Minima of invariant functions: The inverse problem

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Abstract

We determine locally minimizing functions that are invariant with respect to the action of a finite linear group. This resolves a problem which is inverse to one discussed in a seminal paper by Abud and Sartori, and occurs naturally in various physical applications, such as elasticity theory and phase transitions. A general existence result reduces the local problem to elementary computations. Some results are extended to the compact case, and some examples and applications are given.

1 Introduction and overview

Minima of functions that are invariant under a (compact) group action have been in the focus of theoretical as well as applied physics for several decades. In particular, symmetry breaking at critical points was identified as a crucial mechanism to explain a number of physical phenomena. Several fundamental contributions to the underlying mathematical theory culminated in papers by Michel and Radicati [12], Michel [13], Michel and Zhilinskii [14], and Abud and Sartori [1, 2] (the latter will be our basic reference). Abud and Sartori [2] passed to the orbit space (realized as a semi-algebraic variety via the Hilbert map) and resolved the problem of finding minima on strata (which are submanifolds) via the Lagrange multiplier method. In the present paper, we will discuss the inverse problem; viz. to position a minimum of a G-invariant function at some prescribed point. This inverse problem is relevant, for instance, in the modelling and analysis of strain-energy density functions in elasticity theory (see e.g. Coleman and Noll [5], Smith and Rivlin [17])). For readers who are less familiar with this area of research, we give a brief account of the relevant physical and mathematical background.

For materials which are perfectly elastic (hyperelastic), at a given temperature, the strain-energy density is a smooth function over the set of all deformation gradients or rather over all Cauchy-Green strain tensors. The latter are symmetric and positive definite. Critical points of this function correspond to stress-free configurations of the material. At (local) minima, these are stable in the sense that typically lower energies are preferred. If the material possesses symmetry, i.e. the underlying crystal lattice is invariant under certain symmetry transformations, the strain-energy function is invariant under the action of the corresponding symmetry group on the set of all Cauchy-Green strain tensors by conjugacy. This imposes restrictions on the shape of the strain-energy function depending on the type of symmetry. For example, the strain energy density is constant along any orbit of the symmetry group. In particular, all the points of the group orbit of a minimum correspond to energetically equivalent stress-free configurations of the material. The smaller the isotropy of a minimum is, the bigger is the corresponding orbit of the symmetry group, i.e., the more energetically equivalent stress-free configurations of the material exist. The symmetry group for an isotropic elastic material is the full rotation group $SO(3)$, otherwise the material is called anisotropic.

Certain materials can undergo phase transformations. For instance, so-called martensitic phase transformations have been observed in various metals, alloys and even ceramics. They are diffusionless, solid-to-solid phase transformations under applied stresses and/or temperature changes, accompanied by the development of a rich microstructure which might be explained by a change in crystal symmetry during the transition. Martensitic phase transformations can be irreversible, as seen in steels upon quenching, or they can be reversible, such as those observed in shape-memory alloys (cf. Bhattacharya et al. [3]). Generally, under stress free conditions, the highest symmetry of the crystal structure, referred to as the austenite phase, is preferred by these alloys at higher temperatures. At lower temperatures, crystal structures with a relatively lower symmetry, referred to as the martensite phases, have lower energy and are preferred. So, the transition from austenite to martensite phases as the temperature decreases through the transition temperature, is accompanied by a loss of symmetry. At a martensite phase, the micro- as well as the macrostructure of the material is easily deformed by loads. Upon reheating, the deformations disappear. In the framework of thermoelasticity, such phase transitions are frequently described as changes in the internal energy or (Helmholtz) free energy density (cf. Landau [11]). These are functions of the strain as well as of thermodynamic variables such as temperature. So, one might think of them as being parameter-dependent strain-energy functions. At any value of the temperature, they possess minima for each of the stable phases. Since martensite phases do not have maximal isotropy, they are never unique. Again, the inverse problem addressed in the present paper is relevant here. Moreover, questions concerning parameter-dependence of the minima such as symmetry breaking arise (cf. Falk and Konopka [7] and Zimmer [21]).

The plan of the present paper is as follows. Given an action of a finite linear group G on $\mathbb{R}^n$, we first discuss the problem to determine a G-invariant (smooth) function ψ that admits a minimum at a prescribed point z (and hence at its whole orbit), and we solve this problem for locally minimizing functions. As will be shown in Section 2 there actually exist such functions which are robust with respect to any smooth (not necessarily G-invariant) small perturbations, since their Hessians at the local minimum positions are positive definite. (By "robustness" within a given function space we mean here that any small perturbation of the given function will admit a minimum nearby.) Proposition 1 opens an elementary path to a systematic determination of all functions that have the required properties.

In Section 3 we proceed to discuss parameter-dependent G-invariant functions and symmetry breaking. We do not specifically consider physically relevant parameters, i.e., we ignore physical restrictions for the parameter dependence. Rather, our aim is two-fold. On the one hand, we consider critical points emanating smoothly from a point of highest symmetry, and discuss possible geometric restrictions on that curve of critical points due to the group invariance (direct approach). On the other hand, we consider the restriction which a given curve of critical points generally imposes on the parameter-dependence of the underlying G-invariant function (complementary approach).

The final Section 4 is motivated by the modelling and analysis of phase transitions in crystals, particularly shape-memory alloys. We discuss problems considered by Falk and Konopka [7] and Zimmer [21], who presented some partial solutions. Here we will show that, and how, all robust strain-energy functions which have a local minimum at a given position for prescribed

isotropy can be computed, and give explicit examples. There seems to be no such comprehensive discussion in the literature. Moreover we discuss parameter-dependent functions and symmetry-breaking phenomena for the underlying group action.
In this paper we focus on finite groups and, in the spirit of a tutorial, mostly give rather detailed proofs and record elementary steps in computations for that case. In Section 2 generalizations to compact groups are mentioned briefly, with sketches of proofs. The main results of Section 3 are stated and proved for compact linear group actions, but in their geometric interpretation and in examples we consider finite groups only.

2 Locally minimizing functions

We consider a finite group $G \subseteq GL(n, \mathbb{R})$ acting on $\mathbb{R}^n$; we may and will assume that G is a subgroup of the orthogonal group $O(n, \mathbb{R})$. Given a point $z \in \mathbb{R}^n$, our objective is to find group-invariant functions (of sufficiently high differentiability) which admit a local minimum at z, but no other local minima in some neighborhood of z. By invariance, there necessarily exist local minima at all points of the orbit $G \cdot z$.
It is natural to construct such minimizing functions from a Hilbert basis of the polynomial invariant ring; a well-known theorem by G.W. Schwarz [16] implies that by this approach one obtains all smooth minimizing functions (see also Rumberger [15] for the finitely differentiable case). But a priori it seems not obvious that there is a (uniform) degree bound for the Taylor expansion of a group-invariant function that guarantees a minimum for any group-invariant higher-order perturbation. We will show with an elementary argument that there exist group-invariant minimizing functions which are locally robust with respect to arbitrary (not necessarily group-invariant) perturbations, and that Taylor expansion up to degree two suffices. We start with an auxiliary result.

