Invariant sets forced by symmetry

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Dedicated to Tudor Ratiu on the occasion of his sixtieth birthday.

Abstract

Given a linear (algebraic) group G acting on real or complex n-space, we determine all the common invariant sets of G-symmetric vector fields. It turns out that the investigation of certain algebraic varieties is sufficient to characterize these invariant sets forced by symmetry. Toral, compact and reductive groups are discussed in some detail, and examples, including a Couette-Taylor system, are presented.

1 Introduction and preliminaries

In the present paper we discuss ordinary differential equations that are symmetric with respect to some linear group G. The aim is to determine and characterize those invariant sets that are common to all G-symmetric differential equations.

We first fix notation and terminology. Let an ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

be given on an open subset U of $\mathbb{K}^n$, with $\mathbb{K}$ standing for $\mathbb{R}$ or $\mathbb{C}$. The independent variable t will always be assumed real. Our focus will be on polynomial vector fields (and $U = \mathbb{K}^n$). Extensions to analytic and formal power series vector fields are straightforward. We will denote the solution of the initial value problem $\dot{x} = f(x)$, $\quad x(0) = y$ (near $t = 0$) by $\Phi(t, y) = \Phi_f(t, y)$ and refer to Φ_f

as the local flow of f. For fixed y there is a maximal interval I_y of existence for $\Phi(t,y)$, and the set of all $\Phi(t,y)$, with $y \in I_y$, is called the trajectory through y.

We will answer the following principal question: Let $G \subseteq GL(n,\mathbb{K})$ be a linear group. What invariant sets do necessarily exist for any G-symmetric polynomial differential equation; in other words, what invariant sets are forced by the symmetry group?

This question is of interest in its own right, but also a starting point for further investigations of symmetric vector fields. For compact groups and C^∞ vector fields the question was essentially answered e.g. in Field [9], and plays an important role for instance in Krupa's [16] investigation of bifurcations of relative equilibria.

We consider an arbitrary linear algebraic group G acting linearly on n-space. Requiring linearity of the group action does not impose an essential restriction for compact groups, cf. e.g. Guillemin and Sternberg [14], or for semisimple groups, cf. e.g. Kushnirenko [17], in the local analytic setting. Since we are primarily interested in polynomial or analytic vector fields (in view of computations), there is also no loss of generality in discussing only algebraic groups. We note that one drawback seems unavoidable when discussing this broad class of groups: In contrast to compact group actions, a general extension to infinite dimensional systems seems impossible. (One should mention that certain noncompact group actions for infinite dimensional systems have been discussed, and successfully used in applications, by Fiedler et al. [10], Golubitsky et al. [13], and others. But these results were based on additional assumptions, for instance compactness of isotropy groups.)

We use some elementary notions and results from Commutative Algebra in our approach. Our results include a precise characterization of the minimal invariant sets which are common to all G-symmetric vector fields. As it turns out, the Zariski closure of such a minimal invariant set is a vector subspace, and the investigation of common invariant sets amounts to the investigation of certain algebraic varieties (Theorems 1 and 2). Moreover, if an irreducible subvariety of such a variety is invariant for all G-symmetric vector fields and is not a linear space then there exists a common rational first integral for the G-symmetric vector fields on this subvariety (Theorem 3). We discuss toral groups, compact groups and reductive groups in some detail, and provide descriptions of the subspaces spanned by minimal common invariant sets in Propositions 3 and 4, and Theorems 4 and 6. At the end of the paper we present examples, including a Couette-Taylor system and some low-dimensional representations of $SL(2)$.

The results of the present paper form a basis for future work, with two problems to be addressed in particular. First, there exist group representations for which orbit space reduction via group invariants (see e.g. the survey by Chossat [5]) is not applicable (when the invariant algebra is not finitely generated) or not feasible (e.g. when even minimal sets of generators are very large). Here, our results provide a starting point for alternative reduction approaches, e.g. via rational functions. Second, the qualitative behavior of symmetric differential equations on invariant sets, in particular minimal ones, is of special interest, in particular for low-dimensional invariant subsets of high-dimensional representations. In Proposition 10 we provide a first result (assuming no a priori knowledge about the structure of symmetric vector fields on the whole space) concerning the extension of polynomial vector fields on minimal common invariants sets to rational symmetric vector fields on the whole space. This will be taken up in forthcoming work.

2 General properties and known results

We will assume that equation (1) has polynomial right-hand side and is symmetric with respect to a linear group $G \subseteq GL(n, \mathbb{K})$, thus the identities

$$Tf(x) = f(Tx) \tag{2}$$

hold for all $T \in G$. In view of polynomiality (or, more generally, analyticity) of f we may take G to be an algebraic group defined over $\mathbb{K}$, with identity component G^0. Denote the Lie algebra of G by $\mathcal{L}$. We do not require finite generation of the invariant algebra of G, or of any related modules.

Polynomial differential equations (1) correspond to derivations of the algebra $\mathbb{K}[x_1, \ldots, x_n]$, via assigning to $f = (f_1, \ldots, f_n)^t$ the associated Lie derivative

$$L_f = \sum f_i \frac{\partial}{\partial x_i}.$$

Therefore we will identify f with this element of $\mathrm{Der}(\mathbb{K}[x_1, \ldots, x_n])$, and also speak of f as a vector field. To G-symmetric differential equations there correspond G-invariant derivations, i.e., derivations which commute with the group action on polynomials. The set of these will be denoted by

$$\mathcal{D}_G = \mathrm{Der}_G(\mathbb{K}[x_1, \ldots, x_n]).$$

Let us record a few elementary properties, which are easy to prove directly from the symmetry criterion (2). The first two depend essentially on linearity of the group action.

Lemma 1. *Let f and g be G-symmetric vector fields. Then the composite $f \circ g$, the left-symmetric product defined by $(f \bullet g)(x) = Dg(x)f(x)$ and the Lie bracket $[f, g] = f \bullet g - g \bullet f$ are G-symmetric. In particular the G-symmetric vector fields form a Lie algebra.*

At this point it may be appropriate to remark on terminology. In invariant theory, maps satisfying (2) are usually called covariant. In the scenario that we consider (linear group actions on vector spaces) some criteria for maps and vector fields coincide, and thus most results could be stated for covariant maps as well as for symmetric vector fields. The actual notations and notions chosen here reflect the intended applications.

Recall that a subset Y of $\mathbb{K}^n$ is called *invariant* for the polynomial differential equation (1) if for every $y \in Y$ the whole trajectory through y is contained in Y.

It is obvious that set operations (union, intersection, complement) on invariant sets produce invariant sets. Moreover, due to continuous dependence the closure, interior and boundary of an invariant set (with respect to the norm topology) are also invariant. We are interested in sets that are invariant with respect to all G-symmetric vector fields.

Definition 1. (a) *A set $Y \subseteq \mathbb{K}^n$ is called* $\mathrm{Der}_G(\mathbb{K}[x_1, \ldots, x_n])$*-invariant (or $\mathcal{D}_G$-invariant) if it is an invariant set for every G-symmetric differential equation.*
(b) *Given a $\mathcal{D}_G$-invariant set Y and $v \in Y$, we call Y* minimal with respect to *v if $v \in Z$ for some $\mathcal{D}_G$-invariant set Z implies $Y \subseteq Z$.*

Lemma 2. (a) *Unions, intersections and complements of $\mathcal{D}_G$-invariant sets are $\mathcal{D}_G$-invariant. In particular, for every $v \in \mathbb{K}^n$ there exists a minimal $\mathcal{D}_G$-invariant set containing v.*
(b) *The closure, boundary and interior (with respect to the norm topology) of a $\mathcal{D}_G$-invariant set is $\mathcal{D}_G$-invariant.*
(c) *Every connected component (with respect to the norm topology) of a $\mathcal{D}_G$-invariant set is $\mathcal{D}_G$-invariant.*

Proof. Parts (a) and (b) are immediate from the remarks above. To prove part (c), note that for any $v \in \mathbb{K}^n$, the union of all the trajectories through v of G-symmetric differential equations is connected. □

We record a few more simple and mostly well-known, but useful observations.

Lemma 3. *Let H be a (closed) subgroup of G. Then every $\mathcal{D}_H$-invariant set Y is also $\mathcal{D}_G$-invariant.*

Lemma 4. (a) *Let $\dot{x} = f(x)$ admit the symmetry group G. Then for any $T \in G$ and $y \in \mathbb{R}^n$ one has*

$$T\Phi_f(t, y) = \Phi_f(t, Ty) \quad \text{for all } t.$$

(b) *If Y is invariant for some G-symmetric vector field and $T \in G$ then TY is invariant.*

Proof. Since T is a symmetry, $T\Phi_f(t, y)$ is a solution of the differential equation. At $t = 0$ this solution attains the value Ty. The second assertion is an obvious consequence. □

Proposition 1. (a) *All points on a trajectory of $\dot{x} = f(x)$ have the same isotropy subgroup.*
(b) *Given any (closed) subgroup H of G, the fixed point subspace*

$$\mathrm{Fix}(H) := \{z : Tz = z \text{ for all } T \in H\}$$

of H is $\mathcal{D}_G$-invariant.

Proof. Both assertions are direct consequences of Lemma 4; for the first assertion note that flows can be reversed.

□

Corollary 1. *Let G be given, and let $v \in \mathbb{K}^n$. Then the minimal $\mathcal{D}_G$-invariant set with respect to v is contained in the fixed point subspace of the isotropy group G_v. (In particular one has $f(v) \in \mathrm{Fix}(G_v)$ for all G-symmetric vector fields f.)*

3 Rank considerations

We first recall some familiar invariance criteria. (See, e.g. [23]. For the reader's convenience a proof is given in the Appendix.)

Lemma 5. (a) *Given equation (1), let $\psi_1, \ldots, \psi_r \in \mathbb{K}[x_1, \ldots, x_n]$. If there are $\mu_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$ such that*

$$L_f(\psi_j) = \sum_k \mu_{jk}\psi_k, \quad 1 \leq j \leq r \tag{3}$$

then the set Y of common zeros of the ψ_j is invariant for $\dot{x} = f(x)$.
(b) *A vector subspace W of $\mathbb{K}^n$ is invariant for $\dot{x} = f(x)$ if and only if $f(w) \in W$ for all $w \in W$.*

Next we introduce a class of distinguished invariant sets.

Definition 2. *Let $G \subseteq GL(n, \mathbb{K})$.*
(a) *For $v \in \mathbb{K}^n$ denote by*

$$\epsilon_v : \mathcal{D}_G \to \mathbb{K}^n, \quad f \mapsto f(v)$$

the evaluation map.
(b) *For a nonnegative integer s let*

$$Z_s = Z_s(G) := \{y \in \mathbb{K}^n : \dim(\epsilon_y(\mathcal{D}_G)) \leq s\}$$

and $Z^*_{s+1} := Z_{s+1} \setminus Z_s$.

Remarks. (a) The set $\epsilon_v(\mathcal{D}_G)$ is a vector subspace and equal to the set of all $f(v)$, f symmetric with respect to G. The notation ϵ_v is taken from Lehrer and Springer [18] and Panyushev [20], who discuss covariant maps. The symmetry condition shows that $\epsilon_v(\mathcal{D}_G) \subseteq \mathrm{Fix}(G_v)$. Since $h(x) = x$ defines a G-symmetric vector field, one always has $v \in \epsilon_v(\mathcal{D}_G)$.
(b) Obviously $y \in Z_s$ if and only if for all $q \geq 1$ and all G-symmetric vector fields $g_1, \ldots, g_q$ the rank of $(g_1(y), \cdots, g_q(y))$ is not greater than s. Since the points satisfying this rank condition can be described as common zero sets of suitable determinants, one sees that every Z_s is Zariski closed. Moreover, $y \in Z_s$ satisfies $y \in Z^*_s$ if and only if there exist G-symmetric vector fields $h_1, \ldots, h_s$ such that $h_1(y), \ldots, h_s(y)$ are linearly independent in $\mathbb{K}^n$.

