Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(4.1) Exercise: Integral bases.
Let R be a principal ideal domain, let $K := \mathrm{Q}(R)$ be its field of fractions, let $K \subseteq L$ be a separable finite extension of degree $n := [L\colon K]$, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \subseteq L$ be a K-basis contained in the integral closure S of R in L. Show that for any $\alpha \in S$ we have $\alpha = \frac{1}{\delta} \cdot \sum_{j=1}^{n} r_j \alpha_j$, with unique $r_j \in R$ such that $\delta := \mathrm{disc}(\mathcal{B}) \mid r_j^2$.

(4.2) Exercise: Stickelberger's Criterion.
Let K be an algebraic number field of degree $n := [K\colon \mathbb{Q}]$, let $\mathcal{O}$ be its ring of integers, and let $\mathcal{B} := \{\alpha_1, \ldots, \alpha_n\} \subseteq K$ be a $\mathbb{Q}$-basis being contained in $\mathcal{O}$; then $\mathrm{disc}(\mathcal{B}) \in \mathbb{Z}$. Show that $\mathrm{disc}(\mathcal{B}) \equiv \{0, 1\} \pmod 4$. In particular, derive **Stickelberger's Criterion** saying that $\mathrm{disc}(\mathcal{O}) \equiv \{0, 1\} \pmod 4$.

Hint. Use Laplace expansion to compute $\det(\Delta_{\mathcal{B}})$.

(4.3) Exercise: Resultants.
Let R be an integral domain, and $f = \sum_{i=0}^{n} f_i X^i \in R[X]$ and $g = \sum_{j=0}^{m} g_j X^j \in R[X]$, where $f_n, g_m \neq 0$. Then the associated **Sylvester matrix** is defined as

$$S(f,g) := \left[\begin{array}{cccccc} f_n & f_{n-1} & \cdots & f_0 & & \\ & f_n & \cdots & f_1 & f_0 & \\ & & \ddots & & \ddots & \ddots \\ & & & f_n & \cdots \quad \cdots & f_0 \\ \hline g_m & g_{m-1} & \cdots & g_0 & & \\ & g_m & \cdots & g_1 & g_0 & \\ & & \ddots & & \ddots & \ddots \\ & & & g_m & \cdots \quad \cdots & g_0 \end{array}\right] \in R^{(n+m)\times(n+m)}.$$

where the upper and lower halves consist of m and n rows, respectively, and let $\mathrm{res}(f,g) := \det(S(f,g)) \in R$ be the **resultant** of f and g.

a) Let $K := \mathrm{Q}(R)$, let $\overline{K}$ be an algebraic closure of K, and let $f = f_n \cdot \prod_{i=1}^{n}(X - \alpha_i) \in \overline{K}[X]$ and $g = g_m \cdot \prod_{j=1}^{m}(X - \beta_j) \in \overline{K}[X]$. Show that $\mathrm{res}(f,g) = f_n^m \cdot \prod_{i=1}^{n} g(\alpha_i) = (-1)^{nm} g_m^n \cdot \prod_{j=1}^{m} f(\beta_j) = f_n^m g_m^n \cdot \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j)$.

b) Let $f = \sum_{i=0}^{n} f_i X^i \in K[X]$ be irreducible and separable, such that $f_n \neq 0$. Show that we have $\mathrm{disc}(f) = (-1)^{\binom{n}{2}} \cdot \frac{1}{f_n} \cdot \mathrm{res}(f, \partial f) \in K$.

(4.4) Exercise: Rings of integers.
Let $\alpha \in \mathbb{R}$ such that $\alpha^3 = \alpha + 4$. Show that $\{1, \alpha, \frac{1}{2}\alpha(1+\alpha)\}$ is an integral basis of $\mathbb{Q}(\alpha)$, and determine its discriminant.