Lemma 1. *Let G be a finite subgroup of $GL(n, \mathbb{R})$. Given $z \in \mathbb{R}^n$, there exists a G-invariant C^∞ function ψ which has local and global minima at the points of the orbit $G \cdot z$, and there is a neighborhood of the orbit which contains no other local minima. Moreover, the Hessian of ψ at every point of $G \cdot z$ is positive definite; thus the minima are robust with respect to C^2 perturbations.*

Proof. We denote by $\| \cdot \|$ the Euclidean norm, and for $r > 0$ we let $K_r(a)$ be the open ball with center a and radius r. Given $z \in \mathbb{R}^n$, denote by G_z

the isotropy subgroup of z.
(i) There exists $\rho > 0$ (we may assume $\rho < 1$) such that

$$G_x \subseteq G_z \text{ for all } x \in K_\rho(z) \tag{1}$$

and

$$\overline{K_\rho(Tz)} \cap \overline{K_\rho(Sz)} = \emptyset \text{ whenever } T,\, S \in G, \quad Tz \neq Sz. \tag{2}$$

Here (2) is obvious from the finiteness of the orbit. To prove (1), assume that there is a sequence (x_ℓ) converging to z and for every ℓ there exists $T_\ell \in G_{x_\ell} \setminus G_z$. Since G is finite, there is an $S \in G$ such that $T_\ell = S$ for infinitely many ℓ. Taking the limit shows $Sz = z$; a contradiction.
(ii) There exists a C^∞ function $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \sigma(t) = 1 - t, \quad t < \rho^2/2, \\ \sigma(t) = 0, \quad t > \rho^2, \end{aligned}$$

and $\sigma'(t) < 0$ for all $t < \rho^2$. Define

$$\mu(x) := \sigma(\|x - z\|^2) \tag{3}$$

and note that for all $T \in G$ one has

$$\mu(Tx) = \sigma(\|Tx - z\|^2) = \sigma(\|x - T^{-1}z\|^2) \tag{4}$$

by orthogonality; in particular $\mu(Rx) = \mu(x)$ for all $R \in G_z$.
(iii) Now define

$$\psi(x) := -\frac{1}{|G_z|} \sum_{T \in G} \mu(Tx). \tag{5}$$

Then ψ is a G-invariant C^∞ function (by construction) with the following properties:

- $\psi(x) = 0$ for all $x \notin \bigcup_{T \in G} K_\rho(Tz)$.
- If $Sy \in K_\rho(z)$ then

$$\psi(y) = -\frac{1}{|G_z|} \sum_{R \in G_z} \mu(RSy) = -\mu(Sy).$$

- There are robust local minima at the points of the orbit $G \cdot z$, and only there.
- For all $x \in \mathbb{R}^n \setminus G \cdot z$ one has $0 \geq \phi(x) > -1$; therefore the points of $G \cdot z$ are also global minima.

The first and last of these are immediate. For the second one may assume $S = E$, due to G-invariance. Then $Ty \in K_\rho(z)$ if and only if $T \in G_z$, due to (2), then (4) shows the assertion. To prove the third claim, by invariance it suffices to consider $x \in K_\rho(z)$. For these x one finds

$$\psi(x) = -\mu(x) = -\sigma(\|x - z\|^2), \quad D\psi(x) = -2\sigma'(\|x - z\|^2) \cdot (x - z)^{\mathrm{tr}} .$$

Therefore z is the only critical point in $K_\rho(z)$, and $D^2\psi(z) = 2E$ is positive definite.

□

There is an immediate application to constructive polynomial approximation.

Proposition 1. *Let G be a finite subgroup of $GL(n, \mathbb{R})$, with a generator system $\gamma_1, \ldots, \gamma_s$ for the algebra of G-invariant polynomials, denote by $\Gamma := (\gamma_1, \ldots, \gamma_s)^{\mathrm{tr}}$ the corresponding Hilbert map, and let $z \in \mathbb{R}^n$.*

(a) There exist smooth, G-invariant locally minimizing functions with positive definite Hessian at z.

(b) The ansatz

$$\theta = \sum_i \mu_i \left(\gamma_i - \gamma_i(z)\right) + \sum_{i,j} \nu_{ij} \left(\gamma_i - \gamma_i(z)\right) \left(\gamma_j - \gamma_j(z)\right) \qquad (6)$$

with real parameters μ_i and $\nu_{ij} (= \nu_{ji})$ will yield all such smooth locally minimizing functions with $\theta(z) = 0$, modulo terms of order > 2 in $x - z$.

(c) The parameter sets $(\mu_i,\, \nu_{jk})$ which determine a minimum with positive definite Hessian form a nonempty semialgebraic subset of $\mathbb{R}^s \times \mathbb{R}^{(s,s)}$.

Proof. Part (a) follows from Lemma 1. Concerning part (b), let the smooth function ψ admit a minimum with positive definite Hessian at z. Due to Schwarz [16] one has

$$\psi = \eta(\gamma_1 - \gamma_1(z), \ldots, \gamma_s - \gamma_s(z))$$

with a smooth function η in s variables. With $w_i = \gamma_i(z)$, Taylor expansion yields

$$\eta(y) - \eta(w) = \sum_i \mu_i \left(y_i - w_i\right) + \sum_{i,j} \nu_{ij} \left(y_i - w_i\right) \left(y_j - w_j\right) + \widetilde{R}(y).$$

Given a compact and convex neighborhood $\widetilde{K}$ of w there is a positive constant M such that $\|\widetilde{R}(y)\| \leq M \cdot \|y-w\|^3$ for all y in $\widetilde{K}$. Let K be a compact and convex neighborhood of z so that $\Gamma(K) \subseteq \widetilde{K}$, and let $L > 0$ so that $\|\Gamma(x) - \Gamma(x^*)\| \leq L\|x - x^*\|$ for all x, $x^* \in K$. Thus, with $\psi(z) = 0$ one has

$$\psi = \sum_i \mu_i \left(\gamma_i - \gamma_i(z)\right) + \sum_{i,j} \nu_{ij} \left(\gamma_i - \gamma_i(z)\right) \left(\gamma_j - \gamma_j(z)\right) + R$$

with $\|R(x)\| \leq ML^3\|x - z\|^3$ for all $x \in K$. This shows that the gradient and the Hessian of R vanish at z.
To prove part (c), note that the condition

$$0 = \operatorname{grad} \psi(z) = \sum \mu_i \operatorname{grad}\gamma_i(z) \tag{7}$$

defines a system of linear equations for the μ_i. Likewise, the entries of the Hessian at z are linear in the coefficients μ_i and ν_{jk}, so the Hurwitz conditions on positivity of all principal minors yield polynomial inequalities. These equations and inequalities define a semi-algebraic set. □

Remark 1. (a) One thus obtains all robust G-invariant locally minimizing functions (at any fixed z) from an ansatz with degree two polynomials in the γ_i, modulo higher order terms. It should be noted that (6) does not (and is not intended to) provide all G-invariant polynomials of degree $\leq 2M$, where M is the maximal degree of the γ_i. But the ansatz includes all G-invariant polynomials which provide nontrivial contributions to the gradient or the Hessian at z.

