The measurable chromatic number of Euclidean space

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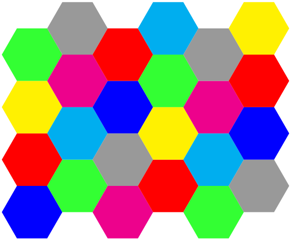
$\chi(\mathbb{R}^n)$

- The chromatic number of Euclidean space $\chi(\mathbb{R}^n)$ is the smallest number of colors needed to color every point of $\mathbb{R}^n$, such that two points at distance apart 1 receive different colors.
- E. Nelson, 1950, introduced $\chi(\mathbb{R}^2)$.
- Dimension 1:

$$\chi(\mathbb{R}) = 2$$

- No other value is known!

$\chi(\mathbb{R}^2) \leq 7$



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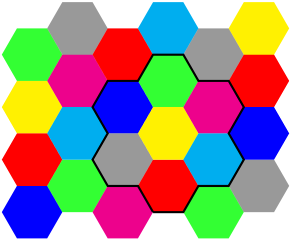

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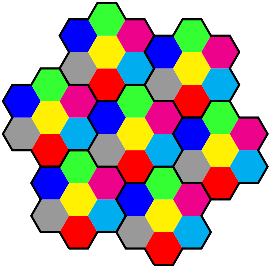


$\chi(\mathbb{R}^2) \geq 4$

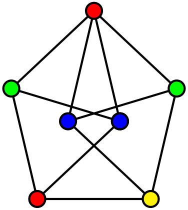


Figure: The Moser's Spindle

The two inequalities:

$$4 \leq \chi(\mathbb{R}^2) \leq 7$$

where proved by Nelson and Isbell, 1950. No improvements since then...

$\chi(\mathbb{R}^n)$

- Other dimensions: lower bounds based on

$$\chi(\mathbb{R}^n) \geq \chi(G)$$

for all finite graph $G = (V, E)$ embedded in $\mathbb{R}^n$ ($G \hookrightarrow \mathbb{R}^n$) i.e. such that $V \subset \mathbb{R}^n$ and $E = \{(x, y) \in V^2 : \|x - y\| = 1\}$.

- De Bruijn and Erdös (1951):

$$\chi(\mathbb{R}^n) = \max_{\substack{G \text{ finite} \\ G \hookrightarrow \mathbb{R}^n}} \chi(G)$$

- Good sequences of graphs: Raiski (1970), Larman and Rogers (1972), Frankl and Wilson (1981), Székely and Wormald (1989).

$\chi(\mathbb{R}^n)$ for large n

$$(1.2 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n$$

- Lower bound : Frankl and Wilson (1981). Use graphs with vertices in $\{0, 1\}^n$ and the “linear algebra method” to estimate $\chi(G)$.
- FW 1.207^n is improved to 1.239^n by Raigorodskii (2000).
- Upper bound: Larman and Rogers (1972). Use Voronoï decomposition of lattice packings.

$\chi_m(\mathbb{R}^n)$

- The measurable chromatic number $\chi_m(\mathbb{R}^n)$: the color classes are required to be measurable.
- Obviously $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$.
- Falconer (1981): $\chi_m(\mathbb{R}^n) \geq n + 3$. In particular

$$\chi_m(\mathbb{R}^2) \geq 5$$

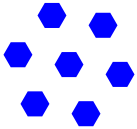
- The color classes are measurable 1-avoiding sets, i.e. contain no pair of points at distance apart 1.

$m_1(\mathbb{R}^n)$

$$m_1(\mathbb{R}^n) = \sup\left\{\delta(S) \ : \ S \subset \mathbb{R}^n,\ S \text{ measurable, avoids } 1\right\}$$

where $\delta(S)$ is the density of S:

$$\delta(S) = \limsup_{r \to +\infty} \frac{\mathrm{vol}(S \cap B_n(r))}{\mathrm{vol}(B_n(r))}.$$



$$\delta = 1/7$$

$m_1(\mathbb{R}^n)$

- Obviously

$$\chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)}$$

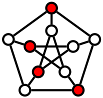
- Problem: to upper bound $m_1(\mathbb{R}^n)$.
- Larman and Rogers (1972):

$$m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|} \quad \text{for all } G \hookrightarrow \mathbb{R}^n$$

where $\alpha(G)$ is the independence number of the graph G i.e. the max number of vertices pairwise not connected by an edge.

Finite graphs

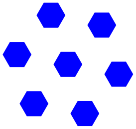
- An independence set of a graph $G = (V, E)$ is a set of vertices pairwise not connected by an edge.
- The independence number $\alpha(G)$ of the graph is the number of elements of a maximal independent set.