Theorem 1. (a) *For every $s \geq 0$ the sets Z_s and Z^*_{s+1} are invariant for every G-symmetric vector field. Moreover, the sets Z_s and Z^*_{s+1} are also invariant with respect to the group action.*
(b) *For every $y \in \mathbb{K}^n$ the subspace $\epsilon_y(\mathcal{D}_G)$ is invariant for every G-symmetric vector field.*

Proof. We first prove part (b). For any G-symmetric f and any $w \in \epsilon_y(\mathcal{D}_G)$ it suffices to show that $f(w) \in \epsilon_y(\mathcal{D}_G)$, due to Lemma 5. But there is a $g \in \mathcal{D}_G$ such that $w = g(y)$, and by Lemma 1 one has $f(w) = (f \circ g)(y) \in \epsilon_y(\mathcal{D}_G)$. Now the $\mathcal{D}_G$-invariance of Z_s and Z^*_{s+1} follows by Lemma 2. The group invariance is straightforward from the definitions: If $g_1(y), \ldots, g_p(y)$ are linearly (in-)dependent then so are $g_1(Ty), \ldots, g_p(Ty)$ for all $T \in G$, due to $g_i(Ty) = Tg_i(y)$. □

Remark. Essentially, part (a) remains true for (nonlinear) algebraic group actions on affine varieties Y and G-symmetric vector fields on Y: For every s the set Z_s is $\mathcal{D}_G$-invariant. Here, ϵ_v should be viewed as a map to the tangent space to Y at v. The proof works with $(s+1) \times (s+1)$ minors of matrices that have columns built from the module elements $g \in \mathcal{D}_G$, and uses the invariance criterion (3) for their common zero set, viz. Z_s, noting that the ideal generated by all such $(s+1) \times (s+1)$ minors is finitely generated by Hilbert's "Basissatz". The argument is an obvious modification of the proof of Theorem 3.1 in [23]. In this sense, part (a) does not depend on the linearity of the action of G.

The following observation may be seen as a weak finiteness result for G-symmetric vector fields, for arbitrary G. Moreover, it is of interest even if the invariant algebra of G and the module of symmetric vector fields are finitely generated.

Proposition 2. *There exist finitely many G-symmetric vector fields $h_1, \ldots, h_q$ such that for every v the space $\epsilon_v(\mathcal{D}_G)$ is spanned by $h_1(v), \ldots, h_q(v)$. Given any G-symmetric f and $v \in Z^*_s$ there exist rational functions σ_j which are defined and G-invariant in a Zariski-open neighborhood of v in Z_s such that*

$$f = \sum_{j=1}^{q} \sigma_j h_j$$

in this neighborhood.

Proof. Given $y \in Z^*_s$, by Cramer's rule there exist symmetric vector fields $g_{y,1}, \ldots, g_{y,s}$ and a Zariski-open neighborhood U_y such that for every $v \in U_y$ the space spanned by $g_1(v), \ldots, g_s(v)$ has dimension s. By quasi-compactness of the Zariski topology, finitely many of these neighborhoods suffice to cover every Z^*_s. The collection of all the associated vector fields satisfies the desired condition. For the last assertion, choose a subset I of $\{1, \ldots, q\}$ such that the $h_j(v)$ with $j \in I$ form a basis of $\epsilon_v(\mathcal{D}_G)$, and set $\sigma_j = 0$ for $j \notin I$. Since the coefficients of the h_j are then uniquely determined, and f and all h_j are G-symmetric, the σ_j are G-invariant. □

Remark. This Proposition suggests to think of the $\mathcal{D}_G$-invariant sets Z_s as the Zariski-closed invariant sets of "general" G-symmetric polynomial vector fields. For instance, there is an open and dense subset $U^* \subset \mathbb{K}^q$ such that for every v the vector fields $\sum \alpha_j h_j(v)$, with $(\alpha_1, \ldots, \alpha_q) \in U^*$, span $\epsilon_v(\mathcal{D}_G)$.

Given a (closed) subgroup H of G and $v \in \mathbb{K}^n$, consider the space $\epsilon_v(\mathcal{D}_H)$ which is spanned by all $g(v)$ with H-symmetric g. We note an elementary property (following from the definitions and Lemma 3) for later use.

Lemma 6. *The space $\epsilon_v(\mathcal{D}_H)$ is $\mathcal{D}_G$-invariant and contains $\epsilon_v(\mathcal{D}_G)$.*

Thus one has a descending chain

$$\mathbb{K}^n = Z_n \supseteq Z_{n-1} \supseteq \cdots \supseteq Z_0$$

of $\mathcal{D}_G$-invariant sets, and $y \in Z_s^*$ is contained in the s-dimensional $\mathcal{D}_G$-invariant vector subspace $\epsilon_y(\mathcal{D}_G)$.

Theorem 2. *Let $y \in Z_r^*$. Then the minimal $\mathcal{D}_G$-invariant set with respect to y is the connected component, in the norm topology, of y in $\epsilon_y(\mathcal{D}_G) \setminus Z_{r-1}$. Moreover, $\epsilon_y(\mathcal{D}_G)$ is the smallest Zariski-closed $\mathcal{D}_G$-invariant set which contains y.*

Proof. It is obviously sufficient to prove the first assertion. Thus let Y be the minimal $\mathcal{D}_G$-invariant set with respect to y.
(i) We already know that $Y \subseteq \epsilon_y(\mathcal{D}_G)$; see Theorem 1. Moreover $Y \subseteq \epsilon_y(\mathcal{D}_G) \setminus Z_{r-1}$, since the latter set is $\mathcal{D}_G$-invariant and contains y. Since every connected component of a $\mathcal{D}_G$-invariant set is itself $\mathcal{D}_G$-invariant (Lemma 2), Y is connected. Let Y^* be the connected component of $\epsilon_y(\mathcal{D}_G) \setminus Z_{r-1}$ which contains Y.
(ii) Let $v \in Y$. Then there exists a neighborhood of v in $\epsilon_y(\mathcal{D}_G) \setminus Z_{r-1}$ which is also contained in Y. To prove this, let $g_1, \ldots, g_r$ be G-symmetric such that the $g_i(v)$ are linearly independent. Consider the analytic map

$$\Psi : \quad (\alpha_1, \ldots, \alpha_r) \mapsto \Phi_{\alpha_1 g_1 + \cdots + \alpha_r g_r}(1, v)$$

which maps some connected neighborhood U of $0 \in \mathbb{K}^r$ to $\epsilon_y(\mathcal{D}_G)$. From the series expansion

$$\Psi(\alpha_1, \ldots, \alpha_r) = v + \alpha_1 g_1(v) + \cdots + \alpha_r g_r(v) + o(|\alpha_1|, \ldots, |\alpha_s|)$$

one sees that the derivative

$$D\Psi(0, \ldots, 0) = (g_1(v), \ldots, g_r(v))$$

has rank r, and therefore induces an analytic diffeomorphism from some (connected) neighborhood of 0 in $\mathbb{K}^r$ to a neighborhood of v in $\epsilon_y(\mathcal{D}_G)$. By Lemma 2 this neighborhood is contained in Y.
(iii) According to (ii), Y is relatively open in Y^*. But the same argument shows relative openness of the complement $Y^* \setminus Y$, and thus $Y = Y^*$ since Y^* is connected. □

Lemma 7. *Let $G \subseteq GL(n,\mathbb{R})$ and $y \in \mathbb{R}^n$, and consider*

$$\begin{array}{rcl} \epsilon_y(\mathcal{D}_G^{\mathbb{R}}) & = & \{f(y);\, f:\mathbb{R}^n \to \mathbb{R}^n \ \textit{is } G-\textit{symmetric}\}\,, \\ \epsilon_y(\mathcal{D}_G^{\mathbb{C}}) & = & \{h(y);\, h:\mathbb{C}^n \to \mathbb{C}^n \ \textit{is } G-\textit{symmetric}\}\,. \end{array}$$

Then the complex space $\epsilon_y(\mathcal{D}_G^{\mathbb{C}})$ is the complexification of the real space $\epsilon_y(\mathcal{D}_G^{\mathbb{R}})$. In particular the respective dimensions are equal.

Proof. Given any complex G-symmetric vector field, its conjugate is also G-symmetric. Therefore $\epsilon_y(\mathcal{D}_G^{\mathbb{C}})$ is equal to its complex conjugate subspace, which implies that if $\epsilon_y(\mathcal{D}_G^{\mathbb{C}})$ is spanned by $w_1,\ldots,w_m$ then it is also spanned by the real and imaginary parts of the w_j. The assertion follows. □

Example. Consider the "diagonal" action of $SO(3,\mathbb{R})$ on

$$\mathbb{R}^6 = \left\{ x = \begin{pmatrix} u \\ v \end{pmatrix} : \, u,\, v \in \mathbb{R}^3 \right\}$$

Thus G consists of all block diagonal matrices of the form

$$\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \quad C \in SO(3).$$

According to [11], Subsection 2.5, Examples (d) and (e), the module $\mathcal{D}_G$ is generated by the elements

$$\begin{pmatrix} u \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u \end{pmatrix}, \begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix}, \begin{pmatrix} u\times v \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u\times v \end{pmatrix}.$$

Therefore, if u_0 and v_0 are linearly independent in $\mathbb{R}^3$ then for $y = \left(\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right)$ the dimension of $\epsilon_y(\mathcal{D}_G)$ is equal to 6. If u_0 and v_0 are linearly dependent but not both equal to zero then $\epsilon_y(\mathcal{D}_G)$ has dimension 2.

This example illustrates the usefulness of Theorem 2 if the module $\mathcal{D}_G$ is sufficiently well known. We note the application to $SO(3)$-symmetric second-order systems in $\mathbb{R}^3$: The only nontrivial invariant sets forced by symmetry are defined by the condition that position and velocity have the same direction. (The six-dimensional vector fields corresponding to the second-order systems are of a particular type, but they generate the full Lie algebra $\mathcal{D}_G$, and this is the relevant structure when discussing common invariant sets.)

Remark. Most of the results obtained thus far also apply to G-symmetric discrete dynamical systems

$$x(t+1) = h(x(t)),$$

as is to be expected, since the conditions for symmetric vector fields and for covariant maps correspond: The symmetry criterion (2) also applies to difference equations, thanks to the linearity of the group action. Thus one may consider the set of all G-symmetric maps from $\mathbb{K}^n$ to itself, which is closed with respect to vector space operations and composition. Mutatis mutandis, the elementary properties from Lemmas 2 (except for the connectedness property), 3 and 4, and from Proposition 1 and its Corollary hold. The principal result, Theorem 2 needs modification only with respect to connectedness: The minimal invariant set containing v is, in general, a union of connected components of $\epsilon_v(\mathcal{D}_G)\setminus Z_{s-1}$.

It should be emphasized that the search for arbitrary $\mathcal{D}_G$-invariant sets has been reduced to investigating the algebraic sets $\epsilon_v(\mathcal{D}_G)$ and Z_s. The next result elucidates the structure of G-symmetric vector fields on Z_s.