(b) As mentioned above, Rumberger [15] showed the existence of a positive integer q (depending only on the group action) such that every G-invariant function ψ of class $C^{m \cdot q}$ admits a representation $\psi = \eta \circ \Gamma$ with a C^m function η. From this result one obtains a version of the Proposition for functions of differentiability class C^{3q}.

Remark 2. A substantial part of the results in Lemma 1 and Proposition 1 remains true for compact linear group actions.Thus let $G \subseteq O(n, \mathbb{R})$ be a compact linear group, with identity component G^o, and $z \in \mathbb{R}^n$ with orbit $G \cdot z$ of dimension d. Since any G-invariant function is constant on orbits, a minimum at z cannot be robust with respect to arbitrary perturbations whenever $d > 0$, as the rank of the Hessian is $\leq n - d$. But there exist G-invariant functions which have a local minimum at z, with positive semidefinite Hessian of rank $n - d$. Thus robustness with respect to G-invariant perturbations holds. Moreover one will find all locally minimizing

functions via the ansatz (6).
As for a sketch of the proof, Lemma 1 can be modified to show the existence of a G-invariant smooth function ψ which has a local and global minimum at $G \cdot z$, with positive semidefinite Hessian of rank $n - d$.
Essential for proving this modification are some facts about orbits of compact groups. For our purpose, the account in Abud and Sartori [2], Section IV is quite appropriate; for more background see Bredon [4]. Given z, one has a well-defined function

$$x \mapsto d(x,\, G^o \cdot z)^2 := \min_{T \in G^o} \|x - Tz\|^2$$

which is smooth (in a neighborhood of $G^o \cdot z$) and G-invariant. For sufficiently small $\rho > 0$ denote by

$$S_\rho(z) := \left\{ x;\, d(x,\, G^o \cdot z)^2 < \rho^2 \right\}$$

a tubular neighborhood of the orbit $G^o \cdot z$. Then analogues to (1) and (2) hold for suitable tubular neighborhoods. Now one may define

$$\mu(x) := \sigma(d(x,\, G^o \cdot z)^2)$$

and furthermore define ψ by summation over a system of representatives for G/G^o, and normalization. The assertion about the rank of the Hessian follows from the slice theorem (see [2], Section IV).

3 Parameter-dependent minimizing functions

In applications, the focus of interest is often on G-invariant functions that depend on a parameter, and one is led to the question how minima (more generally, critical points) evolve with changing parameter values. The results of the previous section resolve the local problem for any fixed parameter value, but special phenomena occur when minima move between strata, concerning e.g. smoothness with respect to the parameter.
We recall some pertinent facts from Abud and Sartori [2]; no elementary proof seems available for these.

Lemma 2. *Let $G \subseteq O(n, \mathbb{R})$ be a finite group. Given a Hilbert basis $\gamma_1, \ldots, \gamma_s$ of the invariant algebra, the fixed point space of G_z is spanned by the gradients of the γ_i, evaluated at z. Denoting by $\Gamma = (\gamma_1, \ldots, \gamma_s)^{\mathrm{tr}}$ the corresponding Hilbert map, one has in particular*

$$U = \{x \in \mathbb{R}^n;\, G_x \text{ is minimal}\} = \{x \in \mathbb{R}^n;\, \operatorname{rank} D\Gamma(x) = n\},$$

and the union of the non-principal strata is the complement of U.

Sketch of proof. This follows from [2] (VIII.P1) on p. 336 together with equation (5.6) on p. 325, noting that all the group dimensions equal zero in our scenario. $\square$

3.1 A direct approach

In this subsection we consider critical points emanating smoothly from a point of highest symmetry and discuss possible geometric restrictions due to group invariance. We consider compact (not necessarily finite) group actions, but specialize the setting a little more via the following assumptions.

- G is a compact subgroup of $O(n,\, \mathbb{R})$ which acts irreducibly on $\mathbb{R}^n$.
- $(\gamma_1, \ldots, \gamma_s)$ is a (minimal) homogeneous generating system for the invariant algebra, ordered by ascending degree.

By irreducibility, the minimal degree is two, and there is, up to scalar multiples, only one invariant of degree two, viz. $\gamma_1(x) = \|x\|^2$. We recall a proof: Let ϕ be a homogeneous invariant of degree two. There is some $\alpha \in \mathbb{R}$ such that the symmetric bilinear form β corresponding to the quadratic form $\alpha\gamma_1 - \phi$ is degenerate, since the symmetric matrix representing ϕ admits real eigenvalues. Thus the subspace

$$W := \{v;\, \beta(x, v) = 0 \text{ for all } x \in \mathbb{R}^n\}$$

is nonzero and G-invariant. Irreducibility forces $W = \mathbb{R}^n$.

We now consider a smooth G-invariant function

$$\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \quad \psi(x,\, t) = \eta(\gamma_1(x), \ldots, \gamma_s(x),\, t) \tag{8}$$

with η a smooth function of $s+1$ variables. Then ψ has a critical point at 0 and this is the only point with isotropy group G, by our assumptions. Let a smooth curve

$$t \mapsto z(t), \quad z(0) = 0, \quad z'(0) := v \neq 0 \tag{9}$$

be defined in some neighborhood of $t = 0$, such that each $z(t)$ is a critical point of $\psi(\cdot,\, t)$. Taking the gradient we obtain a criticality condition that can be written as

$$0 = \operatorname{grad} \psi(z(t),\, t) = \sum_{i=1}^{s} \mu_i(t) \operatorname{grad} \gamma_i(z(t)) \quad \text{for all } t \text{ near } 0, \tag{10}$$

with the abbreviation

$$\mu_i(t) = D_i\eta\,(\gamma_1(z(t)), \ldots, \gamma_s(z(t)), t)\,, \quad 1 \le i \le s,$$

the D_i denoting partial derivatives. There is no a priori restriction on the μ_i at $t = 0$, since all gradients vanish at 0. The condition $z'(0) \neq 0$ may have nontrivial consequences, however.

