# Proving the nonexistence of algebraic solutions of differential equations

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# Part I

# The problem

# Stating the problem

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Solve the system of differential equations

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where $a$ and $b$ are polynomials in $x$ and $y$. More concisely,

$$\dot{X} = F(X),$$

where $X = (x, y)$ and $F = (a, b)$ is a polynomial vector field.

# What does it mean to solve an equation?

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## The canonical definition

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What if the function were also known implicitly?

Suppose we know a function $H = H(x, y)$ whose set of zeros is $C$.

**Question**

*How can we say that the curve is a solution of the system using $H$ instead of the parameterization?*

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which is equivalent to

$$(F \cdot \nabla H)(C(t)) = 0.$$

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### First integral

*A function* $H(x,y)$ *is a first integral of the system* $\dot{X} = F(X)$ *if*

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*as a function of* $x$ *and* $y$*.*

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## Key property

*If* $H$ *is a first integral of* $\dot{X} = F(X)$ *then every integral curve of this system is contained in a level curve of* $H$.

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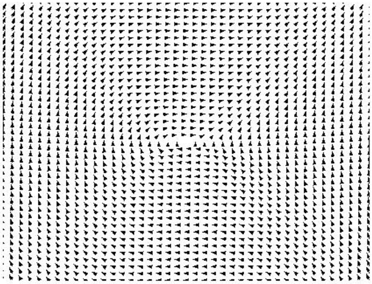

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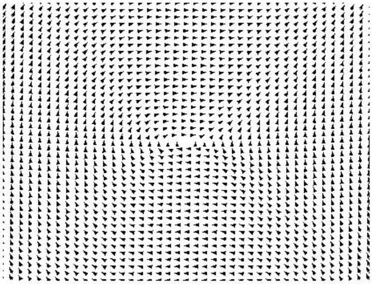

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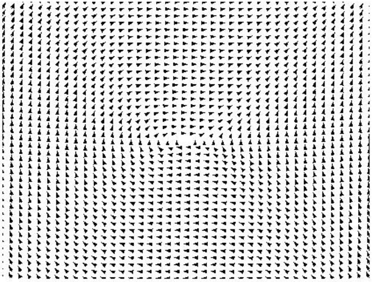

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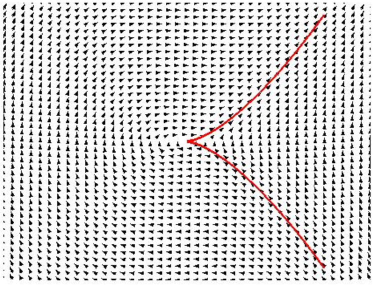

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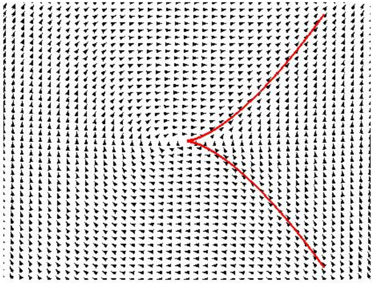

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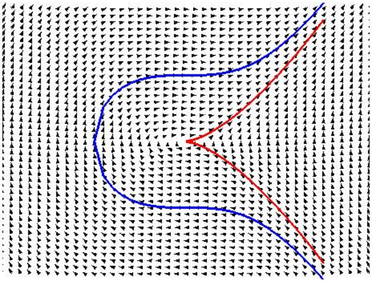
$$H(x, y) = 0 \text{ and } H(x, y) = 1.$$



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- etc.

# Part II

# The 19th century

# C. G. J. Jacobi, 1842

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**1.**

**De integratione aequationis differentialis**

$$(A+A'x+A''y)(x\,dy-y\,dx)$$
$$-(B+B'x+B''y)\,dy+(C+C'x+C''y)\,dx = 0.$$

(Auct. C. G. J. *Jacobi*, prof. ord. Regiom.)

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Solves a differential equation with linear coefficients,

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Solves a differential equation with linear coefficients, with a long calculation.

# Alfred Clebsch, 1872

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Ueber eine Fundamentalaufgabe der Invarianten-theorie.

Von

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Der Königl. Gesellschaft der Wissenschaften überreicht am 2. März 1872.

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Geometric interpretation of differential equations using homogeneous coordinates.

# G. Darboux, 1878

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MÉLANGES.

MÉMOIRE SUR LES ÉQUATIONS DIFFÉRENTIELLES ALGÉBRIQUES DU PREMIER ORDRE ET DU PREMIER DEGRÉ;

PAR M. G. DARBOUX.

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MÉMOIRE SUR LES ÉQUATIONS DIFFÉRENTIELLES ALGÉBRIQUES
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PAR M. G. DARBOUX.

Introduces the method that defined the research line we will pursue in this talk.

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so that,

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If $H$ and $F$ are polynomial, then so is

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Therefore, the conclusion above implies that,

$$F(x,y)\cdot\nabla H = GH,$$

for some polynomial $G = G(x,y)$, called the co-factor of $H$.

