefore, modulo $\mathcal{F}_f^0(d_1, \ldots, d_r)$ the vector field X is congruent to some $f_\ell \cdot X^*$, with $X^* \in \mathcal{F}_f(d_1, \ldots, d_\ell - 1, \ldots, d_r)$. Lemma 2 shows that this principle may be applied repeatedly. We can therefore note:

Lemma 7. *In order to investigate the structure of the quotient space*

$$\mathcal{F}(d_1, \ldots, d_r)/\mathcal{F}_f^0(d_1, \ldots, d_r)$$

one may replace d_i by 1 if d_i is a positive integer, and by $d_i - k_i$ with any positive integer k_i otherwise. In particular we may assume that each d_i has real part ≤ 1.

We use this to outline the proof of Theorem 3. By Lemma 7 one may assume $d_1 = \cdots d_r = 1$. Now step 3 of the proof of Theorem 4.2 of [5] applies. The proof in [5] also uses reduction, under stronger hypotheses, which are not needed in this particular case.

In the general case, one may try to achieve degree reduction for the vector fields involved. But without additional hypotheses it is impossible to keep these degrees under control when passing from X to X^* in Lemma 6. Generally, such a degree reduction strategy will not work, and moreover, Proposition 5 shows that the conclusion of Proposition 4 is not always true.

But there exists a general reduction strategy which relies on a weaker principle.

Proposition 8. *Given $f_1, \dots, f_r$, let $n_j := \delta_y(f_j)$ be the respective degree in y, $n := \sum n_j = \delta_y(f)$, and assume that each $f_j = \alpha_j \cdot y^{n_j} + \dots$ with nonzero constants α_j, $1 \le j \le r$.*

(a) Then any vector field $X \in \mathcal{V}_f^1$ which admits $f = f_1 \cdots f_r$ has a representation

$$X = \sum_{i=1}^{r} \tilde{a}_i \frac{f}{f_i} \cdot X_{f_i} + f \cdot \tilde{\tilde{X}}$$

such that $\delta_y(\tilde{a}_i) < n_i$ for $i = 1, \dots, r$.

(b) Assume that X admits the integrating factor (3), with some $d_\ell \neq 1$.

(b1) If $\delta_y(X) > n + n_\ell - 1$, then

$$X + \frac{1}{d_\ell - 1} \cdot f_1^{d_1} \cdots f_r^{d_r} \cdot Z_{\tilde{a}_\ell} = f_\ell \cdot X^* \quad \text{and} \quad \delta_y(X^*) = \delta_y(X) - n_\ell.$$

(b2) If $\delta_y(X) \le n + n_\ell - 1$, then

$$X + \frac{1}{d_\ell - 1} \cdot f_1^{d_1} \cdots f_r^{d_r} \cdot Z_{\tilde{a}_\ell} = f_\ell \cdot X^* \quad \text{and} \quad \delta_y(X^*) \le n - 1.$$

Proof. Given a representation

$$X = \sum_{i=1}^{r} a_i \frac{f}{f_i} \cdot X_{f_i} + f \cdot \widetilde{X},$$

as in (2), then whenever $\delta_y(a_j) \ge \delta_y(f_j)$ one has $a_j = \tilde{a}_j + b_j \cdot f_j$ with polynomials $\tilde{a}_j$ and b_j, and $\delta_y(\tilde{a}_j) < n_j$. Substituting this in (2) and rearranging terms shows statement (a).

To prove part (b1), from (a) we have that $\delta_y(\tilde{a}_\ell) < n_\ell$. Now observe

$$\delta_y\left(\tilde{a}_\ell \frac{f}{f_i} \cdot X_{f_i}\right) \le n + n_\ell - 2,$$

due to $f_i = \alpha_i y^{n_i} + \cdots$ with α_i constant, and

$$\delta_y\left(f \cdot X_{\tilde{a}_\ell}\right) \le n + n_\ell - 1.$$

Thus

$$\delta_y\left(f_1^{d_1} \cdots f_r^{d_r} \cdot Z_{\tilde{a}_\ell}\right) \le n + n_\ell - 1.$$

The rest of statement (b1) and statement (b2) follows easily. □

Remarks.

(i) We will always assume in the following that the two nondegeneracy conditions (ND1) and (ND2) hold.

(ii) The hypothesis on the f_i can be achieved via a linear transformation of the variables, with $n_i = \delta(f_i)$. In this sense the principle is universally applicable, and the hypothesis will be assumed from now on. (Of course, one will gladly consider cases with $n_i < \delta(f_i)$.)

Corollary 9. *When conditions (ND1) and (ND2) hold and d_ℓ is not a positive integer for some ℓ then the question whether some vector field $X \in \mathcal{F}_f(d_1, \dots, d_r)$ is contained in $\mathcal{F}_f^0(d_1, \dots, d_r)$ is reduced to the analogous question for a vector field $\widehat{X} \in \mathcal{F}_f(d_1, \dots, d_\ell - k_\ell, \dots, d_r)$ with*

$$\delta_y(\widehat{X}) \le n - 1$$

for some integer $k_\ell \ge 0$.

Proof. This follows by repeated application of Proposition 8, Theorem 1, Lemma 6 and the previous remarks. □

The usefulness of the reduction principle becomes manifest next, because there is an approach to handling vector fields of bounded y-degree:

Theorem 10. *Let m_1 and m_2 be positive integers, and consider vector fields*

$$X = P\,\partial/\partial x + Q\,\partial/\partial y$$

with $\delta_y(P) < m_1$ and $\delta_y(Q) < m_2$. Then the following hold:

(a) There exist vector fields

$$Y_i = v_i\,\partial/\partial x + w_i\,\partial/\partial y, \quad 1 \le i \le s,$$

that are linearly independent over $\mathbb{C}[x]$, satisfying the degree conditions $\delta_y(v_i) < m_1$, $\delta_y(w_i) < m_2$, and all Y_i admitting f, with the following property: The vector field X admits f if and only if it has a representation

$$X = u_1(x) \cdot Y_1 + \cdots + u_s(x) \cdot Y_s \tag{7}$$

with $u_1, \ldots, u_s \in \mathbb{C}[x]$. Moreover the polynomials u_i are uniquely determined.

(b) Assume that X is of the form (7). Given $d_1, \ldots, d_r$, there exist matrices

$$V(x) \quad \text{and} \quad B(x) = B_{d_1,\ldots,d_r}(x)$$

with entries in $\mathbb{C}[x]$ with the following property: X is contained in $\mathcal{F}_f(d_1, \ldots, d_r)$ if and only if

$$V(x) \cdot \begin{pmatrix} u_1' \\ \vdots \\ u_s' \end{pmatrix} = B(x) \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_s \end{pmatrix}.$$

The matrices V and B have s columns and at most $\max\{m_1, m_2\} - 1$ rows. The entries of V do not depend on $d_1, \ldots, d_r$.

Proof. (a) The vector fields of y-degree less than m_1, m_2 respectively in their components form a free module over $\mathbb{C}[x]$, and among these the vector fields which admit f obviously form a submodule. Since $\mathbb{C}[x]$ is a principal ideal domain, this submodule is also free.