Proposition 2. *Let $d = \deg(\gamma_2)$ be the second smallest degree among the γ_i, and assume that (9) defines a smooth curve of critical points for the parameter-dependent G-invariant function ψ.*

(a) Then $\mu_1(0) = 0$, and all derivatives of μ_1 up to order $d - 3$ vanish at $t = 0$.

(b) The following identity holds:

$$2\mu_1^{(d-2)}(0) \cdot v + (d-1)! \sum_{i:\, \deg \gamma_i = d} \mu_i(0) \cdot \operatorname{grad} \gamma_i(v) = 0.$$

Proof. For the proof it is convenient to rewrite condition (10) in the equivalent form

$$0 = \sum_{i=1}^{s} \mu_i(t)\, D\gamma_i(z(t)), \quad \text{for all } t \text{ near } 0.$$

(i) We recall a version of Euler's identity: Given a homogeneous polynomial map F between finite dimensional vector spaces, of degree $r \ge 1$, the following properties hold for the k-th derivatives (which are considered as k-multilinear symmetric maps):

$$D^k F(0) = 0 \text{ for all } k < r, \quad D^r F(0)(y, \ldots, y) = r!\, F(y) \text{ for all } y.$$

Using these properties for the $D\gamma_i$, we obtain for γ_j of degree r_j the identites

$$D^{r_j - 1}\gamma_j(0)(y, \ldots, y, \cdot) = (r_j - 1)!\, D\gamma_j(y)$$

and $D^{k-1}\gamma_j(0) = 0$ for all $k < r_j$.
(ii) Differentiation yields

$$0 = \sum_{i=1}^{s} \mu_i'(t) D\gamma_i(z(t)) + \sum_{i=1}^{s} \mu_i(t) D^2\gamma_i(z(t))\,(z'(t), \cdot)$$

Recalling that only γ_1 has degree 2, substitution of $t = 0$, $z(0) = 0$ yields with (i) that

$$0 = \mu_1(0) D\gamma_1(v),$$

hence $\mu_1(0) = 0$ due to $v \neq 0$.
(iii) By further differentiation one obtains

$$\begin{array}{rcl} 0 & = & \sum_{i=1}^s \mu_i''(t)\, D\gamma_i(z(t)) \\ & & +2\sum_{i=1}^s \mu_i'(t)\, D^2\gamma_i(z(t))\,(z'(t),\,\cdot) \\ & & +\sum_{i=1}^s \mu_i(t)\, D^2\gamma_i(z(t))\,(z''(t),\,\cdot) \\ & & +\sum_{i=1}^s \mu_i(t) D^3\gamma_i(z(t))\,(z'(t),\, z'(t),\,\cdot) \end{array}$$

At $t = 0$ the first term on the right-hand side vanishes for degree reasons, and so does the third due to $\mu_1(0) = 0$. The second term reduces to $\mu_1'(0)D\gamma_1(v)$, and by Euler the last term equals

$$2 \sum_{i:\, \deg\gamma_i=3} \mu_i(0) \cdot D\gamma_i(v).$$

In case $d = 3$ one has the assertion of (b) (after rewriting the result for gradients); in case $d > 3$ one sees that $\mu_1'(0) = 0$.
(iv) The assertion for $d > 3$ follows in an analogous manner, by successive differentiation up to order d and comparing terms. □

For the setting of the present subsection, Proposition 2 thus imposes two conditions for a curve of critical points to emanate from a minimum at 0. First, the Hessian at 0 must be trivial for the critical parameter value (loss of robustness for the minimum at 0). Second, since the $\mu_i(0)$, $i > 1$, are a priori arbitrary, the linear dependence of $\operatorname{grad}\gamma_1(v) = 2v$ and the $\operatorname{grad}\gamma_i(v)$ for those i with $\deg\gamma_i = d$ may provide nontrivial restrictions on v. As for examples and applications we return to finite groups.

Example. (a) Let $m > 2$ be an integer and D_m the dihedral group acting naturally on $\mathbb{R}^2$ as symmetry group of the regular m-gon. The invariant algebra is generated by

$$\gamma_1(x) = x_1^2 + x_2^2, \quad \gamma_2(x) = \mathrm{Re}\,(x_1 + ix_2)^m.$$

The nontrivial isotropy subspaces are just the reflection axes, and by Lemma 2 the points on these are characterized by the linear dependence of $\operatorname{grad}\gamma_1(v)$ and $\operatorname{grad}\gamma_2(v)$. According to the Proposition, smooth curves of critical points emanating from 0 will "generically" be tangent to a reflection axis.

(b) On the other hand let C_m be the subgroup of rotations. The invariant algebra is now generated by

$$\gamma_1(x) = x_1^2 + x_2^2, \quad \gamma_2(x) = \mathrm{Re}\,(x_1 + ix_2)^m, \quad \gamma_2(x) = \mathrm{Im}\,(x_1 + ix_2)^m,$$

and while Proposition 2(b) provides nontrivial conditions relating v and the μ_i at 0, there is no general restriction to possible tangent directions.

The reader may be reminded of similar phenomena related to the equivariant branching lemma, in particular the cube deformation example in Golubitsky et al. ([8] XI, §1(b) and XIII §3). The underlying mathematical arguments are quite similar indeed.

Example. The case of finite reflection groups (which includes D_m) is of particular interest. This class of groups is distinguished by the property that their invariant algebras admit an algebraically independent system of n generators (see Springer [19], Ch. 4, Grove and Benson [9], Solomon [18] for their properties). Thus let G be a finite reflection group acting irreducibly on $\mathbb{R}^n$, and let $\psi(x,t)$ and $z(t)$ be as in Proposition 2. Unless $\mu_i(0)=0$ for all i with $\deg\gamma_i = d$, the tangent vector v lies in some nontrivial isotropy subspace. To see this, recall Lemma 2 and note that linear dependence of some $\operatorname{grad}\gamma_i(v)$ already implies non-invertibility of $D\Gamma(v)$, as the number of generators equals the dimension of the vector space. We will return to this example later.

