On Endomorphism Rings And Character Tables

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Contents

Wenn es die Verwirklichung von Urträumen ist, fliegen zu können und mit den Fischen zu reisen, sich unter den Leibern von Bergriesen durchzubohren, mit göttlichen Geschwindigkeiten Botschaften zu senden, das Unsichtbare und Ferne zu sehen und sprechen zu hören, Tote sprechen zu hören, sich in wundertätigen Genesungsschlaf versenken zu lassen, mit lebenden Augen erblicken zu können, wie man zwanzig Jahre nach seinem Tode aussehen wird, in flimmernden Nächten tausend Dinge über und unter dieser Welt zu wissen, die früher niemand gewußt hat, wenn Licht, Wärme, Kraft, Genuß, Bequemlichkeit Urträume der Menschheit sind, — dann ist die heutige Forschung nicht nur Wissenschaft, sondern ein Zauber, eine Zeremonie von höchster Herzens- und Hirnkraft, vor der Gott eine Falte seines Mantels nach der anderen öffnet, eine Religion, deren Dogmatik von der harten, mutigen, beweglichen, messerkühlen und -scharfen Denklehre der Mathematik durchdrungen und getragen wird.

Allerdings, es ist nicht zu leugnen, daß alle diese Urträume nach Meinung der Nichtmathematiker mit einemmal in einer ganz anderen Weise verwirklicht waren, als man sich das ursprünglich vorgestellt hatte. Münchhausens Posthorn war schöner als die fabriksmäßige Stimmkonserve, der Siebenmeilenstiefel schöner als ein Kraftwagen, Laurins Reich schöner als ein Eisenbahntunnel, die Zauberwurzel schöner als ein Bildtelegramm, vom Herz seiner Mutter zu essen und die Vögel zu verstehen schöner als eine tierpsychologische Studie über die Ausdrucksbewegungen der Vogelstimme. Man hat Wirklichkeit gewonnen und Traum verloren. [...]

Man braucht wirklich nicht viel darüber zu reden, es ist den meisten Menschen heute ohnehin klar, daß die Mathematik wie ein Dämon in alle Anwendungen unseres Leben gefahren ist. Vielleicht glauben nicht alle diese Menschen an die Geschichte vom Teufel, dem man seine Seele verkaufen kann; aber alle Leute, die von der Seele etwas verstehen müssen, weil sie als Geistliche, Historiker oder Künstler gute Einkünfte daraus beziehen, bezeugen es, daß sie von der Mathematik ruiniert worden sei und daß die Mathematik die Quelle eines bösen Verstandes bilde, der den Menschen zwar zum Herrn der Erde, aber zum Sklaven der Maschine mache. [...]

In Unkenntnis dieser Gefahren lebten eigentlich nur die Mathematiker selbst und ihre Schüler, die Naturforscher, die von alledem so wenig in ihrer Seele verspürten wie Rennfahrer, die fleißig darauf los treten und nichts in der Welt bemerken wie das Hinterrad ihres Vordermanns. [60, pp.39–40]

0 Introduction

(0.1) Graphs which are related to finite groups are of interest in both algebraic graph theory and group theory. From the group theoretical point of view, graphs on which a given group acts might yield new insights into the structure of the group; a few of the sporadic simple groups have even been discovered as automorphism groups of certain graphs, see [11, Ch.16.3]. From the point of view of algebraic graph theory, the automorphism group of a graph reflects the internal symmetry of the graph. In the present work, we shed some light on two aspects of this interplay between graphs and groups, namely so-called distance-transitive graphs and more generally distance-regular graphs, and so-called Ramanujan graphs; we give the appropriate definitions in Section 7.

Distance-transitivity is a rather strong graph theoretical condition, and in fact intimately relates the graph and its automorphism group. In particular, a distance-transitive graph can be realized as an orbital graph arising from the permutation action of its automorphism group on the vertex set of the graph, where additionally this permutation action turns out to be multiplicity-free. In particular the sporadic simple groups have been used in the construction of certain distance-transitive graphs. In recent years much progress has been made in the attempt to classify the distance-transitive and the distance-regular graphs, see [8]; but for the time being these classification problems are still open. Related to these graph theoretical classification problems is the group theoretical problem of classifying the multiplicity-free permutation actions of finite groups. Much work has been done on this classification problem as well, see the comments in [34], but currently this also is still open.

Ramanujan graphs are characterised by a certain property of their spectrum. Different constructions of series of Ramanujan graphs are known, and in all of them groups play a certain role, see [44, Ch.1,Ch.4.5]. One of these constructions realizes Ramanujan graphs as orbital graphs arising from a multiplicity-free permutation action of a certain finite general linear group, see [76, Ch.II.19]. It seems natural to consider the multiplicity-free permutation actions of other groups as well, in particular those of the sporadic simple groups, and to look for Ramanujan graphs amongst the arising orbital graphs. For the smaller sporadic simple groups such considerations have been made in the thesis [32], which the author has had the opportunity to co-supervise.