(b) Let $K_{i,j}$ be the cofactor of f_j with respect to Y_i. For X as in part (a), the cofactor of f_j is then given by

$$L_j = \sum_i u_i \cdot K_{i,j}.$$

Note that $\delta_y(K_{i,j}) \le \max\{m_1, m_2\} - 2$. Evaluation of the integrating factor condition $\operatorname{div} X = \sum d_j L_j$ gives

$$\operatorname{div} \sum u_i \cdot Y_i = \sum (u_i \cdot \operatorname{div} Y_i + Y_i(u_i)) = \sum (u_i \cdot \operatorname{div} Y_i) + \sum v_i \cdot u_i' = \sum d_j L_j,$$

and hence

$$\sum_i v_i \cdot u_i' = \sum_i u_i \left(\sum_j d_j \cdot K_{i,j} - \operatorname{div} Y_i \right). \tag{8}$$

We compare coefficients of powers of y (note that only powers from 0 to $\max\{m_1, m_2\} - 2$ may occur) to obtain the assertion. □

In the following sections we will use this theorem to investigate particular geometric settings. The principal computational problem that remains is to explicitly determine the vector fields Y_i. We are able to calculate the vector fields Y_i in some settings, see Sections 4, 5 and 6. But a general consequence for the structure of $\mathcal{F}_f/\mathcal{F}_f^0$ is worth noting first.

Theorem 11. *Let the setting of Theorem 10 be given. If some d_ℓ is not a positive integer then*

$$\dim\left(\mathcal{F}_f(d_1,\dots,d_r)/\mathcal{F}_f^0(d_1,\dots,d_r)\right) < \infty.$$

Proof. Under our assumptions we can apply Proposition 8, hence we may assume the setting of Theorem 10 with all $\delta_y(Y_i) \le n-1$.

We will first show that $v_1,\dots,v_s$ are linearly independent over $\mathbb{C}[x]$. Thus let $c_1,\dots,c_s \in \mathbb{C}[x]$ such that $c_1v_1+\cdots+c_sv_s = 0$. Since every Y_i admits f, we have

$$v_i \cdot f_x + w_i \cdot f_y = K_i \cdot f, \quad 1 \le i \le s,$$

with K_i the cofactor corresponding to Y_i. We obtain

$$\left(\sum_i c_i w_i\right) \cdot f_y = \left(\sum_i c_i K_i\right) \cdot f.$$

Due to our basic assumption f and $f_y \neq 0$ are relatively prime, thus f must divide $\sum c_i w_i$. But if this sum is nonzero then it has y-degree less than n; a contradiction. So, we conclude that $\sum c_i w_i = 0$, and therefore $\sum c_i Y_i = 0$. Since the Y_i are linearly independent over $\mathbb{C}[x]$ we find that $c_1 = \ldots = c_s = 0$.

We may now pass to $\mathbb{C}(x)$. In view of (8), the linear independence of $v_1,\dots,v_s$ is equivalent to the fact that the rank $V = s$. Thus one may choose an $s\times s$-subsystem with invertible matrix on the left-hand side, and obtain a meromorphic linear system

$$\begin{pmatrix} u_1' \\ \vdots \\ u_s' \end{pmatrix} = A(x) \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_s \end{pmatrix}. \tag{9}$$

The space of solutions of this system has finite dimension s, and so, a fortiori, contains the subspace of polynomial solutions. The dimension of the latter is an upper bound for the dimension of $\mathcal{F}_f/\mathcal{F}_f^0$.

□

In the following sections we will explicitly determine system (9) for several geometric scenarios, and use it to determine $\mathcal{F}_f/\mathcal{F}_f^0$. We therefore recall some facts about meromorphic linear systems. In $\mathbb{C}^n$ consider a system of linear differential equations

$$w' = A(x)\cdot w$$

with the entries of A meromorphic; thus each singular point x_0 is (at most) a pole, and we have a Laurent expansion

$$A(x) = (x-x_0)^{-\ell}\cdot A_{-\ell} + \cdots \qquad A_{-\ell} \neq 0,$$

with some integer ℓ and a constant matrix $A_{-\ell}$. If there exists a solution that is meromorphic in x_0, i.e.

$$w(x) = (x-x_0)^q\, w_q + \cdots \qquad w_q \neq 0,$$

with some integer $q \geq 0$, then substitution into the differential equation leads to

$$q(x-x_0)^{q-1} \cdot w_q + \cdots = (x-x_0)^{q-\ell} \cdot A_{-\ell} w_q + \cdots$$

For $\ell > 1$ we see that necessarily

$$A_{-\ell} \cdot w_q = 0;$$

thus $A_{-\ell}$ admits the eigenvector w_q with eigenvalue 0. For $\ell = 1$ (the case of a *weak singular point*) we obtain the necessary condition

$$A_{-1} \cdot w_q = q \cdot w_q,$$

thus A_{-1} admits the eigenvector w_q with integer eigenvalue q. If the solution is holomorphic in x_0 then q must be a nonnegative integer. There is an extensive theory for weak singular points of meromorphic linear systems; see for instance Ch. 4 of Coddington and Levinson [9]. In what follows we will only need the fact that the existence of meromorphic solutions at singular points of A has implications for the eigenvalues of $A_{-\ell}$.

The singular point at infinity is of particular interest to us; see Coddington and Levinson [9], Ch. 4, Sec 6. To analyze this point, introduce

$$v = x^{-1}; \quad \frac{dw}{dv} = -\frac{1}{v^2}\frac{dw}{dx}.$$

Hence

$$\frac{dw}{dv} = -v^{-2}A(v^{-1}) \cdot w = (v^{-\ell}B_{-\ell} + \cdots) \cdot w,$$

and one considers the singularity of this system at $v = 0$. If ∞ is a weak singular point of system $w' = A(x) \cdot w$ then the eigenvalues of $A_\infty := B_{-1}$ are of particular significance: If a polynomial solution

$$w = w_0 + x \cdot w_1 + \cdots x^m \cdot w_m$$

exists for $w' = A(x) \cdot w$ then

$$w(v) = v^{-m} \cdot w_m + \cdots + v^{-1} \cdot w_1 + w_0,$$

and the eigenvalue $-m$ of A_∞ is just the negative degree of the polynomial solution in question.

4. One irreducible factor - special curves

We will first apply the reduction principle and Theorem 10 to the case of one irreducible curve. In the present section we will focus on the special class of smooth irreducible curves defined by a polynomial

$$f(x,y) = y^m - p(x) \tag{10}$$

with $m \geq 2$ an integer and p a polynomial in one variable with no multiple roots. This class includes elliptic and hyperelliptic curves. We are interested in $\mathcal{F}_f(d)$ for all nonzero d, and there only remains the case when d is not a positive integer; see Theorem 3. Due to Lemma 7 we may assume that $\mathrm{Re}\, d < 1$. By Corollary 9, modulo $\mathcal{F}_f^0$ it is sufficient to consider the problem for a vector field

$$Y = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for some polynomials b_1 and b_2 of y-degree $\leq m-1$ (and still with $\mathrm{Re}\, d < 1$). We proceed as suggested by Theorem 10, and first determine the vector fields of this type which admit the polynomial f.