- A 1-avoiding set in $\mathbb{R}^n$ is an independent set of the unit distance graph

$$V = \mathbb{R}^n \qquad E = \{(x, y) : \|x - y\| = 1\}.$$

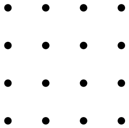
1-avoiding sets versus packings



S avoids $d = 1$

$\delta(S) = \lim \frac{\mathrm{vol}(S \cap B_n(r))}{\mathrm{vol}(B_n(r))}$

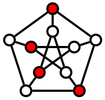
$m_1(\mathbb{R}^n) = \sup_S \delta(S)$?



S avoids $d \in]0, 2[$

$\delta(S) = \lim \frac{|S \cap B_n(r)|}{\mathrm{vol}(B_n(r))}$

$\delta_n = \sup_S \delta(S)$?



S avoids $d = 1$

$\delta(S) = \frac{|S|}{|V|}$

$\frac{\alpha(G)}{|V|} = \sup_S \delta(S)$?

The linear programming method

- A general method to obtain upper bounds for densities of distances avoiding sets.
- For packing problems: initiated by Delsarte, Goethels, Seidel on S^{n-1} (1977); Kabatianskii and Levenshtein on compact 2-point homogeneous spaces (1978); Cohn and Elkies on $\mathbb{R}^n$ (2003).
- For finite graphs: Lovász theta number $\vartheta(G)$ (1979).
- For sets avoiding one distance: B, G. Nebe, F. Oliveira, F. Vallentin for $m(S^{n-1}, \theta)$ (2009). F. Oliveira and F. Vallentin for $m_1(\mathbb{R}^n)$ (2010).

Lovász theta number

- The theta number $\vartheta(G)$ (L. Lovász, 1979) satisfies the Sandwich Theorem:
$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G})$$
- It is the optimal value of a semidefinite program
- Idea: if S is an independence set of G, consider the matrix

$$B_S(x,y) := \mathbf{1}_S(x)\,\mathbf{1}_S(y)/|S|.$$

$$B_S \succeq 0,\ B_S(x,y) = 0 \text{ if } xy \in E,\ |S| = \sum_{(x,y)\in V^2} B_S(x,y).$$

$\vartheta(G)$

- Defined by:

$$\vartheta(G) = \max\left\{ \sum_{(x,y)\in V^2} B(x,y) \quad : \; B \in \mathbb{R}^{V\times V},\; B \succeq 0, \right.$$
$$\sum_{x\in V} B(x,x) = 1,$$
$$\left. B(x,y) = 0 \quad xy \in E \right\}$$

- Proof of $\alpha(G) \leq \vartheta(G)$: Let S be an independent set. $B_S(x,y) = \mathbf{1}_S(x)\,\mathbf{1}_S(y)/|S|$ satisfies the constraints of the above SDP. Thus

$$\sum_{(x,y)\in V^2} B_S(x,y) = |S| \leq \vartheta(G).$$

$\vartheta(\mathbb{R}^n)$

- Over $\mathbb{R}^n$: take $B(x,y)$ continuous, positive definite, i.e. for all k, for all $x_1,\dots,x_k \in \mathbb{R}^n$, $\big(B(x_i,x_j)\big)_{1\le i,j\le k} \succeq 0$.
- Assume B is translation invariant: $B(x,y) = f(x-y)$ (the graph itself is invariant by translation).
- Replace $\sum_{(x,y)\in V^2} B(x,y)$ by

$$\delta(f) := \limsup_{r\to+\infty} \frac{1}{\mathrm{vol}(B_n(r))} \int_{B_n(r)} f(z)dz.$$

$\vartheta(\mathbb{R}^n)$

- Leads to:

$$\vartheta(\mathbb{R}^n) \quad := \sup \Big\{ \delta(f) : \quad f \in \mathcal{C}_b(\mathbb{R}^n),\ f \succeq 0 \\ f(0) = 1, \\ f(x) = 0 \quad \|x\| = 1 \Big\}$$

Theorem

(Oliveira Vallentin 2010)

$$m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$$

The computation of $\vartheta(\mathbb{R}^n)$

- Bochner characterization of positive definite functions:

$$f \in \mathcal{C}(\mathbb{R}^n), f \succeq 0 \iff f(x) = \int_{\mathbb{R}^n} e^{ix\cdot y} d\mu(y),\ \mu \geq 0.$$

- f can be assumed to be radial i.e. invariant under $O(\mathbb{R}^n)$:

$$f(x) = \int_0^{+\infty} \Omega_n(t\|x\|) d\alpha(t),\ \alpha \geq 0.$$

  where

$$\Omega_n(t) = \Gamma(n/2)(2/t)^{(n/2-1)} J_{n/2-1}(t).$$

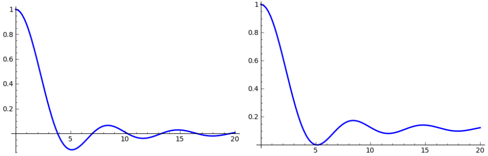
- Then take the dual program.