Theorem 3. *Let $Y \subseteq Z_s$ be an irreducible $\mathcal{D}_G$-invariant subvariety, and $Y \cap Z_s^* \neq \emptyset$. Then either $Y = \epsilon_v(\mathcal{D}_G)$ for some $v \in Y$ or there exists a nonconstant rational first integral of $\mathcal{D}_G$ on Y; i.e., a nonconstant rational function ψ on Y such that*

$$L_f(\psi)(x) = 0 \quad \textit{for all } x \in Y \textit{ and for all } f \in \mathcal{D}_G,$$

whence all level sets of ψ on Y are $\mathcal{D}_G$-invariant. In particular, $Z_n^* = \emptyset$ if and only if there exists a nonconstant rational first integral for $\mathcal{D}_G$ on $\mathbb{K}^n$.

Proof. If there exists a nonconstant rational first integral ψ of $\mathcal{D}_G$ on Y then ψ is defined on an open and dense subset $\widetilde{Y} \subseteq Y$, and every level set $\psi =$ const. on $\widetilde{Y}$ is $\mathcal{D}_G$-invariant. Since every level set has dimension smaller than $\dim Y$, we see that $\dim \epsilon_y(\mathcal{D}_G) < \dim Y$ for all $v \in \widetilde{Y}$, hence for all $v \in Y$.

To prove the reverse direction, let $v \in Y$ such that $\dim \epsilon_v(\mathcal{D}_G) = s$. If $\dim Y = s$ then $Y = \epsilon_v(\mathcal{D}_G)$ by irreducibility. In the following assume that $\dim Y > s$, and let $f_1, \ldots, f_s \in \mathcal{D}_G$ such that $f_1(v), \ldots, f_s(v)$ are linearly independent. In addition, we may take $f_1(x) = x$. By Cramer's rule, every $g \in \mathcal{D}_G$ admits a representation

$$g = \sum_{j=1}^{s} \alpha_j f_j \quad \text{ on } Y,$$

with rational functions α_j.
Given a matrix with s columns $a_1, \ldots, a_s \in \mathbb{K}^n$, denote by $\Delta = \Delta(a_1, \ldots, a_s)$ any $s \times s$ minor of this matrix. Consider the polynomial

$$\rho(x) := \Delta\left(f_1(x), \ldots, f_s(x)\right) \in \mathbb{K}[Y].$$

For any $f \in \mathcal{D}_G$ we have $f \bullet f_j \in \mathcal{D}_G$ by Lemma 1, hence

$$f \bullet f_j = \sum_{j=1}^{s} \alpha_{jk} f_k \quad \text{ on } Y,$$

with rational functions α_{jk}. By the product rule and the alternating property of Δ one finds

$$\begin{aligned} L_f(\rho)(x) &= \textstyle\sum_j \Delta\left(f_1(x), \ldots, f \bullet f_j(x), \ldots, f_s(x)\right) \\ &= \textstyle\sum_{j,k} \alpha_{jk}(x) \Delta\left(f_1(x), \ldots, f_k(x), \ldots, f_s(x)\right) \\ &= \textstyle\sum_j \alpha_{jj}(x) \Delta\left(f_1(x), \ldots, f_j(x), \ldots, f_s(x)\right) \\ &= \left(\textstyle\sum_j \alpha_{jj}(x)\right) \rho(x), \quad \text{ all } x \in Y. \end{aligned}$$

If Δ_1 and Δ_2 are minors such that $\rho_2 \neq 0$ then

$$L_f(\rho_1/\rho_2) = \frac{1}{\rho_2^2}\left(\rho_2 L_f(\rho_1) - \rho_1 L_f(\rho_2)\right) = 0 \text{ on } Y$$

and unless ρ_1/ρ_2 is constant, it is a first integral as asserted.
There remains to show that not all such quotients are constant. Assume, on the contrary, that for any choice of two $s \times s$ minors the quotient will be constant. Denote the rows of the matrix $(f_1(x), \ldots, f_s(x))$ by $z_1(x), \ldots, z_n(x)$. We may assume that $z_1(v), \ldots, z_s(v)$ are linearly independent. Then Cramer's rule implies the existence of rational functions β_{jk} (actually, quotients of $s \times s$ minors) such that

$$z_j = \sum_{k=1}^{s} \beta_{jk} z_k \quad \text{ on } Y; \text{ all } j > s.$$

If ρ_1/ρ_2 is constant for every choice of $s \times s$ minors then all the β_{jk} are constant. Now recall that $f_1(x) = x$, hence in particular

$$x_j - \sum_{k=1}^{s} \beta_{jk} x_k = 0 \text{ on } Y, \quad s < j \leq n.$$

In other words, Y is contained in an s-dimensional vector subspace of $\mathbb{K}^n$. This is a contradiction to the assumption $\dim Y > s$. □

The assumption $Y \cap Z_s^* \neq \emptyset$ in this Theorem involves no loss of generality.

4 Diagonalizable groups

In this section we will discuss connected diagonalizable groups. It seems appropriate to fix terminology first. We call a connected algebraic group (real or complex) an *algebraic torus* if its complexification is isomorphic to some full group of $r \times r$ diagonal matrices. Equivalently, the complexification is connected and diagonalizable (see Humphreys [15]). A *multiplicative one-parameter group* $H \subseteq GL(n, \mathbb{C})$ is the image of a nontrivial homomorphism $\gamma : \mathbb{C}^* \to GL(n, \mathbb{C})$ of algebraic groups; thus the matrix of $\gamma(s)$ with respect to a suitable basis is diagonal with entries s^{k_j}, $k_j \in \mathbb{Z}$, not all zero. By a *real (compact) torus* we mean a real algebraic torus which is compact in the norm topology. (There are other characterizations; see Bröcker and tom Dieck [4].)

Let G be an algebraic torus over $\mathbb{C}$, with Lie algebra $\mathcal{L}$. Then $\mathbb{C}^n$ is the direct sum of weight spaces

$$U_i := \{x : Bx = \omega_i(B) \cdot x, \text{ for all } B \in \mathcal{L}\}$$

for suitable (pairwise distinct) weights $\omega_1, \ldots, \omega_r$. We may assume that $\mathcal{L}$ consists of diagonal matrices, and we may furthermore assume that the elements of

U_i are of the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ z_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad z_i \in \mathbb{C}^{d_i}.$$

Proposition 3. *Let $G \subseteq GL(n,\mathbb{C})$ be an algebraic torus, with notation as above. Given $v \in \mathbb{C}^n$, let*

$$y = \sum_{i=1}^{r} y_i \in \epsilon_v(\mathcal{D}_G) \quad (y_i \in U_i),$$

with a maximal number s of nonzero terms: $y_{i_1} \neq 0, \ldots, y_{i_s} \neq 0$. Then

$$\epsilon_v(\mathcal{D}_G) = U_{i_1} + \cdots + U_{i_s}.$$

In particular one has $Z_n^* \neq \emptyset$.

Proof. Due to Lemma 5(b) and Theorem 1(b), $f(y) \in \epsilon_v(\mathcal{D}_G)$ for all G-symmetric f. Consider, in particular, the linear vector fields that are symmetric with respect to G. According to the assumption above, these are represented by block diagonal matrices of the form

$$C := \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C_r \end{pmatrix}, \quad C_i \in \mathbb{C}^{(d_i,d_i)} \text{ arbitrary.}$$

Now for any $j \in \{i_1, \ldots, i_s\}$ and any $z_j \in U_j$ there is a matrix C_j such that $C_j y_j = z_j$. This shows that $U_{i_1} + \cdots + U_{i_s} \subseteq \epsilon_v(\mathcal{D}_G)$. Equality follows from maximality of the number of nonzero terms for y. □

In view of Lemma 7 we also have:

Corollary 2. *If $\widetilde{G} \subseteq GL(n,\mathbb{R})$ is an algebraic torus then the assertion holds, mutatis mutandis, for all $v \in \mathbb{R}^n$ and the real space*

$$\epsilon_v(\mathcal{D}_G) = \{f(y);\, f : \mathbb{R}^n \to \mathbb{R}^n \text{ is } G-\text{symmetric}\}.$$

Moreover, in the complex setting one can easily characterize the common invariant subspaces of all G-symmetric vector fields. We keep the hypotheses and assumptions from above. The arguments in the proof are quite similar to those used for Poincaré-Dulac normal forms on invariant manifolds; see Bibikov [1], and also [22].

Proposition 4. *Let $G \subseteq GL(n, \mathbb{C})$ be an algebraic torus, with further notation as in Proposition 3. Moreover, let $I = \{i_1, \ldots, i_s\}$ be a proper nonempty subset of $\{1, \ldots, r\}$. Then the subspace*

$$U_I := U_{i_1} + \cdots + U_{i_s}$$

is invariant for every G-symmetric vector field if and only if the following condition is satisfied: For every $k \in \{1, \ldots, r\} \setminus I$ one has

$$\sum_{j \in I} m_j \omega_j \neq \omega_k \tag{4}$$

for all tuples of nonnegative integers m_j such that $\sum m_j \geq 1$.

Proof. Let $e_1, \ldots, e_n$ denote the standard basis, with coordinates $x_1, \ldots, x_n$, and let χ_j be the character corresponding to x_j. We will prove the assertion for the spaces $V_J := \sum_{\ell \in J} \mathbb{C} e_\ell$. By Proposition 3 this will suffice, since every χ_j is among the ω_i's, and vice versa.

The G-symmetric polynomial vector fields are precisely the linear combinations of monomials

$$\prod_{1 \leq i \leq n} x_i^{m_i} e_\ell \tag{5}$$

with nonnegative integers m_i and $1 \leq \ell \leq n$ satisfying

$$\sum m_i \chi_i = \chi_\ell.$$

Assume that for some $k \in \{1, \ldots, n\} \setminus J$ there are nonnegative integers s_j such that

$$\chi_k = \sum_{j \in J} s_j \chi_j, \quad \sum s_j \geq 1.$$

Then the polynomial vector field

$$g(x) = \prod_{j \in J} x_j^{s_j} e_k$$

is G-symmetric, but does not leave V_J invariant.

On the other hand, if condition (4) holds then every G-symmetric vector monomial (5) maps V_J to itself. □

Remark. The proof provides a description of all $\mathcal{D}_G$-symmetric vector monomials, and, as noted, every G-symmetric vector field is a linear combination of these. Likewise, the G-invariants are linear combinations of those monomials $x_1^{d_1} \cdots x_n^{d_n}$ whose exponents satisfy $\sum d_i \chi_i = 0$. This basic observation follows from the diagonalizability of the group action on spaces of functions resp. vector fields (see e.g. [22]), and gives rise to an iterative procedure to determine all monomial invariants and symmetric vector monomials for toral groups. This procedure is outlined in the Appendix, for the reader's convenience.

Corollary 3. *Let assumptions and notation be as in Proposition 4. For $B \in \mathcal{L}$ and a linear form α on $\mathcal{L}$ define*

$$\begin{array}{rcl} I_\alpha^*(B) & = & \{j : 1 \le j \le r \text{ and } \omega_j(B) \cdot \alpha(B) > 0\}, \\ I_\alpha(B) & = & \{j : 1 \le j \le r \text{ and } \omega_j(B) \cdot \alpha(B) \ge 0\}. \end{array}$$

Then U_I is $\mathcal{D}_G$-invariant both for $I = I_\alpha^*(B)$ and for $I = I_\alpha(B)$.

Proof. Let $I = I_\alpha^*(B)$ and assume that there are nonnegative integers m_j and some index $k \notin I$ such that $\sum m_j\omega_j = \omega_k$. Then

$$\sum m_j\omega_j(B)\alpha(B) = \omega_k(B)\alpha(B)$$

leads to a contradiction, as the left-hand side is > 0 and the right-hand side is ≤ 0. Proposition 4 shows the first assertion. The proof of the second assertion is similar. □

There are some applications which we record for later use.