Lemma 12. *A vector field $Y = b_1\,\partial/\partial x + b_2\,\partial/\partial y$ with $\delta_y(b_i) \leq m-1$ for $i = 1, 2$ admits f if and only if*

$$Y = u_0(x)Y_0 + u_1(x)\cdot Y_1 + \cdots + u_{m-1}(x)\cdot Y_{m-1}$$

with polynomials $u_0, u_1, \ldots, u_{m-1}$ in one variable, and

$$Y_0 := \begin{pmatrix} my^{m-1} \\ p'(x) \end{pmatrix}, \qquad Y_j := \begin{pmatrix} mp(x)y^{j-1} \\ p'(x)y^j \end{pmatrix} \qquad \text{for} \quad 1 \leq j \leq m-1.$$

Proof. We start from

$$Y = \Big(u_0(x) + u_1(x)y + \cdots u_{m-1}(x)y^{m-1}\Big)\cdot \begin{pmatrix} my^{m-1} \\ p'(x) \end{pmatrix} + (y^m - p(x))\cdot \begin{pmatrix} \vdots \end{pmatrix}.$$

Rewrite this expression, replacing y^{m+k} by $p\cdot y^k$ in the first summand and changing the second accordingly. This yields

$$u_0(x)\cdot \begin{pmatrix} my^{m-1} \\ p' \end{pmatrix} + u_1(x)\cdot Y_1 + \cdots + u_{m-1}(x)\cdot Y_{m-1} + (y^m - p(x))\cdot \begin{pmatrix} \vdots \end{pmatrix}.$$

The last term must vanish by degree considerations, and the lemma follows. □

Theorem 13. *Let d be nonzero, and not a positive integer. For any polynomial f as in* (10) *one has $\mathcal{F}_f(d) = \mathcal{F}_f^0(d)$.*

Proof. We may assume $\mathrm{Re}\, d < 1$. Let Y be as in Lemma 12. Note that $Y_0 = X_f$. The vector field Y_j admits f with cofactor equal to $K_j := mp'(x)y^{j-1}$, $1 \leq j \leq m-1$, while

$$\begin{aligned} \mathrm{div}\,(u_j\cdot Y_j) &= \Big(mp(x)u_j'(x) + (m+j)p'(x)u_j(x)\Big)\, y^{j-1} \quad (j \geq 1), \\ \mathrm{div}\,(u_0\cdot Y_0) &= mu_0' y^{m-1}. \end{aligned}$$

Therefore the integrating factor condition

$$d\cdot \sum_{j=0}^{m-1} u_j K_j = \sum_{j=0}^{m-1} \mathrm{div}(u_j Y_j)$$

is satisfied if and only if $u_0' = 0$ and

$$dmp'(x)u_j(x) = mp(x)u_j'(x) + (m+j)p'(x)u_j(x), \quad 1 \leq j \leq m-1,$$

as follows from comparing powers of y. Thus u_0 is constant.

If $j \geq 1$ and $u_j \neq 0$ then last equality is equivalent to

$$\frac{u_j'}{u_j} = (d - 1 - j/m)\cdot \frac{p'}{p}, \quad 1 \leq j \leq m-1.$$

Therefore u_j is a constant multiple of a power of p, with the exponent given by the first factor on the right-hand side. Since we assume that $u_j \neq 0$ this exponent must be a nonnegative integer, due to the assumption on p. But this is impossible in view of $\mathrm{Re}\, d < 1$. Hence $u_j = 0$ for $1 \leq j \leq m-1$, and this completes the proof. □

5. One irreducible factor - general cubic curves

As for a second class of examples, we consider smooth irreducible cubic curves determined (with no loss of generality) by a polynomial

$$f = y^3 + p_1(x) \cdot y + p_0(x). \tag{11}$$

We imitate the strategy from the previous section. This leaves us with a vector field X that admits f and can be written in the form

$$X = a \cdot X_f + f \cdot \widetilde{X}, \quad \delta_y(a) \leq 2;\ \delta_y(\widetilde{X}) \leq 1.$$

Lemma 14. *Modulo $\mathcal{F}_f^0(d)$ it is sufficient to investigate vector fields of type $Y = b_1\,\partial/\partial x + b_2\,\partial/\partial y$ with $\delta_y(b_1) \leq 1$ and $\delta_y(b_2) \leq 2$. Additionally, we may assume* $\operatorname{Re} d < 1$.

Proof. We may assume that $\operatorname{Re} d < 1$ by Lemma 7 and Theorem 3, and this property persists in further applications of Lemma 6. Due to Corollary 9 we can reduce the problem to vector fields X with $\delta_y(X) \leq 2$. Additionally, from Proposition 8(a) we have

$$X = aX_f + f\tilde{X}$$

with $\delta_y(a) < 3$. So, $\delta_y(f\tilde{X}) = \delta_y(X - aX_f) \leq 4$, which shows that $\tilde{X} = \tilde{b}_1\partial/\partial x + \tilde{b}_2\,\partial/\partial y$ with $\delta_y(\tilde{b}_1) \leq 1$ and $\delta_y(\tilde{b}_2) \leq 1$.

Moreover $f^d Z_a = fX_a - (d-1)aX_f$ and so we have

$$X = aX_f + f\tilde{X} = \left(\frac{1}{d-1} fX_a - \frac{1}{d-1} f^d Z_a\right) + f\tilde{X},$$

and

$$X + \frac{1}{d-1} f^d Z_a = f\left(\tilde{X} + \frac{1}{d-1} X_a\right).$$

Since $\delta_y(a) < 3$ we can write $a = a_0(x) + a_1(x)y + a_2(x)y^2$ and obtain

$$X_a = \begin{pmatrix} -2a_2(x)y - a_1(x) \\ a_2'(x)y^2 + a_1'(x)y + a_0'(x) \end{pmatrix}.$$

Thus $X_a = c_1\partial/\partial x + c_2\partial/\partial y$ with $\delta_y(c_1) \leq 1$ and $\delta_y(c_2) \leq 2$ and $\tilde{X} + \frac{1}{d-1}X_a$ is as desired. □

Lemma 15. *A vector field $Y = b_1\,\partial/\partial x + b_2\,\partial/\partial y$ with $\delta_y(b_1) \leq 1$ and $\delta_y(b_2) \leq 2$ admits f if and only if*

$$Y = u_1(x) \cdot Y_1 + u_2(x) \cdot Y_2$$

with polynomials u_1, u_2 in one variable and

$$Y_1 := \begin{pmatrix} 3p_0 + 2p_1 y \\ p_0' y + p_1' y^2 \end{pmatrix}, \quad Y_2 := \begin{pmatrix} -2p_1^2 + 9p_0 y \\ 2p_0' p_1 - 3p_0 p_1' - p_1 p_1' y + 3p_0' y^2 \end{pmatrix}.$$

Proof. Computing modulo f we have $y^3 \equiv -p_1 y - p_0$ and $y^4 \equiv -p_1 y^2 - p_0 y$. Thus for suitable polynomials $a_i = a_i(x)$ we get

$$\begin{aligned} Y &\equiv \left(a_0 + a_1 y + a_2 y^2\right) \cdot \begin{pmatrix} -3y^2 - p_1 \\ p_1' y + p_0' \end{pmatrix} \\ &\equiv \begin{pmatrix} (3a_1 p_0 - p_1 a_0) + (2a_1 p_1 + 3a_2 p_0)\, y + (2a_2 p_1 - 3a_0)\, y^2 \\ (p_0' a_0 - p_1' p_0 a_2) + (p_0' a_1 + p_1' a_0 - p_1' p_1 a_2)\, y + (p_0' a_2 + p_1' a_1)\, y^2 \end{pmatrix}. \end{aligned}$$

Considering degrees with respect to y one sees that Y actually must be equal to the last vector field in this chain of equivalences, and moreover that $3a_0 = 2a_2p_1$. Set $a_1 = u_1$, $a_2 = 3u_2$ and rearrange terms to obtain the assertion. □

Modulo $\mathcal{F}_f^0$ there remains the investigation of the vector fields in Lemma 15. Following the strategy suggested by Theorems 10 and 11, we obtain a meromorphic linear system.

