Algorithms for Permutation groups

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Permutation Groups

The Symmetric Group

Let Ω be a finite set.
The Symmetric group, $Sym(\Omega)$, is the group of all bijections from Ω to itself.
A permutation group is a subgroup of $Sym(\Omega)$.

Permutation Groups

- 1960s: the Classification of finite simple groups required to work with large permutation groups.
- 1970s: C. Sims introduced algorithms for working with permutation groups.
- These were among the first algorithms in CAYLEY and GAP.
- 1990s: nearly linear algorithms for permutation groups emerged. These are now in GAP and MAGMA.
- Seress' book.
- A very brief summary.

Notation

From now on:

Let Ω be finite and $G \leq \mathrm{Sym}(\Omega)$.

For $\alpha \in \Omega$ let G_α denote the stabiliser of α in G, i.e.

$$G_\alpha = \{g \in G \mid \alpha^g = \alpha\}.$$

If $\alpha, \beta \in \Omega$ let $G_{(\alpha,\beta)}$ denote the stabiliser of β in G_α, i.e.

$$G_{(\alpha,\beta)} = (G_\alpha)_\beta = \{g \in G \mid \alpha^g = \alpha \text{ and } \beta^g = \beta\}.$$

Bases

Base and Stabiliser Chain

$B = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with $\alpha_i \in \Omega$ is a base for G if $G_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} = \{1\}$.
The chain of subgroups

$$G = G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(k+1)} = \{1\}$$

defined by $G^{(i+1)} = G^{(i)}_{\alpha_i}$ for $1 \leq i \leq k$ is the stabiliser chain for B.

B is irredundant if all the inclusions in the stabiliser chain for B are proper.

Base Images

If G is a permutation group and $B = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ a base for G, then each element $g \in G$ is uniquely determined by $(\alpha_1^g, \alpha_2^g, \ldots, \alpha_k^g)$.
(Since $B^g = B^h$ implies $B^{gh^{-1}} = B$ and thus $gh^{-1} = 1$).

Orbits

Definition

Let $G = \langle X \rangle \leq \mathrm{Sym}(\Omega)$ and $\alpha \in \Omega$. The orbit of α under G, denoted α^G is the set

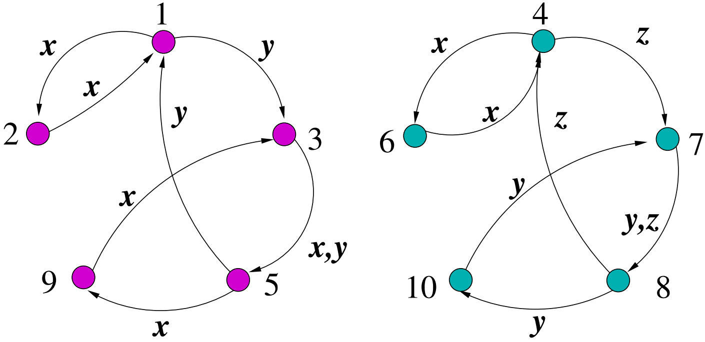
$$\alpha^G := \{\alpha^g \mid g \in G\}.$$

Example

The orbits for $G = \langle x, y, z \rangle$ with

$$x = (1,2)(3,5,9)(4,6),\ y = (1,3,5)(7,8,10),\ z = (4,7,8)$$

on $\Omega = \{1, 2, \dots, 10\}$ are Ω/G are $\Delta_1 = \{1, 2, 3, 5, 9\}$ and $\Delta_2 = \{4, 6, 7, 8, 10\}$.



Definition

Let $G \leq \mathrm{Sym}(\Omega)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ a basis for G. Let $G = G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(k+1)} = \{1\}$ (where $G^{(i+1)} = G^{(i)}_{\alpha_i}$ for $1 \leq i \leq k$) be the stabiliser chain for B.
Then $S \subseteq G$ is a strong generating set for G if for every i with $1 \leq i \leq k+1$ holds $G^{(i)} = \langle S \cap G^{(i)} \rangle$.

