Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(8.1) Exercise: Galois ramification.
Let $K \subseteq L$ be an extension of algebraic number fields, let $\mathfrak{p} \in \mathcal{P}_K$ and $\mathfrak{q} \in \mathcal{P}_L(\mathfrak{p})$.

a) Let $K \subseteq M$ also be an extension of algebraic number fields. Show that if $\mathfrak{p}$ splits completely (is unramified) in both L and M, then $\mathfrak{p}$ splits completely (is unramified) in LM.

Conclude that $\mathfrak{p}$ splits completely (is unramified) in L if and only if $\mathfrak{p}$ splits completely (is unramified) in the normal closure of L.

b) Assume that $K \subseteq L$ is Galois, and that the decomposition group of $\mathfrak{q}$ is normal in $\mathrm{Aut}_K(L)$. For any intermediate field $K \subseteq M \subseteq L$ show that $M \subseteq D_{\mathfrak{q}}$ if and only if $\mathfrak{p}$ splits completely in M.

(8.2) Exercise: Primes in quadratic number rings.
Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be square-free, let $K := \mathbb{Q}(\sqrt{d})$, let $\mathcal{O}$ be its ring of integers, and let $p \in \mathbb{Z}$ be a prime. Determine the ideal factorisation of p in K. In particular, show that p is ramified in K if and only if $p \mid \mathrm{disc}(\mathcal{O})$.

Hint. Distinguish the congruence classes of d modulo 8, the cases $p \mid d$ and $p \nmid d$, and the cases $p = 2$ and p odd.

(8.3) Exercise: Decomposition fields and inertia fields.
Let $K := \mathbb{Q}(\sqrt{15})$ and $L := \mathbb{Q}(\sqrt{3}, \sqrt{5})$, and let $p \in \{2, 5\}$.
a) Compute the ideal factorisation of p in all subfields of L, show that p is non-split in K and L, and determine the decomposition and inertia fields.
b) Show that the unique prime ideal of K lying over p is non-principal, while the unique prime ideal of K lying over p is principal. Relate this to the question of unique factorisation of the element 10 in K and L.

(8.4) Exercise: Rings of integers in cubic fields.
We consider **Dedekind's example** $K := \mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{R}$ is such that $\alpha^3 + \alpha^2 - 2\alpha + 8 = 0$. Let $\mathcal{O}$ be the ring of integers of K.
a) Show that $\{1, \alpha, \frac{1}{2}\alpha(1 + \alpha)\}$ is an integral basis of K, and determine the discriminants $\mathrm{disc}(\mathcal{O})$ and $\mathrm{disc}(\mathbb{Z}[\alpha])$.
b) Show that the prime 2 splits completely in K.
c) Show that the index $[\mathcal{O} \colon \mathbb{Z}[\omega]]$ is even, for any $\omega \in \mathcal{O} \setminus \mathbb{Z}$. Conclude that K does not have an integral basis consisting of powers of a single element.