Proposition 16. *Let f be given as in (11) and assume that d is not a positive integer. The vector field Y in Lemma 15 admits the integrating factor f^{-d}, if and only if the polynomials u_1 and u_2 satisfy*

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{\Delta} \cdot B(x) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{12}$$

with the discriminant $\Delta = 27p_0^2 + 4p_1^3$ of f, and

$$B(x) = \begin{pmatrix} (3d-4)\cdot(9p_0p_0' + 2p_1^2p_1') & (3d-5)\cdot(-3p_1)\cdot(3p_0p_1' - 2p_0'p_1) \\ (3d-4)\cdot(3p_0p_1' - 2p_0'p_1) & (3d-5)\cdot(9p_0p_0' + 2p_1^2p_1') \end{pmatrix}.$$

Moreover we have

$$\dim \mathcal{F}_f(d)/\mathcal{F}_f^0(d) \le 1.$$

Proof. The cofactors of Y_1 and Y_2, respectively, are

$$\begin{aligned} K_1 &= 3p_0' + 3p_1'y \\ K_2 &= -3p_1p_1' + 9p_0'y \end{aligned}$$

while

$$\begin{aligned} \operatorname{div} Y_1 &= 4p_0' + 4p_1'y \\ \operatorname{div} Y_2 &= -5p_1p_1' + 15p_0'y. \end{aligned}$$

Evaluating the integrating factor condition

$$\operatorname{div}\,(u_1Y_1 + u_2Y_2) = d\cdot(u_1K_1 + u_2K_2)$$

yields

$$\begin{pmatrix} 3p_0 & -2p_1^2 \\ 2p_1 & 9p_0 \end{pmatrix} \cdot \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} (3d-4)p_0' & -(3d-5)p_1p_1' \\ (3d-4)p_1' & (3d-5)\cdot 3p_0' \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and inverting the matrix on the left hand side we obtain the meromorphic linear system (12). Therefore $\dim \mathcal{F}_f(d)/\mathcal{F}_f^0(d) \le 2$. We note that the diagonal elements of $B(x)$ are $(3d-4)\cdot\Delta'/6$ and $(3d-5)\cdot\Delta'/6$, respectively and we may suppose that $\operatorname{Re}(d-3/2) < 0$.

Now assume that $\dim \mathcal{F}_f(d)/\mathcal{F}_f^0(d) = 2$. Then system (12) admits a polynomial fundamental system with polynomial Wronskian $w(x)$, and

$$\begin{aligned} w'(x) &= \operatorname{tr} \tfrac{B(x)}{\Delta(x)} \cdot w(x) \\ &= (d-3/2)\, \tfrac{\Delta'(x)}{\Delta(x)} \cdot w(x). \end{aligned}$$

But this implies that $d - 3/2$ is an nonnegative integer; a contradiction. Therefore $\dim \mathcal{F}_f(d)/\mathcal{F}_f^0(d) < 2$ and this completes the proof. □

Remark. The meromorphic linear system has obvious solutions for some special values of the exponent:

$$\begin{array}{ll} Y_1 \in \mathcal{F}_f(4/3) & (u_1 = 1,\, u_2 = 0); \\ Y_2 \in \mathcal{F}_f(5/3) & (u_1 = 0,\, u_2 = 1). \end{array}$$

One verifies directly that Y_1 is of the form (4) with $g = -3y$, and that Y_2 is of the form (4), with $g = -9/2y^2 - 3p_1$. One may proceed to show that

$$\mathcal{F}_f(4/3 + k) = \mathcal{F}_f^0(4/3 + k) \quad \text{and} \quad \mathcal{F}_f(5/3 + k) = \mathcal{F}_f^0(5/3 + k)$$

for every integer $k \geq 0$.

In order to gain a general perspective, we turn to singular points of system (12). These singular points are not necessarily weak singularities: First, the discriminant Δ may have multiple roots even for a smooth and irreducible curve. (One example is given by $f = y^3 + xy + x$, with $p_0 = p_1 = x$.) Second, the upper right entry of the matrix in Proposition 16 may have degree greater than or equal to the degree of Δ, which implies the existence of a pole of order > 1 at infinity. (Generally, given polynomials r and s with leading coefficients $\mathrm{lc}(r)$ and $\mathrm{lc}(s)$, the transform of the rational function $r(x)/s(x)$ will be

$$-v^{-2} r(v^{-1})/s(v^{-1}) = -v^{\delta(s)-\delta(r)-2} \cdot (\mathrm{lc}(r))/\mathrm{lc}(s) + \ldots)$$

with the dots indicating existence of terms of v of positive order).

One may distinguish various cases here, and we discuss two of them:
(i) Assuming that $2\delta(p_0) > 3\delta(p_1)$ (thus $\delta(\Delta) = 2\delta(p_0)$), the upper right entry of $B(x)$ has precise degree $2\delta(p_1) + \delta(p_0) - 1$, and this is smaller than $\delta(\Delta)$ if and only if $2\delta(p_1) \leq \delta(p_0)$. This later condition is necessary and sufficient for the existence of a weak singular point at infinity. (The lower left element of $B(x)$ causes no problem.) For instance, in the case $\delta(p_0) = 5$ and $\delta(p_1) = 3$ the singular point at infinity of the system is not weak.
(ii) Assuming that $2\delta(p_0) < 3\delta(p_1)$ (thus $\delta(\Delta) = 3\delta(p_1)$), the upper right entry of $B(x)$ has precise degree $2\delta(p_1) + \delta(p_0) - 1$, and this is smaller than $\delta(\Delta)$ if and only if $\delta(p_1) \geq \delta(p_0)$. This later condition is necessary and sufficient for the existence of a weak singular point at infinity. (Again, the lower left element of $B(x)$ causes no problem.) For instance, in the case $\delta(p_0) = 4$ and $\delta(p_1) = 3$ the singular point at infinity of the system is not weak.

The remaining case $2\delta(p_0) = 3\delta(p_1)$ is more complicated, since one has to take cancellations of highest-degree terms into account.

We do not aim at a complete discussion of all cubic curves in this section, but we obtain quite definitive results for the cases (i) and (ii) above, assuming that there is a weak singularity at infinity.

Proposition 17. *The following statements hold for the polynomial f be given in* (11).

(a) *If $\delta(p_0) \geq 2\delta(p_1)$ then system (12) admits a first order pole at infinity, with the coefficient matrix of v^{-1} being equal to*

$$A_\infty = \begin{pmatrix} -\frac{3d-4}{3}\delta(p_0) & * \\ 0 & -\frac{3d-5}{3}\delta(p_0) \end{pmatrix}.$$

Therefore, $\mathcal{F}_f(d) = \mathcal{F}_f^0(d)$ whenever d is not a positive integer.

(b) *If $\delta(p_1) \geq \delta(p_0)$ (and $\delta(p_1) > 0$) then system (12) admits a first order pole at infinity, with*

$$A_\infty = \begin{pmatrix} -\frac{3d-4}{2}\delta(p_1) & * \\ 0 & -\frac{3d-5}{2}\delta(p_1) \end{pmatrix}.$$

Therefore, $\mathcal{F}_f(d) = \mathcal{F}_f^0(d)$ whenever d is not a positive integer.

