Übungen zur Algebraischen Zahlentheorie (WS 2023)

PD Dr. Jürgen Müller, **Ausgabe:** 25.01.2024

(15.1) Exercise: Factorial imaginary quadratic number fields.
For $d \in \mathbb{Z} \setminus \{0, 1\}$ square-free let $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$. Show that $\mathcal{O}_d$ is factorial, by showing that it has trivial class group, for

$$d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

(15.2) Exercise: Imaginary quadratic number fields.
For $0 > d \in \mathbb{Z}$ square-free let $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$.

a) If $\mathcal{O}_d$ has trivial class group, show the following:
i) We have $d \equiv 5 \pmod 8$, unless $d \in \{-1, -2, -7\}$.
ii) If $2 \neq p \in \mathcal{P}_{\mathbb{Z}}$ such that $p \nmid d$ and $0 < -d < 4p$, then $\left(\frac{d}{p}\right) = -1$.
iii) If $d < -19$, then $d \equiv \{-43, -67, -163, -403, -547, -667\} \pmod{840}$.

b) Determine all $-2000 < d < 0$ such that $\mathcal{O}_d$ has trivial class group.

(15.3) Exercise: Class numbers of quadratic number fields.
For $d \in \mathbb{Z} \setminus \{0, 1\}$ square-free let $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$, let Cl_d be its class group, and let $h_d \in \mathbb{N}$ be its class number. Show the following:
i) If $d \in \{2, 3, 5, 6, 7, 173, 293, 437\}$, then $h_d = 1$.
ii) If $d \in \{-6, -10, 10\}$, then $h_d = 2$.
iii) If $d \in \{-23, -31, -83, -139, 223\}$, then $h_d = 3$.
iv) If $d \in \{-14, -39\}$, then $h_d = 4$, where Cl_d is cyclic.
v) If $d \in \{-21, -30\}$, then $h_d = 4$, where Cl_d is non-cyclic.
vi) If $d = -103$, then $h_d = 5$.

(15.4) Exercise: Class numbers of cubic number fields.
a) For $m \in \{3, 5, 6, 17, 19\}$ let $K := \mathbb{Q}(\sqrt[3]{m})$, and let $\mathcal{O}_m$ be its rings of integers. Determine $\mathcal{O}_m$, and show that $\mathcal{O}_m$ has trivial class group.
b) Find the ring of integers of $\mathbb{Q}(\sqrt[3]{19})$, and show that it has class number 3.
c) Let $\alpha \in \mathbb{R}$ such that $\alpha^3 = \alpha + 1$, and let $\beta \in \mathbb{R}$ such that $\beta^3 = \beta + 7$. Show that both $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ have trivial class group.