The Schreier-Sims Algorithm

Input: $G \leq \mathrm{Sym}(\Omega)$
Output:

- $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ a basis for G
- $S \subseteq G$ a strong generating set for G
- the orbits $\alpha_i^{G^{(i)}}$ stored in a particular way

Example

- $G := \langle (1,2,3,4,5,6),(2,6)(3,5)\rangle$.
- base: $\{1,2\}$
- strong generating set:
  $S = \{(2,6)(3,5),(1,2,3,4,5,6),(1,3,5)(2,4,6)\}$
- stabiliser Chain:

$$G^{(1)} = G \geq G^{(2)} = \langle (2,6)(3,5)\rangle \geq G^{(3)} = \{1\}.$$

- orbits:

$$1^{G^{(1)}} = \Omega,\ 2^{G^{(2)}} = \{2,6\}.$$

Questions

The data structure of a base and a strong generating set together with the associated stabiliser chain allows us to answer questions about G such as

- what is $|G|$?
- does $g \in \mathrm{Sym}(\Omega)$ satisfy $g \in G$?

Example: Is $g = (1,4)(2,3)(5,6) \in G$?

- $G := \langle (1,2,3,4,5,6), (2,6)(3,5) \rangle$.
- base: $\{1,2\}$
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Example: Is $g = (1,4)(2,3)(5,6) \in G$?

$G := \langle (1,2,3,4,5,6), (2,6)(3,5) \rangle$.

- $1^g = 4$
- Find $h \in G$ with $1^h = 4$.
- $h = (1,4)(2,5)(3,6) \in G$.
- $g \in G$ if and only if $gh^{-1} \in G$.
- $1^{gh^{-1}} = 1$ so $g \in G$ if and only if $gh^{-1} \in G^{(2)}$.
- $gh^{-1} = (2,6)(3,5)$.
- $(2,6)(3,5) \in S$, so $gh^{-1} \in G^{(2)}$.
- Thus $g \in G$.

Schreier's Lemma

SCHREIER'S LEMMA

Let $G = \langle X \rangle$ be a finite group, $H \leq G$ and T set of representatives of the right cosets of H in G such that T contains 1. Denote by $\overline{g}$ the representative of Hg for $g \in G$. Then H is generated by

$$X_H = \{tx(\overline{tx})^{-1} | t \in T, x \in X\}.$$

Essential steps

- Compute the orbits $\alpha_i^{G^{(i)}}$ together with
- T_i set of cosets representatives for cosets of $G^{(i+1)}$ in $G^{(i)}$
- for $\beta \in \alpha_i^{G^{(i)}}$ find representative in T_i
- find generators for $G^{(i+1)}$.

Theorem

Let Ω finite, $n = |\Omega|$ and $G = \langle X \rangle \leq Sym(\Omega)$ a permutation group. Then the complexity of the Schreier -Sims algorithm is

$$O(n^3 \log_2(|G|)^3 + |X| n^3 \log_2(|G|)).$$

Note that $|\mathrm{Sym}(\Omega)| = n! \sim n^n$, so $\log(|\mathrm{Sym}(\Omega)|) \sim n \log(n)$. Therefore, the complexity can be as bad as

$$O(n^6 + |X| n^4).$$

A Remark about $|B|$

Given a basis B for $G = \langle X \rangle \leq \mathrm{Sym}(\Omega)$, with Ω finite. Then $2^{|B|} \leq |G| \leq n^{|B|}$ or

$$\frac{\log(|G|)}{\log(n)} \leq |B| \leq \frac{\log(|G|)}{\log(2)}.$$

Small Base

Let $\mathcal{G}$ be a family of permutation groups. We call $\mathcal{G}$ small-base if for every $G \in \mathcal{G}$ of degree n holds

$$\log |G| < \log^c(n)$$

for a constant c, fixed for $\mathcal{G}$.

Theorem of Liebeck

Theorem

Let $\mathcal{G}$ be a family of permutation groups. Every large-base primitive group in G of degree n involves the action of A_n or S_n on the set of k-element subsets of $\{1, \ldots, n\}$, for some n and $k < n/2$.

These groups are called the giants.

Remark

Let $\mathcal{G}$ be a family of small-base permutation groups, i.e. for every $G \in \mathcal{G}$ of degree n holds

$$\log|G| < \log^c(n)$$

for a constant c, fixed for $\mathcal{G}$.
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This is only slightly more expensive than $O(n^3)$.
If we can limit the length of the basis by n, the complexity is $O(n^6)$.