Proof. (a) The diagonal entries of the matrix are $(3d-4)\Delta'/(6\Delta)$ and $(3d-5)\Delta'/(6\Delta)$. By our assumption, the degree of Δ is equal to $2\delta(p_0)$, the polynomial at the lower left position in the matrix of Proposition 16 has degree less than $2\delta(p_0)-1$ and the polynomial at the upper right has degree no more than $\delta(p_0)-1$. This shows the asserted form of A_∞, and in particular both eigenvalues have positive real parts when (as we may assume) $\mathrm{Re}\,(d) < 1$. By the eigenvalue criterion (see the end of Section 3) the system (12) has no nontrivial polynomial solutions. The proof of statement (b) is analogous. □

Thus for large classes of cubic curves one has $\mathcal{F}_f(d) = \mathcal{F}_f^0(d)$ for any d. It would be interesting to know whether there exist curves and exponents with $\dim\left(\mathcal{F}_f(d)/\mathcal{F}_f^0(d)\right) = 1$.

6. Several factors, all curves are graphs

We now consider a scenario involving several curves, with a quite simple geometric setting: Given polynomials $p_1(x), \ldots, p_r(x)$ in one variable, we let

$$f_i := y - p_i(x),\ 1 \leq i \leq r; \quad f := f_1 \cdots f_r, \tag{13}$$

thus each curve C_i is the graph of some polynomial in x, and condition (ND1) is automatic. We will continue to assume that condition (ND2) holds. Thus for all $i \neq j$, either $p_j - p_i$ is a nonzero constant or $p_j - p_i$ has no common roots with its derivative; moreover for all pairwise different i, j, k the polynomials $p_j - p_i$ and $p_k - p_i$ have no common root. As a first step, we apply Proposition 8 to the vector fields admitting a given integrating factor. Since the case of inverse polynomial integrating factors is known from Theorem 3, we may assume that not all d_i are positive integers. By Lemma 7 we may furthermore assume that $d_i = 1$ or $\mathrm{Re}\,(d_i) < 1$ for all i when we study the dimension of $\mathcal{F}_f(d_1, \ldots, d_r)/\mathcal{F}_f^0(d_1, \ldots, d_r)$.

Lemma 18. *Let f be as in* (13)*, and let $d_1, \ldots, d_r$ be given, not all of them positive integers.*
(a) *The vector field X lies in $\mathcal{F}_f(d_1, \ldots, d_r)$ if and only if there are nonnegative integers $k_1, \ldots, k_r$, polynomials $b_1(x), \ldots, b_r(x)$ in one variable and a polynomial g such that*

$$X = f_1^{k_1} \cdots f_r^{k_r} \left(\sum_i b_i(x) \frac{f}{f_i} \cdot X_{f_i} \right) + f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g,$$

with

$$\widehat{X} := \sum b_i(x) \frac{f}{f_i} \cdot X_{f_i} \in \mathcal{F}_f(d_1 - k_1, \ldots, d_r - k_r).$$

In addition, $X \in \mathcal{F}_f^0(d_1, \ldots, d_r)$ if and only if $\widehat{X} \in \mathcal{F}_f^0(d_1 - k_1, \ldots, d_r - k_r)$.
(b) *Moreover*

$$X^* = \sum_i a_i(x) \frac{f}{f_i} \cdot X_{f_i} \in \mathcal{F}_f(d_1, \ldots, d_r)$$

if and only if

$$-\sum_i a_i'\cdot\frac{f}{f_i}=\sum_{i,\ell:\,i<\ell}((d_\ell-1)a_i-(d_i-1)a_\ell)\cdot\frac{f}{f_if_\ell}\cdot(p_\ell'-p_i').$$

Proof. Applying Proposition 8 we obtain

$$X=f_1^{k_1}\cdots f_r^{k_r}\widehat{X}+f_1^{d_1}\cdots f_r^{d_r}Z_g$$

and by Lemma 6 we get $\widehat{X}\in\mathcal{F}_f^0(d_1-k_1,\dots,d_r-k_r)$. Additionally, from Proposition 8(a) we have

$$\widehat{X}=\sum b_i(x)\frac{f}{f_i}\cdot X_{f_i}+f\tilde{\tilde{X}}$$

due to $\delta_y(b_i)<1$. Now applying Corollary 9 we may assume $\delta_y\widehat{X}<r-1$. Hence, $\tilde{\tilde{X}}=0$ and so $\widehat{X}$ is as desired.

For part (b), note that $X_{f_i}(f_k)=p_k'-p_i'$ and $X_{f_i}(a_i)=-a_i'$. Straightforward computations show

$$\begin{aligned}X^*(f_\ell)&=\left(\sum_{i:\,i\neq\ell}a_i\cdot\frac{f}{f_if_\ell}\cdot(p_\ell'-p_i')\right)\cdot f_\ell=:K_\ell\cdot f_\ell,\\ \operatorname{div}X^*&=-\sum_i a_i'\cdot\frac{f}{f_i}+\sum_{i,\ell:\,i<\ell}(a_i-a_\ell)\cdot\frac{f}{f_if_\ell}\cdot(p_\ell'-p_i'),\end{aligned}$$

and the integrating factor condition follows. $\square$

It is convenient to abbreviate

$$\theta_i:=d_i-1,\quad 1\le i\le r \tag{14}$$

for the following. We may assume $\operatorname{Re}(\theta_i)\le 0$ for all i when discussing $\mathcal{F}_f/\mathcal{F}_f^0$. The particular incarnation of Theorems 10 and 11 in this section is as follows:

Theorem 19. *One has*

$$X^*:=\sum_i a_i(x)\frac{f}{f_i}\cdot X_{f_i}\in\mathcal{F}_f(d_1,\dots,d_r)$$

if and only if the polynomials a_i satisfy the meromorphic linear system

$$a_i'=\sum_{j:j\neq i}(\theta_ja_i-\theta_ia_j)\cdot\frac{(p_j-p_i)'}{p_j-p_i}. \tag{15}$$

The nonzero constant solution $a_1=\theta_1,\dots,a_r=\theta_r$ of system (15) *yields a scalar multiple of $f_1^{d_1}\cdots f_r^{d_r}Z_1$. In particular*

$$\dim\left(\mathcal{F}_f/\mathcal{F}_f^0\right)\le r-1.$$

Proof. To see necessity, start with the condition in part (b) of Lemma 18 and substitute $y\mapsto p_k(x)$ for fixed k. On the left hand side the only nonzero term is

$$a_k'\cdot\prod_{i\neq k}(p_k-p_i).$$

On the right hand side

$$\frac{f}{f_jf_\ell}(x,\,p_k(x))=0\quad\text{if } k\notin\{j,\,\ell\},$$

and there remains

$$\sum_{\ell:\,\ell>k}(\theta_\ell a_k-\theta_k a_\ell)\prod_{i\neq k}(p_k-p_i)\cdot\frac{(p_k-p_\ell)'}{p_k-p_\ell}+\sum_{j:\,j<k}(\theta_k a_j-\theta_j a_k)\prod_{i\neq k}(p_k-p_i)\cdot\frac{(p_j-p_k)'}{p_k-p_j}$$

as asserted.

