Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(14.1) Exercise: Minkowski's Theorem.
We consider the Euclidean space $\mathbb{R}^{2n}$, where $n \in \mathbb{N}$. Show that the limit $\lim_{r\to\infty} \frac{B_r(0)\cap\mathbb{Z}^{2n}}{r}$ exists, and compute its value.

(14.2) Exercise: Four-squares theorem.
a) Show that an integer of shape $4^a \cdot (8k-1)$, where $a \in \mathbb{N}_0$ and $k \in \mathbb{N}$, cannot be written as a sum of three squares in $\mathbb{Z}$. (The converse also holds [Legendre, 1798], but we are not able to show this here.)
b) Show that if $p \in \mathcal{P}_\mathbb{Z}$ is odd, then $4p$ is a sum of four odd squares in $\mathbb{Z}$.

(14.3) Exercise: Finiteness of class groups.
This is an alternative approach to prove the finiteness of class groups of algebraic number fields, without using Minkowski's Theorem:

a) Let K be an algebraic number field, let $\mathcal{O} := \mathcal{O}_K$ be its ring of integers, let $\mathcal{B} \subseteq \mathcal{O}$ be an integral basis, and let $c_K := \prod_{\sigma\in\mathrm{Inj}_\mathbb{Q}(K)}(\sum_{\omega\in\mathcal{B}} \|\omega^\sigma\|) \in \mathbb{R}$. Show that for any ideal $\{0\} \neq \mathfrak{a} \trianglelefteq \mathcal{O}$ there is $0 \neq \alpha \in \mathfrak{a}$ such that $|N_K(\alpha)| \leq c_K \cdot N(\mathfrak{a})$.

b) Show that any ideal class of $\mathcal{O}$ contains an ideal $\mathfrak{a}$ such that $N(\mathfrak{a}) \leq c_K$. Compare c_K with the Minkowski bound $b_K = M_{r,s} \cdot \sqrt{|\mathrm{disc}(K)|}$, where r and s are the number of real and of non-real embeddings of K, respectively.

(14.4) Exercise: Finiteness of class groups.
This is a simplified approach to prove the finiteness of class groups of algebraic number fields, using Minkowski's Theorem but yielding a weaker bound:

a) Let K be an algebraic number field, let $\mathcal{O} := \mathcal{O}_K$ be its ring of integers, and let r and s be the number of real and of non-real embeddings of K, respectively. Show that any ideal $\{0\} \neq \mathfrak{a} \trianglelefteq \mathcal{O}$ possesses an element $0 \neq \alpha \in \mathfrak{a}$ such that $|N_K(\alpha)| \leq (\frac{2}{\pi})^s \cdot \sqrt{|\mathrm{disc}(K)|} \cdot N(\mathfrak{a})$.

b) Conclude that any ideal class of $\mathcal{O}$ contains an ideal $\mathfrak{a}$ such that $N(\mathfrak{a}) \leq (\frac{2}{\pi})^s \cdot \sqrt{|\mathrm{disc}(K)|}$. Compare this with the Minkowski bound $M_{r,s} \cdot \sqrt{|\mathrm{disc}(K)|}$.

Hint for a). Use a subset of $\mathbb{R}^n$ consisting of the vectors $[x_1, \ldots, x_n]$ such that $|x_i| \leq c_i$ for $i \in \{1, \ldots, r\}$, and $x_{r+2j-1}^2 + x_{r+2j}^2 \leq c_{r+j}$ for $j \in \{1, \ldots, s\}$, where $c_1, \ldots, c_{r+s} \in \mathbb{R}$ such that $c_k > 0$ and $\prod_{k=1}^{r+s} c_k \geq (\frac{2}{\pi})^s \cdot \sqrt{|\mathrm{disc}(K)|} \cdot N(\mathfrak{a})$.