"State of the Art"

Seress proves in his book (p. 75, Theorem 4.5.5):

Theorem

Let $G \leq \langle X \rangle \leq \mathrm{Sym}(\Omega)$ with $|\Omega| = n$. Then there exists a Monte-Carlo algorithm, which computes with probability ε for $\varepsilon \leq \frac{1}{n^d}$ (for a positive whole number d, given by the user) a basis and a strong generating system for G in time

$$O(n \log(n) \log(|G|)^4 + |X| n \log(|G|))$$

and uses $O(n \log(|G|) + |X|n)$ space.

For small-base groups this algorithm is nearly linear.

Schreier-Sims for Matrix Groups

One of the first approaches to deal with Matrix Groups (Butler, 1979).
Let $G \leq \mathrm{GL}(n, q)$. Then G acts faithfully as a permutation group on $V = \mathbb{F}_q^n$ via $g : v \mapsto vg$.
Thus we an apply the Schreier-Sims algorithm to this permutation group.

Schreier-Sims for Matrix Groups

Problem

How long can the orbit v^G be? It can be $q^n - 1$.

Example

$q = 3, n = 100$

$$q^n - 1$$
$$= 515377520732011331036461129765621272702107522000.$$

Even

$$3^{20} - 1 = 348678440.$$

Schreier-Sims for Matrix Groups

Problem

- In a permutation group $G \leq S_n$ the length of an orbit is at most n. Hence easy to compute an orbit for n quite large.
- In a matrix group $G \leq \mathrm{GL}(n, q)$ orbits can be $O(q^n)$.

Complexity

- S_n linear in n.
- $\mathrm{GL}(n, q)$ exponential in n.

Schreier-Sims for Matrix Groups

- works well for small *n* and *q*.
- Algorithms developed by Butler (1979)
- Murray & O'Brien (1995) consider the selection of base points
- Lübeck & Neunhöffer (2000) and Müller, Neunhöffer, Wilson (2007) consider large orbits

How can we rule out that our given group is a giant beforehand?

Consider first A_n and S_n in their natural representation.

Definition

An element $g \in S_n$ is called *purple* if it contains in its disjoint cycle decomposition one cycle of prime length p with $n/2 < p < n - 2$.

Theorem (Modification of a theorem of Jordan, 1873)

$G \leq S_n$ acts transitively on $\Omega = \{1, \ldots, n\}$. If G contains a purple element, then G contains A_n.

Theorem

Let p a prime with $n/2 < p < n - 2$. The proportion of purple elements in S_n and A_n is $\frac{1}{p}$.

Bertrand's postulate

The following Theorem was already conjectured by Bertrand (1822-1900) and proved by Chebyshev (1821-1894) in 1850.

Theorem

For a positive integer m with $m > 3$ there exists at least one prime p with $m < p < 2m - 2$.

Proportions in S_n and A_n

The proportion of purple elements in S_n or A_n is $\frac{c}{\log(n)}$ for a small constant c.

Monte-Carlo Test: is $A_n \leq G$?

Algorithm 1: ContainsA$_n$

Eingabe: $G = \langle X \rangle \leq S_n$
Ausgabe: true or false
if *not* IsTransitive(G) **then return** *false*;
for $i = 1 \ldots N$ **do**
    $g := Random(G)$;
    if g *purple* **then return** *true*;
end
return *false*;

Complexity of CONTAINSA_n

The probability that among N independent, uniformly distributed random elements $g \in G$, with $A_n \leq G$, no purple elements were found is $(1 - \frac{c}{\log(n)})^N$. Thus choose N such that $(1 - \frac{c}{\log(n)})^N < \varepsilon$, or

$$N > \log(\varepsilon^{-1}) \log\left(\frac{\log(n)}{\log(n) - c}\right)^{-1}.$$

This is the case, if $N > \log(\varepsilon^{-1})\frac{\log(n)}{c}$.
Thus the complexity is

$$O(\log(\varepsilon^{-1})\log(n)(\rho + n)),$$

where ρ is the cost of a call to RANDOM.

Problems with no known polynomial time Algorithms

Consider the following problems for permutation groups.

- set stabiliser
- centraliser of one group in another
- intersection of permutation groups
- decide whether two elements in a group are conjugate

For Further Reading I

Ákos Seress
Permutation Group Algorithms,
Cambridge Tracts in Mathematics 152,
Cambridge University Press, 2003.