As for sufficiency, assume that (15) holds. Then

$$\begin{aligned}-\sum_i a_i'\cdot\frac{f}{f_i} &= \sum_{i,k:\,i\neq k}(\theta_k a_i-\theta_i a_k)\,\frac{(p_k-p_i)'}{p_k-p_i}\frac{f}{f_i}\\ &= \sum_{i,k:\,i<k}(\theta_k a_i-\theta_i a_k)\,\frac{(p_k-p_i)'}{p_k-p_i}\left(\frac{f}{f_i}-\frac{f}{f_k}\right)\\ &= \sum_{i,k:\,i<k}(\theta_k a_i-\theta_i a_k)\,\frac{(p_k-p_i)'}{p_k-p_i}\frac{f}{f_if_k}\,(f_k-f_i)\end{aligned}$$

and this is the assertion, in view of $f_k-f_i=p_i-p_k$. Finally, the solution space of (15) has dimension r, and it contains a nontrivial element of $\mathcal{F}_f^0$. □

We proceed to investigate the constant solutions of (15) in greater detail. The multiples of the constant solution exhibited in Theorem 19 will be called *trivial constant solutions.*

Proposition 20. *Assume that $r>1$ and $\mathrm{Re}(\theta_i)\leq 0$ for all i, but not all $\theta_i=0$. We define the equivalence relation $\sim$ on pairs of indices by $i\sim j$ if and only if p_j-p_i is constant. Then the following hold for the constant solutions of (15).*

- *(a) If i and j are in different equivalence classes and $\theta_i\neq 0$, then $a_j=a_i\cdot\theta_j/\theta_i$ is uniquely determined by a_i.*
- *(b) If there are two equivalence classes I and J such that there is $i\in I$ with $\theta_i\neq 0$, and $j\in J$ with $\theta_j\neq 0$ then every constant solution of (15) is trivial.*
- *(c) There exist nontrivial constant solutions of (15) if and only if there is an equivalence class I such that $|I|>1$ and $\theta_j=0$ for all $j\notin I$. In this case, the constant solutions are given by $(\alpha_1,\dots,\alpha_r)$ with $\alpha_j=0$ for all $j\notin I$ and α_i arbitrary for $i\in I$. These correspond to the vector fields*

$$\sum_{i\in I}\alpha_i\frac{f}{f_i}\cdot X_{f_i}\in\mathcal{F}_f(d_1,\dots,d_r),$$

and one has

$$\dim\Big(\mathcal{F}_f(d_1,\dots,d_r)/\mathcal{F}_f^0(d_1,\dots,d_r)\Big)=|I|-1.$$

Proof. We first prove that $\theta_j a_i-\theta_i a_j=0$ whenever i and j are in different classes. To verify this, note that

$$0=a_i'=\sum_j(\theta_j a_i-\theta_i a_j)\cdot\frac{(p_j-p_i)'}{p_j-p_i}.$$

If p_j-p_i is not constant then any root x_0 of this polynomial will be a pole of $(p_j-p_i)'/(p_j-p_i)$. Due to condition (ND2) x_0 is not a pole of any $(p_k-p_i)'/(p_k-p_i)$ with $k\neq j$. Therefore the identity can hold only if all coefficients corresponding to nonconstant p_j-p_i vanish, and (a) is proven.

Moreover statement (b) follows easily: a_i determines all a_k with $k\notin I$ because of statement (a). In particular a_j is determined by a_i. But a_j in turn determines all a_ℓ

with $\ell \in I$. Thus there is only one solution, up to scalar multiples. A similar argument proves all but the last assertion of (c).
There remains to prove that only the trivial constant solutions of (15) lie in $\mathcal{F}_f^0(d_1,\ldots,d_r)$. Thus assume that there are constants a_i and a polynomial g such that

$$\sum a_i \frac{f}{f_i} \cdot X_{f_i} = f_1^{d_1}\cdots f_r^{d_r} \cdot Z_g.$$

Now from (4) consider the explicit expression

$$f_1^{d_1}\cdots f_r^{d_r} \cdot Z_g = \left(-f_1\cdots f_r g_y + g\sum_i \theta_i \frac{f}{f_i}\right)\frac{\partial}{\partial x} + \left(f_1\cdots f_r \cdot g_x - g\sum_i \theta_i \frac{f}{f_i} p_i'\right)\frac{\partial}{\partial y},$$

and compare degrees with respect to y. Assuming $g = g_0(x)\cdot y^m + \cdots$ with a nonnegative integer m, the leading term for the first entry of $f_1^{d_1}\cdots f_r^{d_r}\cdot Z_g$ will be equal to

$$-\left(m - \sum \theta_i\right) g_0(x) y^{r+m-1} \neq 0,$$

due to the condition on the real parts of the θ_i. On the other hand, the y-degree of the first entry of $\sum a_i f/f_i \cdot X_{f_i}$ is at most equal to $r-1$. Therefore $g = g_0(x)$, and comparing y-degrees of the second entries shows that g is constant. Thus the remaining assertion in (c) is proven. □

Remark. It is worth looking at the particular case when all $p_j - p_i$ are constant; thus (15) has only constant solutions. Up to an automorphism of the affine plane (given by $x \mapsto x$ and $y \mapsto y - p_1(x)$) we may assume that all p_i are constant. Thus the curves are just parallel straight lines, there are no singular points in the affine plane, and there is only one (albeit quite degenerate) singular point at infinity. Due to Proposition 20(c) the corresponding vector fields are precisely those of the form $q(y)\partial/\partial x$ with $\delta(q) < r$. Given the hypotheses of Proposition 20, one has $\mathcal{F}_f(d_1,\ldots,d_r) \neq \mathcal{F}_f^0(d_1,\ldots,d_r)$ for all $r \geq 2$. Moreover it is easy to construct vector fields that are not integrable in an elementary manner, by adding suitable vector fields of the form $f_1^{d_1}\cdots f_r^{d_r}\cdot Z_g$; see Prelle and Singer [13], and also Singer [17].

From now on we assume that some $p_j - p_i$ is not constant. We turn to analyzing the singular points of system (15). As it turns out, this system admits only weak singularities on the complex projective plane.

Lemma 21. *The following statements hold.*

(a) *Any finite singular point x_0 is a zero of $p_i - p_j$ for exactly one pair of indices. The eigenvalues of A_{-1}, the coefficient matrix of $(x-x_0)^{-1}$, are 0 with multiplicity $r-1$ and $\theta_i + \theta_j$ with multiplicity one.*

(b) *Let $m_{ij} := \delta(p_i - p_j)$ for $i \neq j$. Then the coefficient matrix of v^{-1} for the system at infinity is equal to*

$$A_\infty = \begin{pmatrix} -\left(\sum_{j\neq 1} m_{1j}\theta_j\right) & m_{12}\theta_1 & \cdots & m_{1r}\theta_1 \\ m_{12}\theta_2 & -\left(\sum_{j\neq 2} m_{2j}\theta_j\right) & \cdots & m_{2r}\theta_2 \\ \vdots & \vdots & & \vdots \\ m_{1r}\theta_r & m_{2r}\theta_r & \cdots & -\left(\sum_{j\neq r} m_{rj}\theta_j\right) \end{pmatrix}.$$

(c) *One eigenvalue of A_∞ equals 0, corresponding to the trivial constant solutions in Theorem 19.*

Proof. Fix i and j such that $p_i - p_j$ is nonconstant and denote the zeros by $\beta_1, \ldots, \beta_s$ (each is of multiplicity one). Then

$$\frac{(p_i - p_j)'}{p_i - p_j} = \frac{1}{x - \beta_1} + \cdots + \frac{1}{x - \beta_s}$$

and thus the finite singular points of (15) are just the zeros of the $p_k - p_\ell$. Evaluation at some β_n shows that A_{-1} is a matrix that has nonzero entries only at positions (i,i), (i,j), (j,i), (j,j) and the entries are (in the same order) θ_i, $-\theta_j$, $-\theta_i$, θ_j. This shows part (a). Moreover

$$-v^{-2}\frac{(p_i - p_j)'}{p_i - p_j}(v^{-1}) = -v^{-1} \cdot \left(\frac{1}{1 - \beta_1 v} + \cdots + \frac{1}{1 - \beta_s v}\right).$$

Thus the contribution of the term in brackets to the coefficient of v^{-1} is equal to $s = \delta(p_i - p_j)$, and (b) follows. As for (c) observe that the rows of A_∞ add up to 0. □

Remark. The m_{ij} are related. If i, j and k are distinct then $(p_i - p_j) + (p_j - p_k) + (p_k - p_i) = 0$ and therefore two of the degrees are equal and not smaller than the third.