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Therefore, the conclusion above implies that,

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Assuming that $H$ is reduced, this follows from Hilbert's Nullstellensatz.

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## Invariant curve

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## Invariant curve

*An algebraic curve* $H(x, y) = 0$ *is invariant under the system* $\dot{X} = F(x, y)$ *if*

$$F(x, y) \cdot \nabla H = GH,$$

# Darboux's key Theorem

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## Existence of first integral

*If* $\dot{X} = F(X)$ *has* *more than* $\deg(F)(\deg(F) - 1)/2$ *invariant curves, then it admits a first integral.*

## Degree of a vector field

*If* $F = (a, b)$, *for polynomials* $a$ *and* $b$, *then*

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$$\frac{(\deg(F) - 1)\deg(F)}{2}.$$

# Proof of Darboux's key Theorem

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$p_1, \ldots, p_k$ *be curves invariant under* $\dot{X} = F(X)$*;*

# Proof of Darboux's key Theorem

$$d = \frac{(\deg(F) - 1)\deg(F)}{2}$$

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If $k > d$ then $g_1, \ldots, g_k$ are linearly dependent,

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If $k > d$ then $g_1, \ldots, g_k$ are linearly dependent, so

$$c_1 g_1 + \cdots + c_k g_k = 0 \text{ for scalars } c_1, \ldots, c_k.$$

# Proof of Darboux's key Theorem

Hypotheses:

- $F \cdot \nabla p_j = g_j p_j$, where $1 \leq j \leq k$;
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Define

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then

$$F \cdot \nabla h = F \cdot (c_1 p_1^{c_1 - 1} \cdots p_k^{c_k} \nabla p_1 + \cdots + p_1^{c_1} \cdots c_k p_k^{c_k - 1} \nabla p_k)$$

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$$F \cdot \nabla h = c_1 p_1^{c_1 - 1} \cdots p_k^{c_k} F \cdot \nabla p_1 + \cdots + p_1^{c_1} \cdots c_k p_k^{c_k - 1} F \cdot \nabla p_k$$

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Hypotheses:

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Hence $h$ is a first integral of $F$.

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Suppose that $F = (2x + y + 1, y + 2)$.

Its invariant lines are

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Since $2 \cdot 1 + (-1) \cdot 2 = 0$,

$$h = (y + 2)^2 (2x + 2y + 3)^{-1},$$

is a first integral of $\dot{X} = F(X)$.

# Also in Darboux's paper

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- solutions for some equations with quadratic coefficients;
- a study of the singular points of the differential equations.

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**Singular point**

*The singular points of $\dot{X} = F(X)$ are the points of the plane at which $F$ vanishes.*

From now on we will assume that $F = (a, b)$ with $\gcd(a, b) = 1$. Geometrically, this means that $F$ has finitely many singularities.

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Given a polynomial $h$ of degree $d$ in two variables let $h_d$ be its homogeneous component of degree $d$.

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*A vector field $F = (a, b)$ of degree $n$ that satisfies $ya_n = xb_n$ has, at most, $(n-1)^2 + (n-1) + 1$ singularities.*

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**Note for the experts**

*The condition $ya_n = xb_n$ means that the one-dimensional direction field that $F$ defines in the projective plane has no singularities at infinity.*

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**Introduction.**

Pour reconnaître si une équation différentielle du premier ordre et du premier degré est intégrable algébriquement, il suffit évidemment de trouver une limite supérieure du degré de l'intégrale; il ne reste plus ensuite qu'à effectuer des calculs purement algébriques.

C'est là un problème qui, semble-t-il, aurait dû tenter les géomètres, et cependant ils s'en sont fort peu occupés. Depuis l'œuvre magistrale de M. Darboux, publiée dans le *Bulletin des Sciences mathématiques*, la question a été négligée pendant vingt ans et il a fallu, pour attirer de nouveau sur elle l'attention qu'elle méritait, que l'Académie des Sciences la proposât comme sujet du concours pour le Grand Prix des Sciences mathématiques. Deux Mémoires furent récompensés, M. Painlevé obtint le prix et M. Autonne une mention honorable : l'un de ces deux Mémoires a été publié dans les *Annales de l'École Normale supérieure* et l'autre dans le *Journal de l'École Polytechnique*.

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Also in various textbooks up to the early 20th century.

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"it is evidently sufficient to find an upper limit to the degree of the algebraic invariant curves".

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# One more problem

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## Poincaré's problem

# One more problem

**Poincaré's problem**

*Given a vector field $F$ with polynomial coefficients, find a bound on the degree of the algebraic curves invariant under $F$ as a function of some numerical invariant of $F$.*

Part III

# The 20th century

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- an algebraic curve invariant under a vector field must contain a singularity at least if we include the ones at infinity;
- a general equation of degree higher than 2 does not have any invariant curve.

# Singularities on invariant curves

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## Theorem

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$$F \cdot \nabla h = gh$$

for some polynomial $g$ of degree at most $n-1$.

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Let $h$ be the reduced polynomial in $x$ and $y$ that defines the curve.
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for some polynomial $g$ of degree at most $n-1$ and all singularities $p$ of $F$.

