Übungen zur Algebraischen Zahlentheorie (WS 2023)

PD Dr. Jürgen Müller, **Ausgabe:** 23.11.2023

(7.1) Exercise: Ramification in the Galois case.
Let $K \subseteq L$ be a Galois extension of algebraic number fields, and $G := \mathrm{Aut}_K(L)$.
a) If G is non-cyclic, show that there are only finitely many prime ideals in K which are non-split in L.
b) Let $K \subseteq M \subseteq L$ be a field, such that L is the normal closure of M. Show that a prime ideal in K is completely split in M if and only if it is so in L.

(7.2) Exercise: Ramification in the Abelian Galois case.
Let $K \subseteq L$ be a Galois extension of algebraic number fields, let $G := \mathrm{Aut}_K(L)$, and let $\mathfrak{p} \in \mathcal{P}_K$ and $\mathfrak{q} \in \mathcal{P}_L(\mathfrak{p})$, such that the decomposition group and the inertia group of $\mathfrak{q}$ are normal in G. (In particular, this is fulfilled if G is Abelian.)

Show that $\mathfrak{p}$ splits into $|\mathcal{P}_L(\mathfrak{p})|$ distinct prime ideals in the decomposition field of $\mathfrak{q}$, all of which remain prime ideals in the inertia field of $\mathfrak{q}$, but become an $e_K(\mathfrak{q})$-th power in L.

(7.3) Exercise: Decomposition fields and inertia fields.
Let $K := \mathbb{Q}(i, \sqrt{2}, \sqrt{5})$ and $p := 5$.
a) Show that $\mathbb{Q} \subseteq K$ is Galois, determine $\mathrm{Aut}_{\mathbb{Q}}(K)$, an integral basis of K, its ring of integers, and its discriminant.
b) Compute the factorisation of p in K, determine the associated decomposition and inertia fields, and compute the factorisation of p in these intermediate fields.

(7.4) Exercise: Decomposition fields and inertia fields.
Let $K := \mathbb{Q}(\sqrt[3]{19})$ and $p := 3$.
a) Compute the normal closure $K \subseteq L \subseteq \mathbb{C}$, determine $\mathrm{Aut}_{\mathbb{Q}}(L)$ and the embeddings of K into $\mathbb{C}$, an integral basis of L, its ring of integers, and its discriminant.
b) Compute the factorisation of p in K and in L, determine the associated decomposition and inertia fields, with respect to both $\mathbb{Q}$ and K, and compute the factorisation of p in these intermediate fields.