A natural application of Lemma 21 is to investigate the dimension of $\mathcal{F}_f/\mathcal{F}_f^0$. Sufficient information on the eigenvalues of A_∞ can be obtained in many cases.

Lemma 22. *The following statements hold.*

(a) If $\delta(p_i - p_j) = m > 0$ for all distinct i and j (e.g., when all p_i have the same degree but pairwise distinct leading coefficients), then in case $\theta_1 + \cdots + \theta_r \neq 0$ the eigenvalues of A_∞ are 0 (simple) and $-m(\theta_1 + \cdots + \theta_r)$ (with algebraic and geometric multiplicity $r - 1$).

(b) If $\delta(p_1) > \delta(p_2) > \cdots > \delta(p_r)$ then the eigenvalues of A_∞ are 0 and

$$\begin{aligned}
&-\left(\delta(p_1) \cdot \textstyle\sum_{j=1}^r \theta_j\right), \\
&-\left(\delta(p_1) \cdot \theta_1 + \delta(p_2) \cdot \textstyle\sum_{j=2}^r \theta_j\right), \\
&-\left(\delta(p_1) \cdot \theta_1 + \delta(p_2) \cdot \theta_2 + \theta_3 \cdot \textstyle\sum_{j=3}^r \theta_j\right), \\
&\vdots \\
&-\left(\delta(p_1) \cdot \theta_1 + \cdots + \delta(p_{r-2}) \cdot \theta_{r-2} + \delta(p_{r-1}) \cdot \textstyle\sum_{j=r-1}^r \theta_j\right).
\end{aligned}$$

(c) If all θ_i are real and nonpositive, then all eigenvalues of A_∞ have non negative real parts, and any eigenvalue with real part 0 is zero.

Proof. The proofs of (a) and (b) are elementary and will be omitted.

As for part (c) let B be the transpose of A_∞. We will show that B is an M-matrix in the sense of Berman and Plemmons [1], Ch. 6. According to [1] (Lemma 4.1. of Ch.6, Sec. 4) it suffices to show that for every $\epsilon > 0$ the matrix $B + \epsilon I$ (with I the unit matrix) is an invertible M-matrix. But $B + \epsilon I$ has positive diagonal elements, nonpositive entries outside the diagonal, and for every row the sum of its elements is positive. Hence, from [1] (Theorem 2.3 of Ch.6, Sec. 2, criterion (M_{35}), with $D = I$), we have that $B + \epsilon I$ is an invertible M-matrix.

Since B is an M-matrix, all eigenvalues of B have nonnegative real parts, and any eigenvalue with real part 0 is zero. This also holds for the eigenvalues of the transpose of B, hence the assertion follows. □

Proposition 23. *Let the polynomials p_i be such that the hypothesis of Lemma 22 (a) or (b) holds. If not all exponents $d_1, \ldots, d_r$ are positive integers, then $\mathcal{F}_f(d_1, \ldots, d_r) = \mathcal{F}_f^0(d_1, \ldots, d_r)$.*

Proof. One may use Lemma 7 to assume that all θ_i have real parts ≤ 0. Since the only nonpositive integer eigenvalue of A_∞ is 0, with multiplicity 1, the only polynomial solutions of (15) are the trivial constant solutions. Now Theorem 19 shows the assertion. □

Theorem 24. *Let $d_1, \ldots, d_r$ be arbitrary nonzero real constants, not all of them positive integers. Then the following statements hold for the equivalence relation introduced in Proposition 20.*

(a) *If every equivalence class contains just one element, then*
$$\mathcal{F}_f(d_1, \ldots, d_r) = \mathcal{F}_f^0(d_1, \ldots, d_r).$$
(b) *If for every equivalence class I with $|I| > 1$ there exists an index $j \notin I$ such that d_j is not a positive integer, then*
$$\mathcal{F}_f(d_1, \ldots, d_r) = \mathcal{F}_f^0(d_1, \ldots, d_r).$$
(c) *If there exists an equivalence class I with $|I| > 1$ such that d_j is a positive integer for every $j \notin I$ then*
$$\mathcal{F}_f(d_1, \ldots, d_r) \neq \mathcal{F}_f^0(d_1, \ldots, d_r).$$
The elements of $\mathcal{F}_f(d_1, \ldots, d_r)$ have the explicit form
$$\prod_{i\in I} f_i^{(e_i-1)} \prod_{j\notin I} f_j^{(d_j-1)} \cdot \left(\sum_{i\in I} \alpha_i \frac{f}{f_i} X_{f_i} \right) + f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g$$
with integers $e_i \geq 1$, constants α_i and some polynomial g.

Proof. We may assume that all $\theta_i \leq 0$, due to Lemma 7. Then Lemma 22 shows that the only nonpositive integer eigenvalue of A_∞ is 0 (possibly with multiplicity > 1). Therefore, every polynomial solution of (15) is constant, and the assertion follows with Theorem 19 and Proposition 20. □

It seems likely that the conclusion of Theorem 24 is always true, not only for real exponents. To prove this, one would need more precise information about the eigenvalues of A_∞. In any case, even allowing degeneracies for the geometric setting at infinity, one still obtains $\mathcal{F}_f(d_1, \ldots, d_r) = \mathcal{F}_f^0(d_1, \ldots, d_r)$ in most scenarios. It seems reasonable to conjecture that $\mathcal{F}_f \neq \mathcal{F}_f^0$ will always be an exceptional case.

7. Degenerate geometric settings

In this final section we briefly discuss the setting when the geometric nondegeneracy conditions (ND1) and (ND2) no longer hold. We will only assume from now on that the curves $f_i = 0$ are irreducible and pairwise relatively prime.

As it turns out, variants of the results from the previous sections continue to hold in this more general context if we consider the space of vector fields that are polynomial in y and rational in x. Letting $f = y^n + \cdots = f_1 \cdots f_r$ be a polynomial as before, and given nonzero constants d_i, we now consider the subspaces
$$\mathcal{F}_f^*(d_1, \ldots, d_r) \quad \text{and} \quad \mathcal{F}_f^{*0}(d_1, \ldots, d_r)$$

of this linear space. The former subspace consists of all vector fields admitting the integrating factor (3), while the latter consists of all vector fields of the special form

$$f_1^{d_1}\cdots f_r^{d_r}\cdot Z_g$$

with $g\in\mathbb{C}(x)[y]$. Mutatis mutandis, Lemma 2 obviously holds for these vector fields.