Proposition 5. *Let $G \subseteq GL(n, \mathbb{C})$ be a complex algebraic group.*
a) The set of all v such that $\lim_{s\to 0} \gamma(s)v = 0$ for some multiplicative one-parameter subgroup $H = \{\gamma(s);\, s \in \mathbb{C}^*\}$ is $\mathcal{D}_G$-invariant. In addition, if $\lim_{s\to 0} \gamma(s)v = 0$ then $\lim_{s\to 0} \gamma(s)w = 0$ for all $w \in \epsilon_v(\mathcal{D}_G)$.
b) The set of all v such that $\lim_{s\to 0} \gamma(s)v$ exists for some one-parameter subgroup $H = \{\gamma(s);\, s \in \mathbb{C}^*\}$ is $\mathcal{D}_G$-invariant. In addition, if $\lim_{s\to 0} \gamma(s)v$ exists then $\lim_{s\to 0} \gamma(s)w$ exists for all $w \in \epsilon_v(\mathcal{D}_G)$.

Proof. The limit $\lim_{s\to 0} s^k$ exists if and only if $k \ge 0$, and $\lim_{s\to 0} s^k = 0$ if and only if $k > 0$. We may assume that the matrix of $\gamma(s)$ is diagonal, and consider the generator $B = \mathrm{diag}\,(k_1, \ldots, k_n)$ of this one-parameter group. Then $\lim_{s\to 0} \gamma(s)v$ exists (resp., equals 0) if and only if $v \in U_I$ for $I = I_\alpha(B)$ (resp. $I = I_\alpha^*(B)$), where α sends B to 1. By Corollary 3 and Lemma 3, the subspace U_I is $\mathcal{D}_H$-invariant, hence $\mathcal{D}_G$-invariant, hence contains $\epsilon_v(\mathcal{D}_G)$ by Theorem 2, and the proof is finished. □

Proposition 6. *Let $G \subseteq GL(n, \mathbb{C})$ be the complexification of a real compact torus $\widetilde{G} \subseteq GL(n, \mathbb{R})$, with further notation as in Proposition 3. Then the real $\mathcal{D}_{\widetilde{G}}$-invariant subspaces $U_I \cap \mathbb{R}^n$ correspond to submodules of $\sum \mathbb{Z}\omega_j$ in the following sense:*
$U_I \cap \mathbb{R}^n$ is $\mathcal{D}_{\widetilde{G}}$-invariant if and only if

$$\sum_{j\in I} \mathbb{Z}\omega_j \cap \{\omega_1, \ldots, \omega_r\} = \{\omega_j : j \in I\}$$

Proof. We may assume that $\overline{U}_j = U_{r+1-j}$ and

$$-\omega_j = \overline{\omega}_j = \omega_{r+1-j}$$

for $1 \leq j \leq r$, since all elements of the Lie algebra of $\widetilde{G}$ have purely imaginary eigenvalues. A subspace of $\mathbb{C}^n$ is the complexification of some subspace of $\mathbb{R}^n$ if and only if it is pointwise invariant under conjugation. Thus nonzero weights come in pairs adding to zero, and the criterion in Proposition 4 is equivalent to

$$\omega_k \notin \sum_{j \in I} \mathbb{Z}\omega_j \quad \text{for } k \notin I.$$

□

Example. Let ℓ be any positive integer. Then

$$B_\ell := \mathrm{i} \cdot \mathrm{diag}\,(2\ell, 2\ell-2, \ldots, 2, 0, -2, \ldots, -2\ell+2, -2\ell)$$

spans the Lie algebra of an algebraic torus H which is the complexification of a real torus $\widetilde{H}$, and corresponds to a semisimple element in the complexification of the irreducible $(2\ell+1)$-dimensional representation of $SO(3,\mathbb{R})$. According to Proposition 6, and since $\mathbb{Z}$ is a principal ideal domain, the $\mathcal{D}_{\widetilde{H}}$-invariant subspaces of $\mathbb{R}^n$ are given by

$$V_k := U_I \cap \mathbb{R}^n, \quad I = I_k := \mathrm{i} \cdot 2k\mathbb{Z} \cap \{\mathrm{i} \cdot 2\ell, \ldots, \mathrm{i} \cdot 2, 0, \mathrm{i} \cdot (-2), \ldots, \mathrm{i} \cdot (-2\ell)\}$$

for $0 \leq k \leq \ell$. Thus one obtains precisely ℓ proper $\mathcal{D}_{\widetilde{H}}$-invariant subspaces; viz., all V_k with $k \neq 1$. The dimension of V_k is equal to $1 + 2\left[\frac{\ell}{k}\right]$ if $k > 0$.

5 Compact groups

In view of Proposition 1 and Theorem 2 it is natural to investigate the inclusion $\epsilon_v(\mathcal{D}_G) \subseteq \mathrm{Fix}\,(G_v)$. For real compact groups it is known that equality holds for all v. This is usually proved as a consequence of the Palais slice theorem; see e.g. Field [9], Lemma A. (The basic idea is also used in Michel [19].) We provide an elementary proof here, which takes a different approach.

Theorem 4. *Let G be a compact subgroup of $GL(n,\mathbb{R})$. Then $\epsilon_y(\mathcal{D}_G) = \mathrm{Fix}(G_y)$ for all $y \in \mathbb{R}^n$.*

Proof. (i) We may assume that $G \subseteq O(n,\mathbb{R})$ and that the norm on $\mathbb{R}^n$ is induced by the G-invariant scalar product. Moreover, let μ denote the Haar measure on G. Given any C^∞ function ϕ on $\mathbb{R}^n$ and any $w \in \mathbb{R}^n$, one obtains a G-symmetric C^∞ vector field via

$$g_w^*(x) := \int_G \phi(Tx) T^{-1} w \, \mathrm{d}\mu(T).$$

(ii) Let $y \in \mathbb{R}^n$ and $w \in \mathrm{Fix}(G_y)$. Then for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$||Ty - y|| < \delta \Rightarrow ||Tw - w|| < \epsilon, \quad \text{all } T \in G.$$

To prove this, assume the contrary. Then there exist some $\rho > 0$ and a sequence (T_ℓ) in G such that $T_\ell y \to y$ as $\ell \to \infty$ but all $||T_\ell w - w|| > \rho$. We may assume that $\lim T_\ell =: T^*$ exists. From

$$y = \lim T_\ell y = T^* y$$

one sees that $T^* \in G_y$. But this implies $T^* w = w$; a contradiction.

(iii) Given $y \in \mathbb{R}^n$, $w \in \mathrm{Fix}(G_y)$ and $\epsilon > 0$, there exists a G-symmetric C^∞ vector field g_w such that

$$||g_w(y) - w|| < \epsilon.$$

To see this, choose $\delta > 0$ so that $||Ty - y|| < \delta$ implies $||Tw - w|| < \epsilon$ for all $T \in G$, and choose ϕ as a nonnegative function with support contained in the ball $B_\delta(y)$, with $\phi(y) = 1$. With g_w^* as in part (i), define

$$g_w(x) := \frac{1}{\int_G \phi(Ty)\,\mathrm{d}\mu(T)} \cdot g_w^*(x), \quad x \in \mathbb{K}^n.$$

Then

$$g_w(y) - w = \frac{1}{\int_G \phi(Ty)\,\mathrm{d}\mu(T)} \cdot \int_G \phi(Ty)(T^{-1}w - w)\,\mathrm{d}\mu(T)$$

and

$$||g_w(y) - w|| \leq \frac{1}{\int_G \phi(Ty)\,\mathrm{d}\mu(T)} \cdot \int_G \phi(Ty)(||T^{-1}w - w||\,\mathrm{d}\mu(T) < \epsilon,$$

since $||T^{-1}w - w|| < \epsilon$ whenever $\phi(Ty) \neq 0$.

(iv) Let $(w_1, \ldots, w_r)$ be a basis of $\mathrm{Fix}(G_y)$. Then there exists $\epsilon > 0$ such that every system $(v_1, \ldots, v_r)$ in $\mathrm{Fix}(G_y)$ with $||v_1 - w_1|| < \epsilon, \ldots, ||v_r - w_r|| < \epsilon$ also forms a basis of $\mathrm{Fix}(G_y)$. According to (iii) there exist G-symmetric C^∞ vector fields $g_1, \ldots, g_r$ such that the $g_i(y)$ span $\mathrm{Fix}(G_y)$.

(v) According to a theorem by Poénaru [21], the module of G-symmetric C^∞ vector fields over the algebra of C^∞ G-invariants is generated by polynomial vector fields. □

Remark. At this point it may be appropriate to sketch the relation between the familiar stratification of $\mathbb{K}^n$ by the action of a compact group G and the decomposition induced by the varieties Z_s (see Theorems 1 and 2). For a compact group G and any $v \in \mathbb{K}^n$ the subspaces $\epsilon_v(\mathcal{D}_G)$ and $\mathrm{Fix}(G_v)$ are equal, thus the smallest Zariski-closed $\mathcal{D}_G$-invariant subset which contains v is the fixed point space of the isotropy group. (For general groups $\epsilon_v(\mathcal{D}_G)$ may be a proper subset of the fixed point space.) For compact G and $v \in \mathbb{K}^n$, the stratum of v is by definition the set of all $y \in \mathbb{K}^n$ with isotropy subgroup conjugate to G_v. Since two elements of $\mathbb{K}^n$ have conjugate isotropy groups if they lie on the same orbit, the stratum contains $G \cdot \mathrm{Fix}(G_v)$. Note that $A_v := \overline{G \cdot \mathrm{Fix}(G_v)}$ is the smallest Zariski-closed set which contains v and is both $\mathcal{D}_G$-invariant and G-stable. By properties of algebraic group actions, $G \cdot \mathrm{Fix}(G_v)$ is open and dense in A_v. Therefore the closure of the stratum is the union of a (finite) number of sets A_y. On the other hand, if $\dim(\mathrm{Fix}(G_v)) = r$ then $G \cdot \mathrm{Fix}(G_v)$ is contained in Z_r^*, and so is the stratum. Thus the decomposition into strata may be finer than the decomposition into the Z_s^*, and the $G \cdot \mathrm{Fix}(G_v)$ (resp. the A_v) provide the most refined decomposition. (For general groups this decomposition carries over to $G \cdot \epsilon_v(\mathcal{D}_G)$ and its closure.)

We note a consequence of some practical value for a compact and connected group G. In order to search for points v with $\epsilon_v(\mathcal{D}_G) \neq \mathbb{R}^n$, one may restrict attention to a maximal torus of G.

Corollary 4. *Let $G \subseteq GL(n, \mathbb{R})$ be compact and connected such that $Z_n^*(G) \neq \emptyset$, and let H be a maximal torus of G. If $v \in Z_{n-1}(G)$ then there exists $T \in G$ such that Tv has nontrivial isotropy in H. Conversely, if $w \in Z_{n-1}(H)$ then $Tw \in Z_{n-1}(G)$ for all $T \in G$. In other words,*

$$Z_{n-1}(G) = G \cdot Z_{n-1}(H)$$

Proof. By Theorem 4 there is an element $S \in G$, $S \neq \mathrm{id}$ such that $Sv = v$. By Bröcker and tom Dieck [4], Theorem IV.1.6, S lies in a maximal torus, which in turn is conjugate to H. Thus $TST^{-1} \in H$ for a suitable $T \in G$, and TST^{-1} fixes v. This proves the inclusion "$\subseteq$". The reverse inclusion is elementary: For $T \in G$ and $w \in \mathbb{K}^n$ one has $\epsilon_{Tw} = T\epsilon_w$, hence by Lemma 6 one finds for $w \in Z_{n-1}(H)$:

$$\epsilon_{Tw}(\mathcal{D}_G) \subseteq \epsilon_{Tw}(\mathcal{D}_H) \subseteq T\epsilon_w(\mathcal{D}_H) \neq \mathbb{K}^n.$$

□

Recall that $Z_{n-1}(H)$ can be determined in a systematic and relatively easy manner (Proposition 6). But the description above does not directly provide defining equations for $Z_{n-1}(G)$.

