Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(16.1) Exercise: Class numbers.
Determine the ring of integers of the biquadratic field $K := \mathbb{Q}(\sqrt{2}, \sqrt{-3})$, and its class number. Compare with the class number of its quadratic subfields.

(16.2) Exercise: Totally real number fields.
Let K be an algebraic number field, such that all its embeddings into $\mathbb{C}$ are real, and let $\emptyset \neq S \subset \mathrm{Inj}_{\mathbb{Q}}(K)$. Show that there is a unit $\epsilon \in \mathcal{O}_K^*$ such that $0 < \epsilon^\sigma < 1$ for all $\sigma \in S$, and $1 < \epsilon^\sigma$ for all $\sigma \in \mathrm{Inj}_{\mathbb{Q}}(K) \setminus S$.

(16.3) Exercise: Units in cubic fields.
a) Let K be a cubic number field having a unique real embedding, and $\mathcal{O} := \mathcal{O}_K$. Show that $\mathcal{O}^* = \pm\langle\epsilon\rangle$, where $\epsilon > 1$ is a uniquely defined fundamental unit.
b) Let $\rho \cdot \exp(\pm i\varphi) \in \mathbb{C}$ be the algebraic conjugates of ϵ, where $\rho > 0$ and $0 < \varphi < \pi$. Show that $\epsilon = \rho^{-2}$ and that $\mathrm{disc}(\epsilon) = -4\sin(\varphi)^2 \cdot (\rho^3 + \rho^{-3} - 2\cos(\varphi))$, and conclude that $|\mathrm{disc}(\epsilon)| < 4(\epsilon^3 + \epsilon^{-3} + 6)$.
c) Let $d := |\mathrm{disc}(\mathcal{O})|$. Show that $\epsilon^3 > \frac{d-28}{4}$, where for $d \geq 33$ even $\epsilon^3 > \frac{d-27}{4}$.

(16.4) Exercise: Units in cubic fields.
a) For $m \in \{2, 3, 5, 6, 7\}$ let $K := \mathbb{Q}(\sqrt[3]{m})$. Determine the fundamental unit $\epsilon \in \mathcal{O}_K^*$ such that $\epsilon > 1$.
b) Let $\alpha \in \mathbb{R}$ such that $\alpha^3 + \alpha - 3 = 0$, and let $K := \mathbb{Q}(\alpha)$ Determine its ring of integers $\mathcal{O}$, show that K has only one real embedding, and determine the fundamental unit $\epsilon \in \mathcal{O}^*$ such that $\epsilon > 1$.
c) Let $\beta \in \mathbb{R}$ such that $\beta^3 - 2\beta - 3 = 0$, and let $K := \mathbb{Q}(\beta)$ Determine its ring of integers $\mathcal{O}$, show that K has only one real embedding, and determine the fundamental unit $\epsilon \in \mathcal{O}^*$ such that $\epsilon > 1$.