3.2 A complementary approach

The approach in the previous subsection provides necessary conditions for the curve $z(t)$, given the parameter functions $\mu_i(t)$. Here, we change perspective and consider the restrictions that a given curve $z(t)$ of critical points imposes on the parameter functions μ_i. We assume the parameter functions to be analytic in this subsection; our results are also applicable to the finitely differentiable (with Taylor expansion up to some degree) or smooth case (modulo flat terms), with some restrictions and modifications.
We consider a compact (not necessarily finite) linear group action on $\mathbb{R}^n$, with no further requirements. Letting $\gamma_1,\ldots,\gamma_s$ be a Hilbert basis of the invariants, we restate equation (10) in the form

$$A(t)\cdot M(t) = 0; \tag{11}$$

$$A(t) := \big(\operatorname{grad}\gamma_1(z(t)),\ldots,\operatorname{grad}\gamma_s(z(t))\big),\quad M(t) := \begin{pmatrix}\mu_1(t)\\ \vdots \\ \mu_s(t)\end{pmatrix} \tag{12}$$

Since (10) arises by taking the gradient of (8) one should ascertain that there exist no additional restrictions on the μ_i. To verify this, consider the particular function $\psi(x,\,t) = \sum \mu_i(t)\gamma_i(x)$.

We denote the highest rank of $A(t)$, with t near 0, by r; the case of interest is $\operatorname{rank} A(0) < r$. Moreover we are interested in nonzero M, thus we may assume that $M(0) \neq 0$, after possibly dividing by a power of t. There is an invertible $n \times n$ matrix $Q(t)$ such that

$$Q(t) \cdot A(t) = \begin{pmatrix} E_r & B(t) \\ 0 & 0 \end{pmatrix}$$

with the $r \times r$ identity matrix E_r and an $r \times (n-r)$ matrix

$$B(t) = t^{-m} \left(\sum_{k \geq 0} \beta_{ij}^{(k)} t^k \right)_{r+1 \leq i \leq s,\, 1 \leq j \leq r}$$

for which $m \in \mathbb{Z}$ is chosen minimal, thus some $\beta_{ij}^{(0)} \neq 0$. (A priori we do not exclude the possibilities $r = 0$ or $r = n$; then there are obvious modifications in the notation.) Letting

$$\mu_i(t) = \sum_{k \geq 0} \nu_i^{(k)} t^k, \quad 1 \leq i \leq r,$$

a degree-by-degree evaluation of (11) yields

$$\text{for } 0 \leq k < m: \quad \sum_{j=r+1}^{s} \left(\sum_{d=0}^{k} \beta_{ij}^{(d)} \nu_j^{(k-d)} \right) = 0; \quad 1 \leq i \leq r; \tag{13}$$

$$\text{for } k \geq m: \quad \nu_i^{(k-m)} + \sum_{j=r+1}^{s} \left(\sum_{d=0}^{k} \beta_{ij}^{(d)} \nu_j^{(k-d)} \right) = 0; \quad 1 \leq i \leq r. \tag{14}$$

Now we can quantify the restrictions on the Taylor coefficients $\nu_i^{(k)}$ of the μ_i imposed by relation (11) for given $A(t)$. The proof of the following facts is straightforward.

Proposition 3. *For each $\ell \geq 0$ define*

$$V_\ell := \left\{ \left(\nu_i^{(d)} \right)_{1 \leq i \leq s,\, 0 \leq d \leq \ell} \right\} \cong \mathbb{R}^{s \times (\ell+1)},$$

moreover let W_ℓ be the subspace defined by equations (13) and (14) for $k \leq \ell$, and let $\widetilde{W}_\ell$ be the subspace of V_ℓ defined by equations (14) for $k \leq \ell$. Then $\widetilde{W}_\ell$ has codimension $r \cdot (\ell+1)$ in V_ℓ and W_ℓ has constant codimension in $\widetilde{W}_\ell$ whenever $\ell \geq m$. Thus asymptotically

$$\dim V_\ell / W_\ell \sim r \cdot (\ell + 1) \text{ as } \ell \to \infty.$$

In other words, for a sufficiently high degree of the Taylor expansion, the generic rank r of the matrix A determines the number of "free" parameters $\nu_i^{(k)}$ in the μ_i. (The solution space of (11) will not have finite codimension in the space of all local curves $M(t)$ unless $r = 0$, hence considering the respective codimension of W_ℓ in V_ℓ seems the most natural approach.) From this vantage point it is natural to focus on scenarios with small r. To illustrate the result we continue the reflection group example.

Example. Let G be a finite reflection group, let $\psi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be an analytic function as in (8), and assume that $M(0) \neq 0$. If there is a nonconstant analytic curve $z(t)$, $z(0) = 0$, of critical points for $\psi(\cdot, t)$ then there exists a reflection hyperplane which contains each $z(t)$. The proof is a consequence of the introductory remarks for this section. Indeed the complement of the set U from Lemma 2 is just the union of the reflection hyperplanes; see Solomon [18], proof of Lemma. By analyticity a curve $z(t)$ either has discrete intersection with a reflection hyperplane or is fully contained in it.

4 The cube group action on symmetric matrices

In this section we will consider the group of rotations in $\mathbb{R}^3$ that leave a cube invariant, and focus on the representation of this group given by the action on symmetric 3×3 matrices by conjugation. Our physical motivation lies in the modelling and analysis of anisotropic elastic materials; the approach via invariants of symmetry groups to model strain-energy functions is classical, see e.g. Smith and Rivlin [17]. (Of course, group actions by conjugation are also relevant in many other physical applications.) The minimizing functions we will discuss are specifically motivated by the modelling of shape-memory alloys which were investigated from this perspective by Falk and Konopka [7], Zimmer [20, 21] and others. The problem of finding group-invariant strain-energy functions with prescribed minimum positions (and coefficients to fit to experimental data) naturally leads to an application of Proposition 1. For the group action of interest the minimal number of generators for the invariant algebra is rather large, therefore explicit computations may be seen as cumbersome. Possibly for this reason only solutions with additional restrictions, or geometric descriptions of (not explicitly given) solutions, were determined by Falk and Konopka [7], resp. by Zimmer [21]. (Citing this cumbersomeness as one motive, Hormann and Zimmer [10] even took an alternative approach, departing from invariant polynomials.) In the present section we will review some of the physically interesting scenarios discussed in the literature, and illustrate that the approach via Proposition 1 provides a transparent and computationally simple way to construct minimizing functions. Using the strategies developed in Section 3, we also consider the parameter-dependent scenario.