Example. We continue the example at the end of Section 4, with the irreducible $(2\ell+1)$-dimensional representation G of $SO(3, \mathbb{R})$. The Lie algebra of a maximal torus $\widetilde{H}$ is spanned by

$$B_\ell := \mathrm{i} \cdot \mathrm{diag}\,(2\ell, 2\ell-2, \ldots, 2, 0, -2, \ldots, -2\ell+2, -2\ell)\,,$$

and the nontrivial $\mathcal{D}_{\widetilde{H}}$-invariant subspaces are just V_0, V_2, $\ldots$, V_ℓ. Now assume $\ell > 2$. One obtains $\mathcal{D}_G$- invariant subsets $G \cdot V_k$, of dimension $\leq 3 + \ell$, since $\dim V_k = 1 + 2\,[\ell/k]$, G is three-dimensional and the stabilizer of V_k contains H. Therefore $Z^*_{2\ell+1} \neq \emptyset$, and by Corollary 4, $Z_{2\ell}$ is just the union of the $G \cdot V_k$, $k \neq 1$.

There is a shortcut to determine the varieties Z_s for a compact group G from those of the connected identity component G^0.

Proposition 7. *Let $G \neq G^0$ be compact.*
(a) *There exist homogeneous polynomials $\phi_1, \ldots, \phi_r$ which generate the invariant algebra of G^0 and are $\mathbb{K}$-linearly independent, such that the vector space $\mathbb{K}\phi_1 + \cdots + \mathbb{K}\phi_r$ is stable with respect to the G action. Define*

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}$$

Then for every $T \in G$ there exists a unique $\widetilde{T} \in GL(r, \mathbb{K})$ such that the identity

$$\Phi(Tx) = \widetilde{T}\Phi(x)$$

holds. $\widetilde{T}$ depends only on the class of T mod G^0, thus one has an induced action of G/G^0 on $\mathbb{K}^r$.
(b) *If $Sv = v$ for some $S \in G$ then $\widetilde{S}\Phi(v) = \Phi(v)$, and conversely. Thus the image of $\mathrm{Fix}\,(G_v)$ is equal to the fixed point space of $(G/G^0)_{\Phi(v)}$.*
(c) *The following equality holds:*

$$\epsilon_v(\mathcal{D}_G) = \epsilon_v(\mathcal{D}_{G^0}) \cap \left\{x : \widetilde{S}\Phi(x) = \Phi(x), \text{ for all } \widetilde{S} \in G_v/(G^0 \cap G_v)\right\}.$$

Proof. For every $T \in G$ and every homogeneous G^0-invariant ϕ, $\phi \circ T$ is a G^0-invariant which is homogeneous of the same degree. Given a homogeneous system of generators $\phi_1, \ldots, \phi_s$ for the invariant algebra of G^0, the $\phi_j \circ T$, $T \in G$ will therefore span a finite dimensional vector space. Extending the system $\phi_1, \ldots, \phi_s$ to a basis of this vector space will yield a system $\phi_1, \ldots, \phi_r$ that satisfies part (a). Moreover this system separates G^0-orbits, due to compactness. The nontrivial assertion of part (b) follows: If $\widetilde{S}\Phi(v) = \Phi(v)$ for some $S \in G$ then, due to the separation property, Sv is on the same G^0-orbit as v, thus $Sv = Tv$ for some $T \in G^0$, whence $S^{-1}Tv = v$. $\square$

6 Reductive groups

This section is devoted to extending some of the results for compact groups to complex or real reductive groups. (Recall that the complexification of a real compact group is reductive.) For reductive groups one cannot expect the equality $\epsilon_y(\mathcal{D}_G) = \mathrm{Fix}(G_y)$ to hold for all y. Counterexamples exist even for algebraic tori.
Example. Let G consist of all diagonal matrices with entries a, a^2, a^3, with $a \in \mathbb{C}^*$. The remark following Proposition 4 shows that $\mathcal{D}_G$ is generated by the vector fields

$$\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1^3 \end{pmatrix}.$$

Consider $v = (0,\,1,\,1)^{\mathrm{tr}}$: Since $a^2 = a^3 = 1$ implies $a = 1$, one has $\mathrm{Fix}(G_v) = \mathbb{C}^3$, whereas the image of ϵ_v is two-dimensional.

But for reductive groups there exists a good criterion for surjectivity of the evaluation map.

Theorem 5. *(Panyushev [20].) For a complex reductive group G, the evaluation map*

$$\epsilon_v : \mathcal{D}_G \to \mathrm{Fix}\,(G_v), \quad f \mapsto f(v)$$

is surjective, and thus equality $\epsilon_v(\mathcal{D}_G) = \mathrm{Fix}\,(G_v)$ holds, whenever the Zariski closure of the orbit $G \cdot v$ is a normal variety and $\overline{G \cdot v} \setminus G \cdot v$ has codimension > 1 in $\overline{G \cdot v}$. In particular equality holds when the orbit of v is Zariski closed.

Proposition 8. *(Hilbert-Mumford criterion; Birkes [2], Theorem 4.2.) Let $G \subseteq GL(n,\,\mathbb{C})$ be reductive, and $v \in G$ such that the orbit Gv is not closed. Then there exists a nontrivial multiplicative one-parameter group $\{\gamma(s);\, s \in \mathbb{C}^*\}$ such that $\lim_{s\to 0} \gamma(s)v$ exists.*

Remark. Due to Proposition 5 the set of all v which satisfy such a limit condition for some multiplicative one-parameter subgroup is $\mathcal{D}_G$-invariant.

We obtain a precise description of Z_n^* for complex reductive groups.

Theorem 6. *Let $G \subseteq GL(n,\,\mathbb{C})$ be reductive, and $Z_n^* \neq \emptyset$. Then for all $v \in \mathbb{C}^n$ the following hold:*
a) If the orbit $G \cdot v$ is closed and G_v is trivial then $v \in Z_n^*$.
b) If $G \subseteq SL(n,\,\mathbb{C})$ and $v \in Z_n^*$ then $G \cdot v$ is closed and G_v is trivial.

Proof. The proof of part a) is immediate from Panyushev's theorem: Since the orbit of v is closed, one has $\epsilon_v(\mathcal{D}_G) = \mathrm{Fix}\,(G_v)$, and $\epsilon_v(\mathcal{D}_G) = \mathbb{K}^n$ from trivial isotropy.

To prove part b), we first show that $G \cdot v$ is closed. Assume that this is not the case. Then by the Hilbert-Mumford criterion there is a nontrivial multiplicative one-parameter group $\{\gamma(s);\, s \in \mathbb{C}^*\}$ such that $\lim_{s\to 0} \gamma(s)v$ exists. By Proposition 5, $\lim_{s\to 0} \gamma(s)w$ exists for all $w \in \epsilon_v(\mathcal{D}_G) = \mathbb{K}^n$. Thus, if

$$\gamma(s) = \mathrm{diag}(s^{k_1}, \ldots, s^{k_n}); \quad k_i \in \mathbb{Z}$$

(as may be assumed) then all $k_i \geq 0$ and $\sum k_i > 0$; a contradiction to $\gamma(s) \in SL(n)$.
By Theorem 5, again, one has $\mathbb{K}^n = \epsilon_v(\mathcal{D}_G) = \mathrm{Fix}\,(G_v)$ and therefore G_v is trivial. □

For real reductive groups Panyushev's theorem has the following consequence:

Theorem 7. *Let $G \subseteq GL(n,\mathbb{R})$ be reductive, with complexification ${}_{\mathbb{C}}G$, and let $y \in \mathbb{R}^n$ such that the G-orbit of y is closed in the norm topology. Then the ${}_{\mathbb{C}}G$-orbit of y is Zariski closed and*

$$\epsilon_y(\mathcal{D}_G) = \mathrm{Fix}\,(({}_{\mathbb{C}}G)_y \cap G)\,.$$

Proof. Zariski-closedness of the orbit follows from Birkes [2], Corollary 5.3, and then surjectivity of the map ϵ_y follows from Theorem 5. $\square$

Example. In general one has $G_y \neq (({}_{\mathbb{C}}G)_y \cap G)$. Let

$$G = \left\{\mathrm{diag}(a^3,\, a,\, a^{-1},\, a^{-3}) : a \in \mathbb{R}^*\right\}$$

and $v = (1,\, 0,\, 0,\, 1)^t$. Then ${}_{\mathbb{C}}G$ is defined by the same conditions with $a \in \mathbb{C}^*$. The isotropy groups are defined by the condition $a^3 = 1$. This forces trivial isotropy (and $\mathrm{Fix}(G_v) = \mathbb{R}^4$) in the real case, while one may let a be a primitive third root of unity and obtain a two-dimensional fixed point space for ${}_{\mathbb{C}}G$.

We close this section with some remarks on a natural *extension problem*: Given $G \subseteq GL(n, \mathbb{K})$ and a $\mathcal{D}_G$-invariant variety Y, under what circumstances can a polynomial vector field $\tilde{f}$ on Y be extended to a G-symmetric vector field on $\mathbb{K}^n$? This is of special interest for low-dimensional $\mathcal{D}_G$-invariant sets of high-dimensional systems, for which a complete discussion is not feasible.

The following necessary condition is obvious: Let $H \subseteq G$ be the stabilizer subgroup of Y. The restriction of a G-symmetric vector field on $\mathbb{K}^n$ to Y has symmetry group $H|_Y$, and therefore $\tilde{f}$ must admit this symmetry group.

If G is reductive and the variety Y is also G-stable then $H = G$ and the above condition is also sufficient. See e.g. the Lemma in Panyushev [20] for the following well-known result.

Proposition 9. *Let $G \subseteq GL(n, \mathbb{K})$ be reductive, and Y a G-stable subvariety of $\mathbb{K}^n$. Then every G-symmetric polynomial vector field on Y extends to a G-symmetric polynomial vector field on $\mathbb{K}^n$.*

If Y is not G-stable then, in general, H-symmetry is not even sufficient to ensure well-definedness of an extension as a map. Let $W \subseteq \mathbb{K}^n$ be an irreducible affine $\mathcal{D}_G$-invariant subvariety, and $\tilde{f} : W \to \mathbb{K}^n$ a polynomial vector field. (Linear subspaces are of particular interest, due to Theorem 2 and Proposition 1.) Let V be the Zariski closure of $G \cdot W$ and note that $G \cdot W$ contains a Zariski-open and dense subset of V. If there is an extension f of $\tilde{f}$ to a G-symmetric vector field on V then necessarily the following well-definedness condition holds:

(6) If $w \in W$, $T \in G$ are such that $Tw \in W$ then $\tilde{f}(Tw) = T\tilde{f}(w)$.

As one easily verifies, this condition is necessary and sufficient for the existence of a G-symmetric extension map $f : G \cdot W \to \mathbb{K}^n$ of $\tilde{f}$, which must be given by

$$f(Tx) = T\tilde{f}(x), \quad x \in W, T \in G. \tag{7}$$

The problem is to decide whether (7) defines a polynomial f. Without further assumptions, the following result seems to be the best possible.