The computation of $\vartheta(\mathbb{R}^n)$

- Leads to:

$$\vartheta(\mathbb{R}^n) = \inf \{ z_0 : \begin{array}{l} z_0 + z_1 \geq 1 \\ z_0 + z_1 \Omega_n(t) \geq 0 \quad \text{for all } t > 0 \end{array} \}$$

- Explicitly solvable. For $n = 4$, graphs of $\Omega_4(t)$ and of the optimal function $f_4^*(t) = z_0^* + z_1^* \Omega_4(t)$:



The minimum of $\Omega_n(t)$ is reached at $j_{n/2,1}$ the first zero of $J_{n/2}$.

The computation of $\vartheta(\mathbb{R}^n)$

- We obtain

$$f_n^*(t) = \frac{\Omega_n(t) - \Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})} \qquad \vartheta(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})}.$$

- Resulting upper bound for $m_1(\mathbb{R}^n)$ (OV 2010):

$$m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})}$$

- Decreases exponentially but not as fast as Frankl Wilson Raigorodskii bound (1.165^{-n} instead of 1.239^{-n}). A weaker bound, but with the same asymptotic, was obtained in BNOV 2009 through $m(S^{n-1}, \theta)$.

$\vartheta_G(\mathbb{R}^n)$

- To summarize, we have seen two essentially different bounds:

$$m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|} \quad \text{with FW graphs and lin. alg. bound}$$

$$m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n) \quad \text{morally encodes } \vartheta(G) \text{ for every } G \hookrightarrow \mathbb{R}^n$$

- The former is the best asymptotic while the later improves the previous bounds in the range $3 \leq n \leq 24$.
- It is possible to combine the two methods, i.e to insert the constraint relative to a finite graph G inside $\vartheta(\mathbb{R}^n)$.
  Joint work (in progress) with F. Oliveira and F. Vallentin.

$\vartheta_G(\mathbb{R}^n)$

Let $G \hookrightarrow \mathbb{R}^n$, for $x_i \in V$, let $r_i := \|x_i\|$.

$$\vartheta_G(\mathbb{R}^n) := \inf\{z_0 + z_2 \frac{\alpha(G)}{|V|} : \quad \begin{array}{l} z_2 \geq 0 \\ z_0 + z_1 + z_2 \geq 1 \\ z_0 + z_1\Omega_n(t) + z_2(\frac{1}{|V|}\sum_{i=1}^{|V|}\Omega_n(r_i t)) \geq 0 \\ \text{for all } t > 0\}. \end{array}$$

Theorem

$$m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$$

Sketch of proof

- $\vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$ is obvious: take $z_2 = 0$.
- Sketch proof of $m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n)$: let S a measurable set avoiding 1. Let

$$f_S(x) := \frac{\delta(\mathbf{1}_{S-x}\mathbf{1}_S)}{\delta(S)}.$$

f_S is continuous bounded, $f_S \succeq 0$, $f_S(0) = 1$, $f_S(x) = 0$ if $\|x\| = 1$. Moreover $\delta(f_S) = \delta(S)$.
- Thus f_S is feasible for $\vartheta(\mathbb{R}^n)$, which proves that $\delta(S) \leq \vartheta(\mathbb{R}^n)$.

Sketch of proof

- If $V = \{x_1, \dots, x_M\}$, for all $y \in \mathbb{R}^n$,

$$\sum_{i=1}^{M} \mathbf{1}_{S-x_i}(y) \leq \alpha(G).$$

- Leads to the extra condition:

$$\sum_{i=1}^{M} f_S(x_i) \leq \alpha(G).$$

- Design a linear program, apply Bochner theorem, symmetrize by $O(\mathbb{R}^n)$, take the dual.

$\vartheta_G(\mathbb{R}^n)$

- Bad knews: cannot be solved explicitly (we don't know how to)
- Challenge: to compute good feasible functions.
- First method: to sample an interval $[0, M]$, solve a finite LP, then adjust the optimal solution (OV, $G =$ simplex).