We first fix a Hilbert basis of invariants; such systems of generators can be found in Smith and Rivlin [17], and in Zimmer [21] (which we will use). For notational convenience we identify the matrix

$$\begin{pmatrix} x_1 & x_4 & x_6 \\ x_4 & x_2 & x_5 \\ x_5 & x_2 & x_3 \end{pmatrix}$$

with the vector $x \in \mathbb{R}^6$, and denote the corresponding group acting on $\mathbb{R}^6$ by G. Zimmer [21] computed a Hilbert basis of invariants using SINGULAR [6]; the following eleven generators can be taken from his Theorem 5 (where the x_i are named e_i):

$$\begin{array}{rcl}
\gamma_1 & = & x_1 + x_2 + x_3 \\
\gamma_2 & = & x_1^2 + x_2^2 + x_3^2 \\
\gamma_3 & = & x_1^3 + x_2^3 + x_3^3 \\
\gamma_4 & = & x_4^2 + x_5^2 + x_6^2 \\
\gamma_5 & = & x_4 x_5 x_6 \\
\gamma_6 & = & x_4^4 + x_5^4 + x_6^4 \\
& & \\
\gamma_7 & = & x_3 x_4^2 + x_1 x_5^2 + x_2 x_6^2 \\
\gamma_8 & = & x_3^2 x_4^2 + x_1^2 x_5^2 + x_2^2 x_6^2 \\
\gamma_9 & = & x_3 x_4^4 + x_1 x_5^4 + x_2 x_6^4 \\
\gamma_{10} & = & x_3^2 x_4^4 + x_1^2 x_5^4 + x_2^2 x_6^4 \\
\gamma_{11} & = & x_1^2 x_2 x_4^4 x_5^2 + x_2 x_3^2 x_4^2 x_5^4 + x_1 x_2^2 x_4^4 x_6^2 \\
& & \quad + x_2^2 x_3 x_5^4 x_6^2 + x_1 x_3^2 x_4^2 x_6^4 + x_1^2 x_3 x_5^2 x_6^4
\end{array} \tag{15}$$

Here $\gamma_1, \ldots, \gamma_6$ are the primary invariants, which form an algebraically independent set. For reasons of convenience we slightly modified the secondary invariants $\gamma_7, \ldots, \gamma_{10}$ given in [21]; for instance Zimmer's list contains $\widetilde{\gamma}_7 = \gamma_1\gamma_4 - \gamma_7$ instead of γ_7.

4.1 Tetragonal isotropy

As in Zimmer [21], Subsection 4.1, we wish to construct G-invariant functions which admit a minimum at

$$z := (a,\, b,\, a,\, 0,\, 0,\, 0)^{\mathrm{tr}} \text{ with } a \neq b.$$

We will determine a G-invariant polynomial ψ that admits a robust local minimum at each point in the orbit of z. According to Proposition 1, an ansatz with a quadratic polynomial in the $\gamma_i - \gamma_i(z)$ will work. One has

$$\gamma_1(z) = 2a + b,\ \gamma_2(z) = 2a^2 + b^2,\ \gamma_3(z) = 2a^3 + b^3, \quad \gamma_i(z) = 0 \text{ for } i > 3,$$

and moreover

$$\operatorname{grad}\gamma_1(z) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \operatorname{grad}\gamma_2(z) = \begin{pmatrix} 2a \\ 2b \\ 2a \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \operatorname{grad}\gamma_3(z) = \begin{pmatrix} 3a^2 \\ 3b^2 \\ 3a^2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with all other gradients having value zero at z. The degree one part of ψ therefore has the form

$$\psi^{(1)} = \sum_{i=1}^{11} \mu_i \left(\gamma_i - \gamma_i(z)\right) \tag{16}$$

with real coefficients μ_i subject to the condition

$$\mu_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2a \\ 2b \\ 2a \end{pmatrix} + \mu_3 \begin{pmatrix} 3a^2 \\ 3b^2 \\ 3a^2 \end{pmatrix} = 0,$$

equivalently

$$\mu_1 = 3ab\,\mu_3, \quad \mu_2 = -\frac{3}{2}(a+b)\,\mu_3.$$

We restate this observation:

Lemma 3. *The vector space of linear combinations of the $\gamma_i - \gamma_i(z)$ with zero gradient at z is spanned by*

$$\rho_1 := 3ab\,(\gamma_1 - \gamma_1(z)) - \frac{3}{2}(a+b)\,(\gamma_2 - \gamma_2(z)) + (\gamma_3 - \gamma_3(z))$$

and

$$\rho_j := \gamma_{j+2}, \quad 2 \le j \le 9.$$

The Hessian of $\psi^{(1)}$ at z is a block diagonal matrix, with upper left block equal to $3\mu_3 C_0$, where

$$C_0 := \begin{pmatrix} a-b & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & a-b \end{pmatrix}$$

and lower right block equal to

$$2\mu_4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2\mu_7 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} + 2\mu_8 \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}$$

The lower right block is positive definite e.g. for $\mu_7 = \mu_8 = 0$ and any $\mu_4 > 0$, but the upper left block is indefinite for any choice of $\mu_3 \neq 0$. Therefore quadratic terms of the form $(\gamma_i - \gamma_i(z))(\gamma_j - \gamma_j(z))$ are necessary to obtain a positive definite Hessian. An evaluation, using the relation

$$\mathrm{Hess}\,[(\gamma_i - \gamma_i(z))(\gamma_j - \gamma_j(z))]\,|_z = \mathrm{grad}\,\gamma_i(z)\mathrm{grad}\,\gamma_j(z)^{\mathrm{tr}} + \mathrm{grad}\,\gamma_j(z)\mathrm{grad}\,\gamma_i(z)^{\mathrm{tr}}$$

and $\mathrm{grad}\,\gamma_k(z) = 0$ for all $k > 3$ shows that such a Hessian is nonzero at most for $1 \leq i, j \leq 3$. All of these are block diagonal matrices with zero lower right 3×3 block, and upper left block of the form

$$\begin{pmatrix} u & v & u \\ v & w & v \\ u & v & u \end{pmatrix}. \tag{17}$$

Every such matrix is singular, with eigenvector $(1,\, 0,\, -1)^{\mathrm{tr}}$ for eigenvalue 0. In particular the upper left block of $\mathrm{Hess}\,(\gamma_1 - \gamma_1(z))^2\,|_z$ is equal to

$$C_1 := \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

and the upper left block of $\mathrm{Hess}\,(\gamma_2 - \gamma_2(z))^2\,|_z$ is equal to

$$C_2 := \begin{pmatrix} 4a^2 & 4ab & 4a^2 \\ 4ab & 4b^2 & 4ab \\ 4a^2 & 4ab & 4a^2 \end{pmatrix}.$$

Proposition 4. *There exist scalars α_0, α_1, α_2 such that for any $\beta > 0$, the G-invariant function*

$$\psi^* := \alpha_0 \cdot \rho_1 + \alpha_1(\gamma_1 - \gamma_1(z))^2 + \alpha_2(\gamma_2 - \gamma_2(z))^2 + \beta\gamma_4$$

admits a robust local minimum at the orbit of z.