Proposition 10. *Let G be connected, let W be an irreducible affine subvariety, and $\tilde{f} : W \to W$ a polynomial vector field which satisfies the well-definedness condition (6). Then there exists a rational G-symmetric vector field f on the Zariski closure V of $G \cdot W$ such that $f|_W = \tilde{f}$.*

Proof. The map

$$\Phi: G \times W \to V, \quad (T, x) \mapsto Tx$$

is a dominant morphism of irreducible varieties by construction. Now consider

$$F: G \times W \to \mathbb{K}^n, \quad F(T, x) := T\tilde{f}(x).$$

By the well-definedness condition (6), this map is constant on the fibers of Φ. According to a theorem by Chevalley cited by Borel [3] (Proposition on p. 43), applied to every entry of F, there exists a rational f on V such that $F = f \circ \Phi$; in other words, $T\tilde{f}(x) = f(Tx)$ for all $T \in G$ and all x in a nonempty Zariski-open subset of W. □

Remarks. (a) Condition (6) forces $\tilde{f}$ to be symmetric with respect to the stabilizer subgroup H of W.
(b) Condition (6) forces all intersections of W with isotropy fixed point subspaces of $\mathbb{K}^n$ to be $\tilde{f}$-invariant. Indeed, $Tv = v$ for $T \in G$, $v \in W$ implies $T\tilde{f}(v) = \tilde{f}(v)$. It is not clear whether the analogous statement for the subspaces $\epsilon_v(\mathcal{D}_G)$ holds generally.
(c) In case $y \in W := \mathrm{Fix}(G_v)$ for some v, one can characterize the group elements T which satisfy the premise of (6): One has $y \in W$ if and only if $G_v \subseteq G_y$. In view of $G_{Ty} = TG_yT^{-1}$, one sees that $Ty \in W$ is equivalent to

$$T^{-1}G_vT \subseteq G_y.$$

In case $G_y = G_v$ this condition characterizes the normalizer of G_v, and is equivalent to $T(W) = W$.

7 Examples

7.1 Couette-Taylor symmetry

The previous sections provide, among other results, the tools to discuss groups with toral identity component, such as the symmetry group G of a Couette-Taylor system. While this system has been studied extensively, it may be of some interest to see that and how our approach facilitates computations and increases transparency. Our aim is not to discuss a particular system but to investigate the invariant sets of a general system admitting the symmetry group. (See the Remark following Proposition 2.) We will show that all the relevant information can be obtained in a few pages, starting from scratch.

The Couette-Taylor system under consideration here lives on a six-dimensional real phase space. (We follow the presentation and notation in Gatermann [12], Ch. 4.) Via complexification one turns to to $\mathbb{C}^6$, with coordinates denoted by $z_0, \ldots, z_5$, and the real phase space V is defined by $z_{i+3} = \overline{z}_i$ for $0 \le i \le 2$. The connected identity component G^0 of the symmetry group G is a two-dimensional torus whose Lie algebra $\mathcal{L}$ is spanned by iC_1 and iC_2, with

$$\begin{aligned} C_1 &:= \mathrm{diag}\,(1,\,2,\,0,\,-1,\,-2,\,0) \\ C_2 &:= \mathrm{diag}\,(0,\,1,\,1,\,0,\,-1,\,-1)\,. \end{aligned}$$

The full group is generated by G^0 and the involution ("reflection") R which exchanges z_0 and z_3, z_1 and z_2, z_4 and z_5, respectively. All conjugates of R are also involutions, with three-dimensional fixed point spaces.

Proposition 11. (a) *The invariant algebra of G^0 is generated by*

$$\psi_1 = z_1 z_4, \quad \psi_2 = z_2 z_5, \quad \psi_3 = z_0 z_3, \quad \psi_4 = z_0^2 z_2 z_4, \quad \psi_5 = z_1 z_3^2 z_5$$

with the single relation $\psi_4\psi_5 - \psi_1\psi_2\psi_3^2 = 0$. On the real subspace V one has

$$\begin{array}{rcl} \overline{\psi}_k & = & \psi_k, \quad 1 \leq k \leq 3; \\ \overline{\psi}_4 & = & \psi_5. \end{array}$$

(b) *A vector field is G^0-symmetric if and only if it has the form*

$$f(z) = \begin{pmatrix} \sigma_1 \cdot z_0 & + & \sigma_2 \cdot z_1 z_3 z_5 \\ \sigma_3 \cdot z_1 & + & \sigma_4 \cdot z_0^2 z_2 \\ \sigma_5 \cdot z_2 & + & \sigma_6 \cdot z_1 z_3^2 \\ \sigma_7 \cdot z_3 & + & \sigma_8 \cdot z_0 z_2 z_4 \\ \sigma_9 \cdot z_4 & + & \sigma_{10} \cdot z_3^2 z_5 \\ \sigma_{11} \cdot z_5 & + & \sigma_{12} \cdot z_0^2 z_4 \end{pmatrix}$$

with the σ_j polynomials in $\psi_1, \ldots, \psi_5$. The vector field stabilizes the real subspace V if and only if $\sigma_{6+j} = \overline{\sigma}_j$ on V, $1 \leq j \leq 6$.

Proof. This is a straightforward consequence of the remark following Proposition 4 (see also the Appendix). We sketch only the computations for the invariants, starting with the invariants of C_2. A monomial

$$z_0^{m_0} \cdots z_5^{m_5}$$

is invariant for C_2 if and only if

$$m_1 + m_2 - m_4 - m_5 = 0.$$

Therefore (compare the 1 : 1 - resonance, e.g. in [22]) a generator system for the invariant algebra of C_2 is given by

$$\phi_1 := z_0; \quad \phi_2 := z_3; \quad \phi_3 := z_1 z_4; \quad \phi_4 := z_1 z_5; \quad \phi_5 := z_2 z_4; \quad \phi_6 := z_2 z_5.$$

The ϕ_j are mapped to scalar multiples of themselves by L_{C_1}; one finds

$$L_{C_1}\left(\phi_1^{d_1} \cdots \phi_6^{d_6}\right) = (d_1 - d_2 + 2d_4 - 2d_5) \cdot \phi_1^{d_1} \cdots \phi_6^{d_6}$$

hence G^0-invariant monomials in the ϕ_j are characterized by

$$d_1 - d_2 + 2d_4 - 2d_5 = 0.$$

(This corresponds to the 1 : 2 - resonance; see e.g. [22]) . Thus we obtain generators

$$\phi_3; \quad \phi_6; \quad \phi_1\phi_2; \quad \phi_1^2\phi_5; \quad \phi_2^2\phi_4,$$

since $\phi_4\phi_5 = \phi_3\phi_6$ may be discarded. The remaining computations are similar. The assertion about vector fields stabilizing the real subspace V follows from the necessary and sufficient condition that any entry with index $j+3$ must have a value conjugate to the entry with index j when applied to an element of V. □

Corollary 5. *The nontrivial invariant subspaces of V common to all real vector fields with G^0 symmetry are given by*

$$\begin{array}{lcl} Y_1 & := & \{z \in V;\, z_0 = z_3 = 0\}\,; \\ Y_2 & := & \{z \in V;\, z_0 = z_3 = z_1 = z_4 = 0\}\,; \\ Y_3 & := & \{z \in V;\, z_0 = z_3 = z_2 = z_5 = 0\}\,; \\ Y_4 & := & \{z \in V;\, z_1 = z_4 = z_2 = z_5 = 0\}\,. \end{array}$$

Proof. We use the criterion in Proposition 6. Thus let the weights ω_j on $\mathcal{L}$ be defined by

$$\begin{array}{ll} \omega_1(C_1) = 1, & \omega_1(C_2) = 0, \\ \omega_2(C_1) = 2, & \omega_2(C_2) = 1, \\ \omega_3(C_1) = 0, & \omega_3(C_2) = 1. \end{array}$$

Since ω_2 and ω_3 are not contained in $\mathbb{Z}\cdot\omega_1$, one finds invariance of Y_4; and invariance of Y_3 and Y_2 follow by consideration of $\mathbb{Z}\cdot\omega_2$ and $\mathbb{Z}\cdot\omega_3$, respectively. The relation $2\omega_1 - \omega_2 + \omega_3 = 0$ shows that $\omega_2 \in \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_3$ as well as $\omega_3 \in \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$; hence no nontrivial $\mathcal{D}_{G^0}$-invariant subspaces correspond to these submodules. But $\omega_1 \notin \mathbb{Z}\cdot\omega_2 + \mathbb{Z}\cdot\omega_3$ shows the invariance of Y_1. □

Now we turn to the full group G. The reflection R acts on G^0-invariants as follows:

$$\psi_1 \circ R = \psi_2; \quad \psi_3 \circ R = \psi_3; \quad \psi_4 \circ R = \psi_5.$$

Thus $\mathbb{K}\psi_1 + \cdots + \mathbb{K}\psi_5$ is stable with respect to this action. We define for a polynomial σ in five variables:

$$\sigma^*(\psi_1, \ldots, \psi_5) := \sigma(\psi_2, \psi_1, \psi_3, \psi_5, \psi_4);$$

in other words, $\sigma^* = \sigma \circ \widetilde{R}$.

Proposition 12. *A vector field is G-symmetric if and only if it has the form*

$$f(z) = \begin{pmatrix} \sigma_1 \cdot z_0 & + & \sigma_2 \cdot z_1 z_3 z_5 \\ \sigma_3 \cdot z_1 & + & \sigma_4 \cdot z_0^2 z_2 \\ \sigma_3^* \cdot z_2 & + & \sigma_4^* \cdot z_1 z_3^2 \\ \sigma_1^* \cdot z_3 & + & \sigma_2^* \cdot z_0 z_2 z_4 \\ \sigma_9 \cdot z_4 & + & \sigma_{10} \cdot z_3^2 z_5 \\ \sigma_9^* \cdot z_5 & + & \sigma_{10}^* \cdot z_0^2 z_4 \end{pmatrix}$$

The vector field stabilizes the real subspace V if and only if

$$\begin{array}{ll} \sigma_1^* = \overline{\sigma}_1, & \sigma_2^* = \overline{\sigma}_2; \\ \sigma_9 = \overline{\sigma}_3, & \sigma_{10} = \overline{\sigma}_4 \end{array}$$

hold on V.

Proof. If f is G^0-symmetric then $f + R^{-1} \circ f \circ R$ is G-symmetric, and every G-symmetric vector field is obtained in this way. All assertions now follow from routine calculations. □

Proposition 13. *The set*

$$Y_5 := \{w \in V;\ \psi_1(w) = \psi_2(w) \text{ and } \psi_4(w) = \psi_5(w)\}$$

is invariant for every G-symmetric (real) vector field. Moreover G_v is not a subset of G^0 only if $v \in Y_5$, which is the union of the three-dimensional fixed-point spaces of R and its conjugates; thus $Y_5 \subseteq Z_3$.

The varieties Z_s are determined from $Y_1, \ldots, Y_5$ and unions and intersections of these sets.

Proof. The subspace $\mathbb{K}\psi_1 + \cdots \mathbb{K}\psi_5$ is G/G^0-invariant; now apply Proposition 7 and Theorem 2. □

The fixed point spaces of the conjugates of R and their intersections with Y_4 resp. Y_1 are determined by straightforward computations: For $v \in Y_5$ one finds

$$\begin{array}{rcl} C_v & := & \{z \in V;\ z_0v_3 = z_3v_0,\ z_1v_2 = z_2v_1,\ z_4v_5 = z_5v_4\} \\ B_v & := & \{z \in V;\ z_0v_3 = z_3v_0,\ z_1 = z_2 = z_4 = z_5 = 0\} \\ A_v & := & \{z \in V;\ z_0 = z_3 = 0,\ z_1v_2 = z_2v_1,\ z_4v_5 = z_5v_4\} \end{array}$$

In particular the fixed point space of R equals C_v with $v_0 = \cdots v_5 = 1$.

