The MeatAxe

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Let $\mathbb{F}$ be a field and $\mathbb{F}^{d \times d}$ the set of $d \times d$-matrices.

Definition ($\mathbb{F}$-algebra, matrix algebra)

An $\mathbb{F}$-algebra is a ring $\mathcal{A}$ with identity together with a ring homomorphism $\iota : \mathbb{F} \to C(\mathcal{A})$ into the centre of $\mathcal{A}$.

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- $\mu(\mu(v, X), Y) = \mu(v, XY)$ for all $v \in V$ and $X, Y \in \mathcal{A}$.

Example (Natural module)

If $\mathcal{A} \leq \mathbb{F}^{d \times d}$ is a matrix algebra, then $V := \mathbb{F}^{1 \times d}$ is a right $\mathcal{A}$-module with $\mu(v, X) := v \cdot X$. It is called the natural module.

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A composition series for V is a chain of submodules

$$\{0\} = V_{\ell+1} < V_\ell < V_{\ell-1} < \cdots < V_1 = V$$

such that all V_i/V_{i+1} are irreducible.

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To describe this situation to a computer, it is enough to choose an $\mathbb{F}$-basis $(v_1, \ldots, v_d)$ of V and store one $d \times d$-matrix for each A_i.

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Fundamental Problem

Given an $\mathcal{A}$-module on a computer, decide irreducibility and compute a composition series.

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Problem (Module generated by a vector)

Given $0 \neq v \in V$, *find a basis for*

$$\begin{aligned} v\mathcal{A} &:= \{vX \mid X \in \mathcal{A}\} \\ &:= \text{intersection of all } \mathcal{A}\text{-submodules containing } v \end{aligned}$$

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5. Append y to the end of $\mathcal{B}$
6. Set $i := i + 1$

Let $\mathcal{A} = \langle A_1, \ldots, A_k \rangle_{\text{Alg}} \leq \mathbb{F}^{d \times d}$ be a matrix algebra and $B \in \mathcal{A}$ a singular element. Let $\mathcal{A}^t := \langle A_1^t, \ldots, A_k^t \rangle_{\text{Alg}}$.

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At least one of the following holds:

1. *There is a* $0 \neq v \in \ker B$ *such that* $v\mathcal{A} \neq V$*.*
2. *For all* $v \in \ker B^t$ *holds* $v\mathcal{A}^t \neq V$*.*
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Let $T := (w_1, \dots, w_e, v_1, \dots, v_{d-e})$ and $B' := TBT^{-1}$.

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6. If $v\mathcal{A}^t = V$, we have proved V to be irreducible, stop

Assume we are given an $\mathcal{A}$-module $V = \mathbb{F}^{1\times d}$ by matrices $A_1, \ldots, A_k \in \mathbb{F}^{d\times d}$.

"Chopping" means computing a composition series.

The MeatAxe basically does the following:

A basic step of "Chop"

1. Find an element $B \in \mathcal{A}$ with small, non-trivial kernel
2. Compute $\ker B$
3. Spinup all $0 \neq v \in \ker B$
4. If some $v\mathcal{A} < V$, we found a submodule, goto 7
5. Otherwise spinup one $0 \neq v \in \ker B^t$ under $\mathcal{A}^t$
6. If $v\mathcal{A}^t = V$, we have proved V to be irreducible, stop
7. If $0 < W < V$ is invariant, compute action on W and V/W and recurse (with smaller dimensions!)

The result of “Chop” is a composition series

$$\{0\} = V_{\ell+1} < V_\ell < V_{\ell-1} < \cdots < V_1 = V$$

such that all V_j/V_{j+1} are irreducible.

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Actually, we find a base change $T \in \mathbb{F}^{d\times d}$, such that all matrices TA_iT^{-1} for $1 \le i \le k$ look like this:

$$TA_iT^{-1} = \begin{bmatrix} M_\ell^{(i)} & 0 & \cdots & 0 \\ * & M_{\ell-1}^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & M_1^{(i)} \end{bmatrix}$$

and the matrices $M_j^{(i)}$ describe the action of $\mathcal{A}$ on V_j / V_{j+1}.

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A more detailed analysis shows that the MeatAxe can identify isomorphism types of irreducible modules.

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The MeatAxe can do the following for you:

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- Find homomorphism spaces from an irreducible module to another one.

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- Compute the socle and radical series.

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- Compute the submodule lattice.

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- Compute homomorphism spaces between arbitrary modules.

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- Compute cohomology groups.
- Compute condensed modules.

Definition of class $\mathcal{C}_5$

$G \leq \mathrm{GL}_d(\mathbb{F}_q)$ lies in $\mathcal{C}_5$ if

- the natural module V is absolutely irreducible and
- there is a proper subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$ and $T \in \mathrm{GL}_d(\mathbb{F}_q)$ and $(\beta_g)_{g \in G}$ with $\beta_g \in \mathbb{F}_q$ such that

$$\beta_g \cdot T^{-1} g T \in \mathrm{GL}_d(\mathbb{F}_{q_0}) \text{ for all } g \in G.$$

Let $G := \langle g_1, \ldots, g_m \rangle \leq \mathrm{GL}_d(\mathbb{F}_q)$ with $q = p^f$, assume that the natural module V is irreducible and let $e = \dim_{\mathbb{F}_q}(\mathrm{End}_{\mathbb{F}_q G}(V))$ (degree of splitting field).

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Algorithm to decide $tGt^{-1} \leq \mathrm{GL}_d(\mathbb{F}_{q_0})$ for some $t \in \mathrm{GL}_d(\mathbb{F}_q)$ and a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$:

1. Choose a uniformly distributed random element $c \in \mathbb{F}_p G$ in its action on V and compute $\ker_V(c)$. Repeat this until $\dim_{\mathbb{F}_q}(\ker_V(c)) = e$ or fail after $O(\log \delta^{-1})$ tries.
2. Take $0 \neq w \in \ker_V(c)$ and spin up w with the generators $g_1, \ldots, g_m$ using $\mathbb{F}_q$-linear independence to a basis $\mathcal{B}$.
3. Let $t \in \mathrm{GL}(d, \mathbb{F}_q)$ have the vectors in $\mathcal{B}$ as rows, and find the smallest subfield of $\mathbb{F}_q$ containing all entries of all tg_it^{-1}. If this is $\mathbb{F}_q$ then output “No”. Otherwise return that field and t.

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