Proof. We will show that a suitable linear combination $\alpha_0 C_0 + \alpha_1 C_1 + \alpha_2 C_2$ is positive definite. This, in conjunction with the considerations above, proves the claim.

The quadratic form which is defined by (17) in the standard basis, is represented by the matrix

$$\begin{pmatrix} 4u & 2v & 0 \\ 2v & w & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the new basis

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

In particular, the representing matrix $C_2 + \frac{\delta}{2} C_1$ will be transformed to

$$\widetilde{C}_2 + \frac{\delta}{2}\,\widetilde{C}_1 = \begin{pmatrix} 16a^2 + 4\delta & 8ab + 2\delta & 0 \\ 8ab + 2\delta & 4b^2 + \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to this new basis. The upper left 2×2 minor of this matrix is equal to $16\delta\,(a-b)^2$, and therefore the matrix admits two positive eigenvalues for all $\delta > 0$. Moreover, the base change transforms C_0 to

$$\widetilde{C}_0 = (a-b) \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

hence for $\delta > 0$ and ε of appropriate sign with $|\varepsilon|$ sufficiently small the matrix

$$\varepsilon \cdot (a-b)\, C_0 + \frac{\delta}{2}\, C_1 + C_2$$

will be positive definite. $\square$

We did not attempt to write down the most general minimizing function given by Proposition 1. Indeed this would require determination of the inequalities which characterize the semialgebraic set from part (c) of this Proposition, and this is far from trivial. Instead we we exhibited one such function that can be expressed with the smallest number of generators. Compared to previous approaches, this task is straightforward, requiring only some linear algebra.

4.2 Orthorhombic isotropy

As in Zimmer [21], Subsection 4.2, the primary task here is to construct G-invariant functions which have a minimum at

$$z := (a,\, b,\, a,\, 0,\, 0,\, c)^{\mathrm{tr}} \ \text{ with } a \neq b \text{ and } c \neq 0.$$

Falk and Konopka [7] discussed a minimizing function of degree six. Zimmer did not write down an explicit minimizer in this case, but rather gave a discussion in geometric terms. We will determine a minimizing function ψ via the ansatz (6); as will be seen, little effort is needed. We note some values:

$$\begin{array}{ccc} \gamma_1(z) = 2a+b, & \gamma_2(z) = 2a^2+b^2, & \gamma_3(z) = 2a^3+b^3, \\ \gamma_4(z) = c^2, & \gamma_5(z) = 0, & \gamma_6(z) = c^4, \\ \gamma_7(z) = bc^2, & \gamma_8(z) = b^2c^2, & \gamma_9(z) = bc^4, \\ \gamma_{10}(z) = b^2c^4, & \gamma_{11}(z) = 0. & \end{array}$$

The homogeneous linear part of ψ has the form

$$\psi^{(1)} = \sum_{i=1}^{11} \mu_i \left(\gamma_i - \gamma_i(z)\right) \tag{18}$$

with real coefficients μ_i subject to the conditions derived from

$$\sum \mu_i \operatorname{grad} \gamma_i(z) = 0.$$

We will not go through the details of the computations, which are similar to those in the previous subsection (elementary but more lengthy). The result is as follows.

Lemma 4. *In the orthorhombic isotropy case, the space of linear combinations of the $\gamma_i - \gamma_i(z)$ with zero gradient at z is spanned by:*

$$\begin{array}{lcl} \sigma_1 & := & 3ab\,(\gamma_1 - \gamma_1(z)) - \frac{3}{2}(a+b)\,(\gamma_2 - \gamma_2(z)) + (\gamma_3 - \gamma_3(z)) \\ \sigma_2 & := & \gamma_5 - \gamma_5(z) \\ \sigma_3 & := & -c^2\,(\gamma_4 - \gamma_4(z)) + (\gamma_6 - \gamma_6(z)) \\ \sigma_4 & := & 2ac^2\,(\gamma_1 - \gamma_1(z)) - c^2\,(\gamma_2 - \gamma_2(z)) - b(b-a)\,(\gamma_4 - \gamma_4(z)) \\ & & \quad +(b-a)\,(\gamma_7 - \gamma_7(z)) \\ \sigma_5 & := & 2abc^2\,(\gamma_1 - \gamma_1(z)) - 2bc^2\,(\gamma_2 - \gamma_2(z)) - b^2(b-a)\,(\gamma_4 - \gamma_4(z)) \\ & & \quad +(b-a)\,(\gamma_8 - \gamma_8(z)) \\ \sigma_6 & := & 2ac^4\,(\gamma_1 - \gamma_1(z)) - c^4\,(\gamma_2 - \gamma_2(z)) - 4bc^2(b-a)\,(\gamma_4 - \gamma_4(z)) \\ & & \quad +(b-a)\,(\gamma_9 - \gamma_9(z)) \\ \sigma_7 & := & 2abc^4\,(\gamma_1 - \gamma_1(z)) - bc^4\,(\gamma_2 - \gamma_2(z)) - 2b^2c^2(b-a)\,(\gamma_4 - \gamma_4(z)) \\ & & \quad +(b-a)\,(\gamma_{10} - \gamma_{10}(z)) \\ \sigma_8 & := & \gamma_{11} \end{array}$$

We list only some of the Hessians here. As in the previous subsection, Hess (σ_1) is block diagonal with upper left block

$$3\begin{pmatrix} a-b & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & a-b \end{pmatrix}$$

and lower right block zero. The Hessian of σ_3 is block diagonal, with upper left block zero and lower right block

$$\begin{pmatrix} -2c^2 & 0 & 0 \\ 0 & -2c^2 & 0 \\ 0 & 0 & 2c^2 \end{pmatrix},$$

and the Hessian of σ_8 is block diagonal, with upper left block zero and lower right block

$$\begin{pmatrix} 2a^3c^4 & 0 & 0 \\ 0 & 2a^3c^4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, a suitable linear combination of the latter two matrices will be positive definite. Combining this observation with the arguments to prove Proposition 4, we have:

Proposition 5. *There exist scalars α_0, α_1, α_2 and β_1, β_2 such that the G-invariant function*

$$\psi^* := \alpha_0 \cdot \sigma_1 + \alpha_1(\gamma_1 - \gamma_1(z))^2 + \alpha_2(\gamma_2 - \gamma_2(z))^2 + \beta_1\sigma_3 + \beta_2\sigma_8$$

admits a robust local minimum at the orbit of z.

Again, we did not attempt to write down the most general minimizing function but rather determined one involving the smallest number of generators.