Since Y_1 through Y_4 are also group invariant, the dynamics of G-symmetric vector fields on these spaces is straightforward. As for Y_5, the situation is different, but it is sufficient to consider $Y_5^* := Y_5 \setminus (Y_1 \cup Y_4)$. Note that all $z_i \neq 0$ on Y_5^*. The following result gives a complete characterization of their structure.

Proposition 14. *The restriction of every G-symmetric differential equation to $Y_5^* = Y_5 \setminus (Y_1 \cup Y_4)$ admits the first integrals z_3/z_0, z_2/z_1 and z_4/z_5. One may rewrite the differential equation in the form*

$$\begin{array}{rcl} \dot{z}_0 & = & (\tau_1 + \rho_1\tau_2)\, z_0 \\ \dot{z}_1 & = & (\tau_3 + \rho_2\tau_4)\, z_1 \\ \dot{z}_2 & = & (\tau_3 + \rho_2\tau_4)\, z_2 \\ \dot{z}_3 & = & (\tau_1 + \rho_1\tau_2)\, z_3 \\ \dot{z}_4 & = & (\tau_9 + \rho_3\tau_{10})\, z_4 \\ \dot{z}_5 & = & (\tau_9 + \rho_3\tau_{10})\, z_5. \end{array}$$

with polynomials τ_k depending on ψ_1, ψ_3, ψ_4 only, and $\tau_1 = \overline{\tau}_1$, $\tau_2 = \overline{\tau}_2$, $\tau_9 = \overline{\tau}_3$ and $\tau_{10} = \overline{\tau}_4$, and

$$\begin{array}{rcccl} \rho_1 & = & z_1z_3z_5/z_0 & = & z_0z_2z_4/z_3 \\ \rho_2 & = & z_0^2z_2/z_1 & = & z_1z_3^2/z_2 \\ \rho_3 & = & z_3^2z_5/z_4 & = & z_0^2z_4/z_5, \end{array}$$

On a level set $z_3/z_0 = \alpha_1$, $z_2/z_1 = \alpha_2$ and $z_4/z_5 = \alpha_3$ of the first integrals one has $\psi_4 = \alpha_2/\alpha_1 \cdot \psi_1\psi_3$, and

$$\rho_1 = \alpha_1\alpha_3\psi_1, \quad \rho_2 = \frac{\alpha_2}{\alpha_1}\psi_3, \quad \rho_3 = \frac{\alpha_3}{\alpha_1}\psi_3,$$

which shows that the system on the level set is reducible to a two-dimensional system by $(\psi_1,\, \psi_3)^{\mathrm{tr}}$.

Proof. The first integrals can be determined as in the proof of Theorem 3: From Proposition 12 one obtains, for instance, module elements

$$\begin{pmatrix} 0 \\ z_1 \\ z_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ z_4 \\ z_5 \end{pmatrix}, \begin{pmatrix} 0 \\ z_0^2 z_2 \\ z_1 z_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

On $Y_5 \subseteq Z_3$ we consider 3×3 minors of the matrix with these columns. The minor with rows $1, 2$ and 4 equals $z_1^2 z_3^2 z_4$, while the minor with rows $1, 2$ and 5 equals $z_1^2 z_3^2 z_5$. Their quotient z_4/z_5 is a first integral. The remaining first integrals are obtained similarly. Moreover, on Y_5 one has $\psi_1 = \psi_2$ and $\psi_4 = \psi_5$, and on Y_5^* this implies

$$z_1^2 z_3^2 z_4 z_5 = z_0^2 z_2^2 z_4 z_5 \Rightarrow z_1^2 z_3^2 = z_0^2 z_2^2$$

by cancellation. By this and similar computations the identities asserted for the ρ_i hold.
From the first integral z_0/z_3 one obtains additional conditions on the polynomials in Proposition 12: On Y_5^* one has

$$\begin{aligned} 0 &= z_3 \cdot (\sigma_1 z_0 + \sigma_2 z_1 z_3 z_5) - z_0(\sigma_1^* z_3 + \sigma_2^* z_0 z_2 z_4) \\ &= (\sigma_1 - \sigma_1^*) z_0 z_3 + (\sigma_2 - \sigma_2^*) z_1 z_3^2 z_5 \end{aligned}$$

which implies $\sigma_1 = \sigma_1^*$ and $\sigma_2 = \sigma_2^*$ on Y_5^* and on Y_5. Similarly one finds $\sigma_j = \sigma_j^*$ for the remaining indices. The assertions now follow from straightforward computations. □

Remark. One may compare these results to the list of isotropy fixed point spaces in Gatermann [12], Table 4.5. (This table, as usual, lists representatives modulo conjugation.) The first item in this list corresponds to $\{0\}$, the second item to B_v with $v = (1, 1, 0, \ldots, 0)^{\mathrm{tr}}$. Item 3 corresponds to Y_2 and Y_3, while item 4 corresponds to A_v with $v = (0, 1, 1, 0, 1, 1)^{\mathrm{tr}}$. Both items 5 and 7 correspond to C_v, with $v = (1, 1, 1, 1, 1, 1)^{\mathrm{tr}}$ resp. $v = (i, 1, 1, -i, 1, 1)^{\mathrm{tr}}$. Item 6 corresponds to Y_1, and the last item corresponds to V. The approach taken here seems more transparent and avoids nontrivial as well as unpleasant tasks such as the determination of all isotropy subgroups.

7.2 Representations of $SL(2)$

The representations of $SL(2)$ are well-understood: For every dimension d there is one and only one irreducible representation, up to isomorphism, which can be realized by the linear action on forms of degree $d-1$ in two variables. Elliott [8] is a classical source. We will recall some facts on irreducible representations, for the reader's convenience and also to fix notation (which is not standardized).

We consider the space of forms of degree $d-1$ in two variables u and v with basis

$$e_1 := u^{d-1},\, e_2 := u^{d-2}v, \ldots, e_d := v^{d-1}.$$

(This differs from Elliott by some scaling factors, but is compatible with Cushman and Sanders [7], and also with [11], from which we will quote some results.) The action of $SL(2)$ on forms is as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts via } \left\{ \begin{array}{lcl} u & \mapsto & du - bv \\ v & \mapsto & -cu + av \end{array} \right. \tag{8}$$

We will denote by G the group of matrices which represent this action with respect to the basis $e_1, \ldots, e_d$.

With increasing dimension the generators and relations even for the invariant algebra become intractable. In this subsection we will discuss low-dimensional irreducible representations, viz., the four-dimensional and the five-dimensional irreducible representation. Both of these have been thoroughly investigated by Elliott [8], p. 97 ff. and p. 213 ff.. (See also Cushman and Sanders [7], from which invariants and covariants are taken.) But some properties of symmetric vector fields, their invariant sets and the extension problem seem worth mentioning.

A. First, we consider the four-dimensional irreducible representation of $SL(2)$ (see also [11], Example 2.5 (c)). The invariant algebra is generated by

$$\phi = 18x_1x_2x_3x_4 - 27x_1^2x_4^2 - 4x_1x_3^3 + x_2^2x_3^2 - 4x_2^3x_4,$$

and the module of symmetric vector fields is generated by

$$f_1(x) = x, \quad f_2(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

with

$$\begin{array}{lcl} \psi_1(x) & = & -27x_1^2x_4 + 9x_1x_2x_3 - 2x_2^3, \\ \psi_2(x) & = & -27x_1x_2x_4 + 18x_1x_3^2 - 3x_2^2x_3, \\ \psi_3(x) & = & 27x_1x_3x_4 + 3x_2x_3^2 - 18x_2^2x_4, \\ \psi_4(x) & = & 27x_1x_4^2 + 2x_3^3 - 9x_2x_3x_4. \end{array}$$

Therefore $\mathbb{C}^4 = Z_2$, and from Theorem 3 one can determine rational first integrals. Computing the minors of $(f_1(x),\, f_2(x))$ one obtains, for instance,

$$\det \begin{pmatrix} x_1 & \psi_1 \\ x_2 & \psi_2 \end{pmatrix} = 2(3x_1x_3 - x_2^2)^2,$$

and continuing one finds by straightforward computations that Z_1 is the two-dimensional variety defined by $\rho_1(x) = \rho_2(x) = \rho_3(x) = 0$, with

$$\begin{array}{rcl} \rho_1 : & = & 3x_1x_3 - x_2^2 \\ \rho_2 : & = & 9x_1x_4 - x_2x_3 \\ \rho_3 : & = & 3x_2x_4 - x_3^2 \end{array}$$

and that every G-symmetric system admits rational first integrals ρ_i/ρ_j.

By Proposition 8, Proposition 5 and the fact that all semisimple elements of $\mathcal{L}$ are conjugate to a scalar multiple of B, the null cone (the zero set of ϕ) is equal to $\{G \cdot x : x_3 = x_4 = 0\}$. Every element v not in the null cone has closed orbit (Proposition 8), whence the minimal $\mathcal{D}_G$-invariant subspace equals the fixed point space of G_v by Panyushev's theorem.

Modulo the ideal generated by the ρ_i one has $x_2^2 \equiv 3x_1x_3$ and $x_3^2 \equiv 3x_2x_4$, hence

$$\begin{array}{rcl} \phi & \equiv & 18x_1x_2x_3x_4 - 27x_1^2x_4^2 - 4x_1x_3^3 + 3x_1x_3^3 - 12x_1x_2x_3x_4 \\ & \equiv & x_1\left(6x_2x_3x_4 - 27x_1x_4^2 - x_3^3\right) \\ & \equiv & x_1\left(3x_2x_3x_4 - 27x_1x_4^2\right) \\ & \equiv & -3x_1x_4\rho_2 \equiv 0 \end{array}$$

Therefore Z_1 is contained in the null cone, the subspace $\epsilon_v(\mathcal{D}_G)$ is two-dimensional for every v not in the null cone, and equals the level set of $(\rho_{i-1}/\rho_i,\, \rho_{i+1}/\rho_i)$ provided that $\rho_i(v) \neq 0$ (with indices modulo 3).

As a representative for a two-dimensional subspace contained in the null cone consider the $\mathcal{D}_G$-invariant space Y defined by $x_3 = x_4 = 0$. We will discuss the well-definedness condition (6). If $0 \neq w = (\sigma,\, \tau,\, 0,\, 0)^{\mathrm{tr}} \in Y$ and $T \in G$ then $Tw \in Y$ is equivalent to

$$b^2(-b\sigma + a\tau) = 0 \quad \text{and} \quad 3b^2d\sigma - (b^2c + 2abd)\tau = 0.$$

In any case, this forces $b = 0$, and therefore $Tw \in Y$ if and only if T is an element of the stabilizer $\widehat{G}$ of Y, which is characterized by the condition $b = 0$. A differential equation on Y is $\widehat{G}$-symmetric if and only if it is as follows, with constants α and β:

$$\begin{array}{rcl} \dot{x}_1 & = & \alpha x_1 - 2\beta x_2^3 \\ \dot{x}_2 & = & \alpha x_2 \end{array}$$

Inspection shows that every such equation can be extended to a G-symmetric differential equation on $\mathbb{C}^4$.