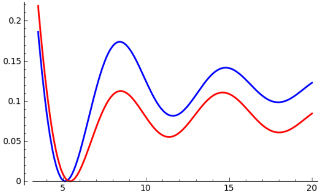


Figure: $f_4^*(t)$ (blue) and $f_{4,G}^*(t)$(red) for $G =$ simplex

$\vartheta_G(\mathbb{R}^n)$

- Observation: the optimal has a zero at $y > j_{n/2,1}$.
- Idea: to parametrize $f = z_0 + z_1\Omega_n(t) + z_2\Omega_n(rt)$ with y: $f(y) = f'(y) = 0$, $f(0) = 1$ determines f.
- We solve for:

$$\begin{cases} z_0 + z_1 + z_2 = 1 \\ z_0 + z_1\Omega_n(y) + z_2\Omega_n(ry) = 0 \\ z_1\Omega_n'(y) + rz_2\Omega_n'(ry) = 0 \end{cases}$$

- Then, starting with $y = j_{n/2,1}$, we move y to the right until $f_y(t) := z_0(y) + z_1(y)\Omega_n(t) + z_2(y)\Omega_n(rt)$ takes negative values.

Numerical results : upper bounds for $m_1(\mathbb{R}^n)$

n	previous	$\vartheta(\mathbb{R}^n)$ [OV 2010]	$\vartheta_{\text{simplex}}(\mathbb{R}^n)$ [OV 2010]	$\vartheta_{\text{FW}}(\mathbb{R}^n)$
2	0.279069	0.287120	0.268412	
3	0.187500	0.178466	0.165609	
4	0.128000	0.116826	0.112937	
5	0.0953947	0.0793346	0.0752845	
6	0.0708129	0.0553734	0.0515709	
7	0.0531136	0.0394820	0.0361271	
8	0.0346096	0.0286356	0.0257971	
9	0.0288215	0.0210611	0.0187324	
10	0.0223483	0.0156717	0.0138079	
11	0.0178932	0.0117771	0.0103166	
12	0.0143759	0.00892554	0.00780322	
13	0.0120332	0.00681436	0.00596811	
14	0.00981770	0.00523614	0.00461051	
15	0.00841374	0.00404638	0.00359372	0.00349172
16	0.00677838	0.00314283	0.00282332	0.00253343
17	0.00577854	0.00245212	0.00223324	0.00188025
18	0.00518111	0.00192105	0.00177663	0.00143383
19	0.00380311	0.00151057	0.00141992	0.00102386
20	0.00318213	0.001191806	0.00113876	0.000729883
21	0.00267706	0.000943209	0.00091531	0.000524659
22	0.00190205	0.000748582	0.00073636	0.000392892
23	0.00132755	0.000595665	0.00059204	0.000295352
24	0.00107286	0.000475128	0.00047489	0.000225128
25		0.000379829		0.000173756
26		0.000304278		0.000135634
27		0.000244227		0.000103665
28		0.000196383		0.0000725347
32		0.0000834258		0.00003061037
36		0.00003621287		0.000010504745
44		0.000007168656		0.0000013007413
52		0.0000014908331		0.00000016991978

Numerical results : lower bounds for $\chi_m(\mathbb{R}^n)$

n	previous	$\vartheta_G(\mathbb{R}^n)$	G
2	5		
3	6	7	Simplex
4	8	9	
5	11	14	
6	15	20	
7	19	28	
8	30	39	
9	35	54	
10	48	73	
11	64	97	
12	85	129	
13	113	168	
14	147	217	
15	191	287	FW
16	248	395	
17	319	532	
18	408	698	
19	521	977	
20	662	1371	
21	839	1907	
22	1060	2546	
23	1336	3386	
24	1679	4442	

Questions, comments

- Exponential behavior of $\vartheta_{FW}(\mathbb{R}^n)$?
- Further improvements for small dimensions: change the graph, consider several graphs. For $n = 2$, several triangles lead to 0.268412 (OV); several Moser spindles to 0.262387 (F. Oliveira 2011).
- Can we reach $m_1(\mathbb{R}^2) < 0.25$? (conjectured by Erdös; would give another proof of $\chi_m(\mathbb{R}^2) \geq 5$).
- Applies to other spaces, e.g. $m(S^{n-1}, \theta)$ (BNOV 2009).
- In turn, a bound for $m_1(S(0, r))$ can replace a finite graph G in $\vartheta_G(\mathbb{R}^n)$.
- The Lovász theta method was successfuly adapted to $\mathbb{R}^n$. What about the linear algebra method (Gil Kalai) ?