Our starting point, in lieu of Theorem 1, is now as follows:

Lemma 25. *If a polynomial vector field X admits f then*

$$X=\sum_i a_i\frac{f}{f_i}\cdot X_{f_i}+f\cdot\widetilde{X}, \tag{16}$$

where the a_i lie in $\mathbb{C}(x)[y]$, and $\widetilde{X}$ is rational in x and polynomial in y. More precisely, there is $h\in\mathbb{C}[x]$, depending only on f, such that all a_i can be chosen in $\mathbb{C}\left[x,\,y\right]\left[\frac{1}{h}\right]$.

Proof. The proof follows essentially from Theorem 6.12 of [8]. (See also [5], Theorem 3.4.) In this theorem it was shown that for any polynomial vector field admitting f, and for every element h of a certain ideal $\mathfrak{I}$ defined via f, there exist polynomials $\hat{a}_i$ and a polynomial vector field $\widehat{X}$ such that

$$h\cdot X=\sum_i \hat{a}_i\frac{f}{f_i}\cdot X_{f_i}+f\cdot\widehat{X}.$$

By definition the ideal $\mathfrak{I}$ contains the polynomials

$$f,\,\frac{f}{f_i}\cdot f_{iy}\quad(1\le i\le n)$$

(see [8] or [5]) which do not have a common factor in view of $f=y^n+\cdots$. Hence the zero set of $\mathfrak{I}$ is finite, and there exists a polynomial $h(x)\in\mathfrak{I}\cap\mathbb{C}[x]$. $\square$

Now Lemmas 6 and 7, Proposition 8 and Corollary 9 carry over to the present setting with obvious modifications. (In order to apply Lemma 25, one may have to multiply by powers of h, but this is harmless.) If all d_i are positive integers then we end up with the case $d_1=\cdots=d_r=1$, for which [5], Proposition 4.3 shows that

$$X=\sum_i \beta_i\frac{f}{f_i}\cdot X_{f_i}+f\cdot X_g \tag{17}$$

with constants β_i and some rational function g. (This may be seen as a counterpart to Theorem 3.) For the remaining exponents we can reduce to the case of vector fields X with $\delta_y(X)\le n-1$.

Theorem 26. *If some d_ℓ is not a positive integer then*

$$\dim\left(\mathcal{F}_f^*(d_1,\ldots,d_r)/\mathcal{F}_f^{*0}(d_1,\ldots,d_r)\right)\le n-1.$$

Proof. Let $X\in\mathcal{F}_f^*(d_1,\ldots,d_r)$, and $\delta_y(X)\le n-1$ with no loss of generality.

We start with some preliminary considerations. Let $\mathbb{K}\supseteq\mathbb{C}(x)$ denote a splitting field of $f\in\mathbb{C}(x)[y]$; thus we have

$$f=(y-q_1(x))\cdots(y-q_n(x))=:g_1\cdots g_n$$

over $\mathbb{K}$, with algebraic functions $q_1,\ldots,q_n$. Since f has no multiple prime factors over $\mathbb{C}(x)$, and we are in characteristic zero, the q_i are pairwise distinct, whence

$f/g_1, \dots, f/g_n$ are relatively prime. By Lagrange interpolation, every $s \in \mathbb{K}[y]$ with degree $\leq n-1$ has a unique representation

$$s = \sum_i c_i \cdot \frac{f}{g_i}$$

with $c_i \in \mathbb{K}$. We note that

$$X_{g_i} = \begin{pmatrix} -1 \\ -q_i'(x) \end{pmatrix}$$

and so the vector field X can be written in the form

$$X = \sum_i \hat{a}_i(x) \frac{f}{g_i} \cdot X_{g_i} + \sum_i \hat{b}_i(x) \frac{f}{g_i} \frac{\partial}{\partial y}. \tag{18}$$

If X admits f then $X(g_k)$ is a multiple of g_k for each k, thus

$$0 = X(g_k)\,(q_k) = \sum_i \hat{b}_i(x) \frac{f}{g_i}(q_k) = \hat{b}_k(x) \frac{f}{g_k}(q_k)$$

and therefore $\hat{b}_k = 0$. So from (18) we obtain

$$X = \sum_i \hat{a}_i(x) \frac{f}{g_i} \cdot X_{g_i} \tag{19}$$

similar to the setting of Lemma 18 and Theorem 19. Now an imitation of the proof of Theorem 19 shows that the $\hat{a}_i$ satisfy the differential equation

$$\hat{a}_i' = \sum_{j:j\neq i} (\theta_j \hat{a}_i - \theta_i \hat{a}_j) \cdot \frac{(q_j - q_i)'}{q_j - q_i}, \tag{20}$$

analogous to equation (15). Here $\theta_i + 1$ equals the exponent of the factor f_ℓ of f which is divided by g_i. As in Theorem 19, the nonzero constant solution $\hat{a}_i = \theta_i$ $(1 \leq i \leq n)$ yields a scalar multiple of $f_1^{d_1} \cdots f_r^{d_r} Z_1 \in \mathcal{F}_f^{*0}$. In summary, the dimension of the quotient space is bounded by $n - 1$. □

For the case of polynomial vector fields, which is of principal interest to us, this theorem provides a strong structural result, but some questions remain open. One problem in this approach is to identify the polynomial vector fields in $\mathcal{F}_f^{*0}(d_1, \dots, d_r)$; another problem is to utilize the differential equation (20) beyond dimension estimates. Our final result illustrates that the differential equation can be useful for finding vector fields in some cases. We consider the case of a "degenerate hyperelliptic curve" (compare Theorem 13 with $m = 2$).

Proposition 27. *For $f = y^2 - p(x)$ the dimension of $\mathcal{F}_f^*(d)/\mathcal{F}_f^{*0}(d)$ is at most 1. Furthermore, if d is not a positive integer then the most general first integral and $X \in \mathcal{F}_f^*(d)$ with $\delta_y(X) = n - 1$ is the sum of a Darboux function and a term of the form $\int^{y/\sqrt{p}} (Z^2 - 1)^{-d} \, \mathrm{d}Z$.*

Proof. It is straight forward to work back through the reduction process to show that the first integral of the original system is of a similar form. The first part is just Theorem 26 if d is not a positive integer, and a consequence of [5], Proposition 4.3 otherwise. For the second part we use that X has a representation (19), and from (20) we obtain

$$\hat{a}_1' = (d-1)(\hat{a}_1 - \hat{a}_2)\frac{p'}{2p}, \qquad \hat{a}_2' = (d-1)(\hat{a}_2 - \hat{a}_1)\frac{p'}{2p},$$

noting $q_1 = \sqrt{p} = -q_2$. Thus $\hat{a}_1 + \hat{a}_2$ is a constant, which we may take to be 0 in view of the last argument in the proof of Theorem 26. There remains

$$\hat{a}_1' = 2(d-1)\hat{a}_1 \frac{p'}{ph},$$

which gives

$$\hat{a}_1 = \ell p^{d-1}, \qquad \hat{a}_2 = -\ell p^{d-1}$$

for some constant ℓ. Going back to (19), we find

$$\frac{1}{(y^2 - p(x))^d} \cdot X = \frac{\ell p^{d-1}}{(y^2 - p(x))^{d-1}} \cdot X_w \tag{21}$$

with

$$w := \ln\left(\frac{y - \sqrt{p}}{y + \sqrt{p}}\right).$$

This vector field admits the first integral

$$\frac{\ell}{2} \int^{y/\sqrt{p}} \frac{\mathrm{d}Z}{(Z^2 - 1)^d}.$$

□

We note that this candidate for an exceptional vector field can only work when p^{d-1} is a rational function.

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