As a representative for a two-dimensional $\mathcal{D}_G$-invariant subspace not contained in the null cone consider the common zero set W of ρ_1 and ρ_3, which is obviously determined by $x_2 = x_3 = 0$. Again we consider the extension problem and the compatibility condition. If $0 \neq w = (\sigma,\, 0,\, 0,\, \tau)^{\mathrm{tr}} \in W$ and $T \in G$ then $Tw \in W$ is equivalent to

$$\begin{pmatrix} -3bd^2 & 3ac^2 \\ 3b^2d & -3a^2c \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Elements of the stabilizer $\widehat{G}$ of W are therefore characterized by either $b = c = 0$ (and $ad = 1$) or $a = d = 0$ (and $bc = -1$). The $\widehat{G}$-symmetric differential equations on W are given by

$$\begin{array}{rcl} \dot{x}_1 & = & \left(\mu(x_1^2x_4^2) + x_1x_4\nu(x_1^2x_4^2)\right)x_1 \\ \dot{x}_4 & = & \left(\mu(x_1^2x_4^2) - x_1x_4\nu(x_1^2x_4^2)\right)x_4 \end{array}$$

with arbitrary polynomials μ and ν in one variable. In this case, there exist $T \notin \widehat{G}$ which send some nonzero element of W to W: Indeed, the determinant of the matrix in the defining condition is equal to $9abcd$, and therefore every T with one entry zero works. For a corresponding w, one the other hand, one always finds $\sigma = 0$ or $\tau = 0$, and condition (6) for such T and w turns out to be satisfied by every $\widehat{G}$-symmetric vector field on W. Again, inspection shows that every such vector field is the restriction of some G-symmetric vector field on $\mathbb{C}^4$.

B. Second, we consider the five-dimensional irreducible representation of $SL(2)$, using results on invariants from [7] and the procedure outlined in [11]. The invariant algebra is generated by the two polynomials

$$\begin{array}{rcl} \phi_1 & = & 12x_1x_5 - 3x_2x_4 + x_3^2 \\ \phi_2 & = & 72x_1x_3x_5 - 27x_1x_4^2 - 2x_3^3 - 27x_2^2x_5 + 9x_2x_3x_4, \end{array}$$

and the module of $\mathcal{D}_G$-invariant vector fields is generated by the two elements

$$f_1(x) = x; \quad f_2(x) = \begin{pmatrix} 8x_1x_3 - 3x_2^2 \\ 24x_1x_4 - 4x_2x_3 \\ 48x_1x_5 + 6x_2x_4 - 4x_3^2 \\ 24x_2x_5 - 4x_3x_4 \\ 8x_3x_5 - 3x_4^2 \end{pmatrix}$$

Therefore $\mathbb{C}^5 = Z_2$. From the minors of $(f_1(x),\, f_2(x))$ one finds the polynomials

$$\begin{array}{rcl} \rho_1(x) & = & 4x_3x_4x_5 - x_4^3 - 8x_2x_5^2 \\ \rho_2(x) & = & -2x_2x_4x_5 + 4x_3^2x_5 - x_3x_4^2 - 16x_1x_5^2 \\ \rho_3(x) & = & -8x_1x_4x_5 + 4x_2x_3x_5 - x_2x_4^2 \\ \rho_4(x) & = & -x_1x_4^2 + x_2^2x_5 \\ \rho_5(x) & = & 8x_1x_2x_5 - 4x_1x_3x_4 + x_2^2x_4 \\ \rho_6(x) & = & 2x_1x_2x_4 - 4x_1x_3^2 + x_2^2x_3 + 16x_1^2x_5 \\ \rho_7(x) & = & -4x_1x_2x_3 + x_2^3 + 8x_1^2x_4 \end{array}$$

whose common zero set is the three-dimensional variety Z_1, and which provide rational first integrals ρ_i/ρ_j. By Proposition 8, Proposition 5 and the fact that all semisimple elements of $\mathcal{L}$ are conjugate to a scalar multiple of B, the null cone (the common zero set of ϕ_1 and ϕ_2) is equal to $\{G \cdot x : x_3 = x_4 = x_5 = 0\}$, and the set of elements which satisfy the conclusion of Proposition 8 is equal to $\{G \cdot x : x_4 = x_5 = 0\}$. Every element v not in this set has closed orbit, whence the minimal $\mathcal{D}_G$-invariant subspace equals the fixed point space of G_v by Panyushev's theorem.

We also discuss the dynamics on some $\mathcal{D}_G$-invariant subspaces. All elements on $W_1 = \{x : x_4 = x_5 = 0\}$ satisfy the conclusion of Proposition 8, and the restriction of any symmetric differential equation to W_1 is "triangular", thus

$$\begin{aligned}\dot{x}_1 &= \mu_1(x_1, x_2, x_3)\\ \dot{x}_2 &= \mu_1(x_2, x_3)\\ \dot{x}_3 &= \mu_1(x_3)\end{aligned}$$

with suitable μ_i. On the other hand, the restriction of a G-symmetric differential equation to $W_2 = \{x : x_2 = x_4 = 0\}$ has the form

$$\begin{aligned}\dot{x}_1 &= \sigma \cdot x_1 + \tau \cdot 8x_3x_1\\ \dot{x}_3 &= \sigma \cdot x_3 + \tau \cdot (48x_1x_5 - 4x_3^2)\\ \dot{x}_5 &= \sigma \cdot x_5 + \tau \cdot 8x_3x_5\end{aligned}$$

with polynomials σ and τ in ϕ_1 and ϕ_2 (restricted to W_2). The identity component $\widehat{G}_0$ of the stabilizer subgroup of W_2 consists of all transformations induced by the diag$(a, a^{-1}) \in SL(2)$, and therefore is one-dimensional. By (ϕ_1, ϕ_2) one obtains reduction to dimension two. If one considers the minimal subspace $\epsilon_v(\mathcal{D}_G)$, e.g. for $v = e_1 + e_5$, one finds that it is two-dimensional with finite stabilizer subgroup. Therefore symmetry induces no further reduction of dimension.

Finally, we look at the $\mathcal{D}_G$-invariant one-dimensional space $V = \langle e_3 \rangle$. By a straightforward calculation, an element of G maps a nonzero element of V to V if and only if either $b = c = 0$ (and $ad = 1$) or $a = d = 0$ (and $bc = -1$). The same condition defines the stabilizer subgroup $\widehat{G}$ of V. Moreover the restriction of $\widehat{G}$ acts trivially on V, whence every vector field on V is $\widehat{G}|_V$-symmetric. Since nonzero constant vector fields on V cannot be extended to G-symmetic vector fields on $\mathbb{C}^5$, the symmetry condition alone on V is not sufficient for extendability. But the compatibility condition (6) also implies that $T\tilde{f}(0) = \tilde{f}(0)$ for all $T \in G$, and therefore $\{0\}$ is an invariant set for every extendable vector field. Inspection shows that every vector field on V which stabilizes 0 is the restriction of some G-symmetric vector field on $\mathbb{C}^5$.

Appendix

A. *Proof of Lemma 5.* Given $\psi_1, \ldots, \psi_r \in \mathbb{K}[x_1, \ldots, x_n]$ and $\mu_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$ such that

$$L_f(\psi_j) = \sum_k \mu_{jk}\psi_k, \quad 1 \le j \le r$$

we have to show that the set Y of common zeros of the ψ_j is invariant for $\dot{x} = f(x)$. Thus let $v \in Y$, and abbreviate $z(t) = \Phi(t, v)$. Then for $1 \le j \le r$ one has

$$\frac{d}{dt}\psi_j(z(t)) = L_f(\psi_j)(z(t) = \sum_k \mu_{jk}(z(t))\psi_k(z(t))$$

and thus $(\psi_1(z(t)), \ldots, \psi_r(z(t)))^{\mathrm{tr}}$ satisfies a homogeneous linear system of differential equations with matrix $(\mu_{jk}(z(t)))$ and initial value 0. By the uniqueness theorem, this solution is identically zero, whence $z(t) \in Y$ for all t. Part (a) is therefore proven, and one direction of part (b) is an immediate consequence. The reverse direction of part (b) follows from the expansion

$$\Phi(t, w) = w + t \cdot f(w) + \cdots,$$

whence $\Phi(t, w) \in W$ for all t implies $f(w) \in W$.

B. *Invariants and symmetric vector monomials for algebraic tori:* Let G be a complex algebraic torus, with Lie algebra $\mathcal{L}$.
(i) Auxiliary result: Given complex numbers $\lambda_1, \ldots, \lambda_s$ and α, consider all tuples $(m_1, \ldots, m_s)$ of nonnegative integers such that

$$m_1\lambda_1 + \cdots + m_s\lambda_s = 0 \tag{9}$$

and all tuples $(n_1, \ldots, n_s)$ of nonnegative integers such that

$$n_1\lambda_1 + \cdots + n_s\lambda_s = \alpha. \tag{10}$$

For problem (9) there exist finitely many solutions such that every solution is a nonnegative integer linear combination of these. For problem (10), provided it is solvable, there exist finitely many solutions such that every solution is a sum of one of these and an arbitrary solution of Problem (9). (See [22], Proposition 1.6. This fact is related to Dickson's Lemma in Commutative Algebra; see e.g. Cox et al. [6].)
(ii) Let $C_1, \ldots, C_r$ be a vector space basis of $\mathcal{L}$. From the eigenvalues $\lambda_1, \ldots, \lambda_n$ of C_1 determine monomial generators for the C_1-invariant algebra via $\sum d_i\lambda_i = 0$ (Problem (9)). Assuming that monomial generators $\phi_{t,1}, \ldots, \phi_{t,\ell_t}$ for the invariant algebra of $\mathbb{C}C_1 + \cdots \mathbb{C}C_t$ are known, these will be eigenfunctions for $L_{C_{t+1}}$ with eigenvalues $\lambda_{t+1,1}, \ldots, \lambda_{t+1,\ell_t}$. If a monomial $x_1^{m_1} \cdots x_n^{m_n}$ is a joint invariant of $C_1, \ldots, C_{t+1}$ then, being a joint invariant of $C_1, \ldots, C_t$, it can be written in the form

$$\phi_{t,1}^{d_1} \ldots \phi_{t,\ell_t}^{d_{\ell_t}},$$

and this is an invariant of C_{t+1} if and only if $\sum d_i\lambda_{t+1,i} = 0$, which again leads to Problem (9). Thus one finds generating monomials for the invariant algebra.
(iii) For vector monomials, consider $x_1^{m_1} \cdots x_n^{m_n} e_j$, with j fixed. For C_1 the condition is then $\sum d_i\lambda_i = \lambda_j$. By Problem (10) there are finitely many monomials $\psi_{1,1}e_j, \ldots, \psi_{1,m_1}e_j$ which generate the module over the algebra of C_1-invariants. Assuming that monomial generators $\phi_{t,1}, \ldots, \phi_{t,\ell_t}$ for the invariant algebra and $\psi_{t,1}e_j, \ldots, \psi_{t,m_t}e_j$ for the corresponding module with respect to $\mathbb{C}C_1 + \cdots \mathbb{C}C_t$ are given, these will be eigenfunctions for $L_{C_{t+1}}$ with eigenvalues $\lambda_{t+1,1}, \ldots, \lambda_{t+1,\ell_t}$, resp. $\mu_{t+1,1}, \ldots, \mu_{t+1,m_t}$. Now a vector monomial

$$\phi_{t,1}^{d_1} \ldots \phi_{t,\ell_t}^{d_{\ell_t}} \psi_{t+1,k} e_j$$

is symmetric for $C_1, \ldots, C_{t+1}$ if and only if $\sum d_i\lambda_{t+1,i} = -\mu_{t+1,k} + \alpha_{t+1,j}$, with $\alpha_{t+1,j}$ the eigenvalue of C_{t+1} corresponding to e_j. This again leads to Problem (10).

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