4.3 Parameter-dependent functions

In this final subsection we consider parameter-dependent functions and discuss critical points emanating from a point with highest isotropy. In contrast to Section 3 the cube group action on symmetric matrices is not faithful; the points with maximal isotropy form the line $\mathbb{R} \cdot (1, 1, 1, 0, 0, 0)^{\mathrm{tr}}$. But while the arguments in the proof of Proposition 2 (which utilize homogeneity) do not carry over to the present setting, the underlying strategy still works.

Taking the direct approach we consider a smooth G-invariant function which depends smoothly on a parameter t, and assume there is a smooth curve

$$t \mapsto z(t), \quad z(0) = \overline{z} := a \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ with } a \neq 0, \quad z'(0) = v.$$

Since our interest lies in critical points with smaller isotropy, we require that $(1,\,1,\,1,\,0,\,0,\,0)^{\mathrm{tr}}$ and v are linearly independent.
We thus arrive at a condition

$$\sum_{i=1}^{11} \mu_i(t) \mathrm{grad}\, \gamma_i(z(t)) = 0$$

with the γ_i given by (15), to be evaluated at $t = 0$. By straightforward (even though somewhat lengthy) computations one finds the zero order condition (in t)

$$\big(\mu_1(0) + 2a\mu_2(0) + 3a^2\mu_3(0)\big) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0; \tag{19}$$

the first-order condition

$$\begin{aligned} 0 \quad = \quad & \big(\mu_1'(0) + 2a\mu_2'(0) + 3a^2\mu_3'(0)\big) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (2\mu_2(0) + 3a\mu_3(0)) \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ & + \big(2\mu_4(0) + 2a\mu_7(0) + 2a^2\mu_8(0)\big) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}; \end{aligned} \tag{20}$$

and the second-order condition

$$
\begin{aligned}
0 \quad = \quad & \left(\mu_1''(0) + 2a\mu_2''(0) + 3a^2\mu_3''(0)\right) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
& +2\,(2\mu_2('0) + 3a\mu_3'(0)) \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3\mu_3(0) \cdot \begin{pmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
& +2\left(2\mu_4'(0) + 2a\mu_7'(0) + 2a^2\mu_8'(0)\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} \\
& +2\mu_5(0) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_5 v_6 \\ v_6 v_4 \\ v_4 v_5 \end{pmatrix} + 2\,(\mu_7(0) + a\mu_8(0)) \cdot \begin{pmatrix} v_5^2 \\ v_6^2 \\ v_4^2 \\ 2v_3 v_4 \\ 2v_1 v_5 \\ 2v_2 v_6 \end{pmatrix}.
\end{aligned}
\tag{21}
$$

We note some consequences of these identities.

Proposition 6. *(a) The order zero condition* (19) *implies*

$$\mu_1(0) + 2a\mu_2(0) + 3a^2\mu_3(0) = 0;$$

this is a nontrivial (necessary and sufficient) condition for the criticality of a point with highest symmetry.

(b) The order one condition (20) *yields a splitting into two cases.*

- *If* $(v_4,\, v_5,\, v_6)^{\mathrm{tr}} = 0$ *then (given the linear independence requirement)*

$$\mu_1(0) = 0, \quad \mu_1'(0) + 2a\mu_2'(0) + 3a^2\mu_3'(0) = 0,$$

which characterizes degeneracy of the upper left block in the Hessian. The order two condition (21) *then forces*

$$\mu_2(0) = \mu_3(0) = 0 \ \textit{or} \ \det \begin{pmatrix} 1 & v_1 & v_1^2 \\ 1 & v_2 & v_2^2 \\ 1 & v_3 & v_1^3 \end{pmatrix} = 0.$$

The Vandermonde determinant vanishes (this represents the generic case) if and only if two of the v_i are equal. This corresponds to tetragonal isotropy.

- *If* $(v_4, v_5, v_6)^{\mathrm{tr}} \neq 0$ *then the order one condition forces*

$$\mu_4(0) + a\mu_7(0) + a^2\mu_8(0) = 0$$

which is equivalent to degeneracy of the lower right block of the Hessian. Generically one has linear dependence of $(1, 1, 1)^{\mathrm{tr}}$ *and* $(v_1, v_2, v_3)^{\mathrm{tr}}$*, and furthermore the top three entries of the order two condition* (21) *imply generically that*

$$\begin{pmatrix} v_5^2 \\ v_6^2 \\ v_4^2 \end{pmatrix} \in \mathbb{R} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

whence v_4, v_5 and v_6 differ at most in their signs.

We have to give a precise meaning to the term "generic" here. Using (with obvious modifications) the notions introduced in Proposition 3, we call a subset of V_2 generic if it is (Zariski-) open and dense in a maximal vector subspace defined by conditions (19) through (21).

Proposition 6 provides necessary conditions for the entries of v, assuming genericity. Each of these necessary conditions yields nontrivial curves of critical points, as can be seen via the complementary approach from subsection 3.2. The proof of the following involves some straightforward computations (such as the determination and evaluation of various gradients) which we will not reproduce here.

Proposition 7. *(a) Let b_1, b_2 and b_3 be real numbers, exactly two of which are equal. Then the curve*

$$z(t) = \overline{z} + t \cdot (b_1, b_2, b_3, 0, 0, 0)^{\mathrm{tr}}$$

defines a matrix $A(t)$ via (12)*, with rank $r = 2$. In particular there exist parameter-dependent G-invariant functions which admit $z(t)$ as a curve of critical points; this describes a passage from cubic to tetragonal isotropy.*

(b) Let b and $c \neq 0$ be real numbers, and $\epsilon_i \in \{1, -1\}$ for $1 \leq i \leq 3$. Then the curve

$$z(t) = \overline{z} + t \cdot (b,\, b,\, b,\, \epsilon_1 c,\, \epsilon_2 c,\, \epsilon_3 c)^{\mathrm{tr}}$$

defines a matrix $A(t)$ via (12), with rank $r = 2$. In particular there exist parameter-dependent G-invariant functions which admit $z(t)$ as a curve of critical points.

Since the rank of $A(t)$ is equal to one only for curves contained in the line of maximal isotropy, the case $r = 2$ represents the "most" free parameters in the μ_i (in the sense of subsection 3.2). Thus the passage from cubic to tetragonal isotropy may be explained as a generic symmetry breaking phenomenon from a point of highest symmetry, but this is not the case for the passage from cubic to orthorhombic isotropy. (The latter observation is in agreement with remarks in Falk and Konopka [7], who use arguments from physics.)
Case (b) of the Proposition corresponds to an isotropy group called D_3 in [21], Fig. 1, when not all ϵ_i are equal, and an isotropy group not listed in [21] when all the ϵ_i are equal. The common feature is that the dimensions of the corresponding fixed point spaces are equal to two.

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