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Let $h$ be the reduced polynomial in $x$ and $y$ that defines the curve.
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Part IV

# The existence problem

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Since no efficient necessary and sufficient criterion for the existence of such curves is known, we will settle for a probabilistic test.

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## Key point

*The $x$-coordinates of the singularities of $F$ are all of them roots of $p$.*

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We will proceed by contradiction, assuming that the field $F$ has an invariant algebraic curve with rational coefficients.

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$$g \cdot (x_0, q(x_0)) = (g(x_0), q(g(x_0))) \text{ for any } \quad g \in G.$$

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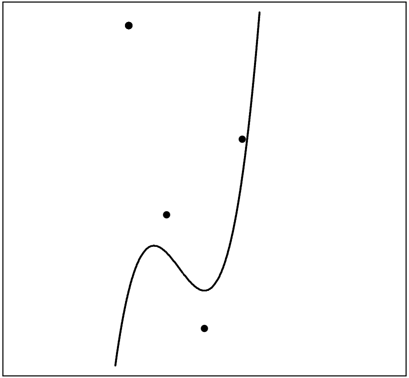

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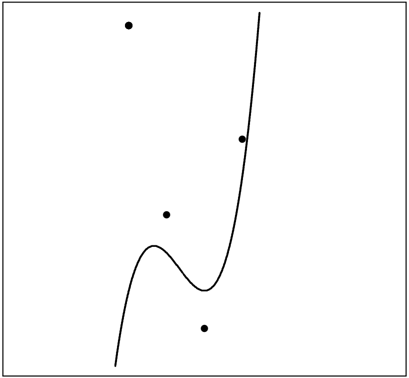

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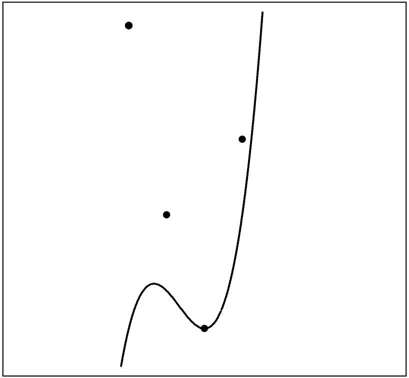

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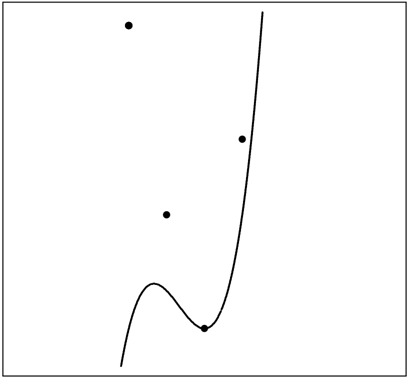

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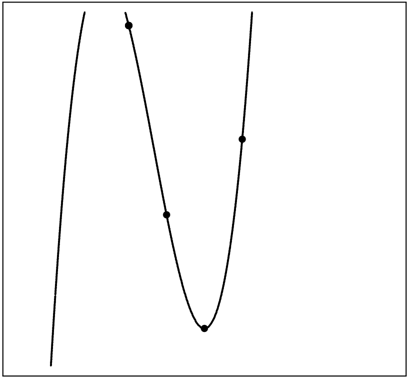

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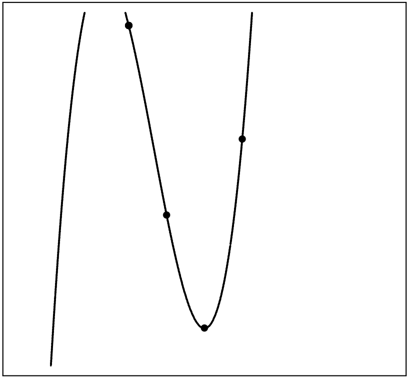

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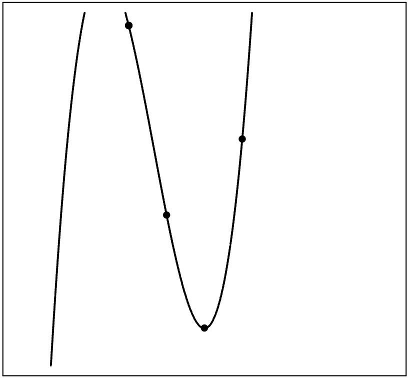

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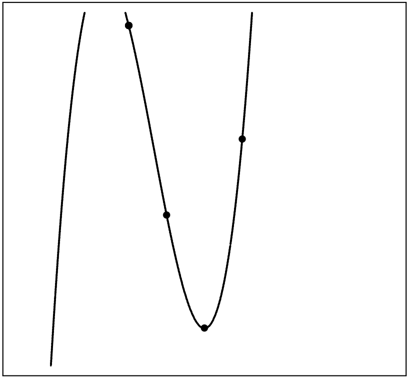

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In both cases the curve cannot contain all the singularities of $F$, hence the contradiction.

# Further developments

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With Menasché Schechter

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With M. da Silva Ferreira.

# Part V

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