It seemed worth-while to compile a database containing as many as possible explicit results concerning the orbital graphs arising from permutation actions of the sporadic simple groups. As far as a multiplicity-free permutation action is concerned, the spectra of the arising orbital graphs are completely determined by, and indeed straightforwardly derived from, the character table of the endomorphism ring of the underlying permutation module. Thus the kind of information to be stored in a database is the character tables of these endomorphism rings. The database [7] is available electronically in GAP-readable format, in

http://www.math.rwth-aachen.de/~Juergen.Mueller/mferctbl/mferctbl.html.

The multiplicity-free permutation actions of the sporadic simple groups, their automorphism groups, their Schur covering groups and their bicyclic extensions have been classified in [6, 43, 5]. The work of systematically computing the character tables of the corresponding endomorphism rings, and related information, has been begun in [68]. In the thesis [32] these and other earlier results, scattered in the literature, have been collected and the remaining cases of multiplicity-free permutation actions of the sporadic simple groups and their automorphism groups on up to 10^7 points have been dealt with. We have now been able to compute the character tables for all but one (currently) of the cases of multiplicity-free permutation actions of the sporadic simple groups, their automorphism groups and their Schur covering groups on more than 10^7

points; these are listed in Section (11.1), see Table 7. An examination of the multiplicity-free permutation actions of the bicyclic extensions of the sporadic simple groups currently is under way.

The techniques used to compute the character tables of the endomorphism rings have been derived from methods of computational representation theory, so-called condensation techniques, which in the first place have been developed to determine decomposition numbers, in particular for the sporadic simple groups and related groups. It has turned out that suitable modifications of these methods can be used as computational workhorses for the present tasks. In particular, we have developed new efficient techniques to deal computationally with transitive group actions on large sets, and thus to enumerate long orbits or at least substantial parts thereof.

(0.2) The overall outline of the present work is as follows.

Part I deals with the more theoretical aspects. We take a slightly more general point of view as would be necessary to consider only permutation actions, inasmuch as we consider monomial representations of finite groups. In Section 1 we introduce the first main actor, the endomorphism ring of a transitive monomial representation of a finite group. We state *the* basic theorem revealing its structure, Schur's Theorem, and we introduce the notions necessary to describe its structural properties, in particular its regular representation. In Section 2 the representation theory of the endomorphism ring is related to the representation theory of the underlying group, the relevant notion being the Fitting correspondence. In Section 3 we introduce the second main actor, the character table of an endomorphism ring. We discuss its structural properties as well as its relation to the character table of the underlying group. In Section 4 we introduce another structure an endomorphism ring of a permutation module is endowed with, the Hadamard product. It is related to the tensor product structure on the characters of the underlying group. The material in Sections 1–4 is inspired by different expositions existing in the literature, where usually only the case of permutation representations is treated. But it seems worth-while to treat the slightly more general case of monomial representations in detail; in particular, we make use of the description of the general situation later on.

In Section 5 we consider the case where we have two transitive monomial representations such that there is an epimorphism from one of these to the other. This causes relations between the character values of the two corresponding endomorphism rings. The exposition is inspired by observations the author has made while compiling the above-mentioned database, where cases of two permutation actions being related as above indeed occur. In turn the theoretical description of this situation helps to compute a few of the character tables in the database. In Section 6 we take a more general point of view by considering arbitrary condensation functors. Condensation techniques, which are explicit computational applications of so-called condensation functors, have proven to be efficient workhorses for different tasks of computational representation theory, including the tasks we are faced with in the present work. It seems worth-while to know as much as possible about the general properties of condensation functors, formulated in terms of suitable module categories. In Section 7 we show how the information collected in the database indeed can be used to describe properties of the corresponding orbital graphs. We introduce the necessary notions from algebraic graph theory, such as the notions of distance-transitive and distance-regular graphs as well as Ramanujan graphs, and we indicate how the relevant properties can be checked using the database. In particular, we provide complete lists of imprimitive distance-transitive orbital graphs as well as non-distance-transitive but distance-regular orbital graphs arising from multiplicity-free permutation actions of the sporadic simple groups, their automorphism groups and their Schur covering groups, up to the above-mentioned exception. While the case of primitive distance-transitive orbital graphs for these groups has been dealt with in [34], the imprimitive case has been open so far, up to the knowledge of the author. Finally, we comment on Ramanujan orbital graphs.

Part II is concerned with the computational techniques which have been used to actually compute the character table of an endomorphism ring, where we restrict ourselves to the commutative case. In Section 8 we describe a technique, related to the Dixon-Schneider technique for the group algebra case, to compute the character table of an endomorphism ring if enough information on its regular representation is known. Furthermore, we introduce the notion of table automorphisms, and indicate how this is related to the problem of determining the Fitting correspondence for an explicitly given example. In Section 9 we consider practical aspects of condensation techniques. In particular we place the regular representation of an endomorphism ring into this context. We address the problem, arising in many practical applications of condensation methods, that we usually are not able to compute the full algebra acting on a condensed module, and present new ideas to circumvent this. In Section 10 we describe the ideas which have led to a new efficient technique to enumerate long orbits and discuss a few of the technical details. In particular, under certain circumstances this technique not only allows to enumerate an orbit, but also uses Schreier-Sims techniques to collect group theoretic information, for example on the point stabilizer. An implementation of this method has been used to deal with two of the largest examples in the database.

Part III gives the details of the computations necessary to compile the above-mentioned database, and gives two other applications of the techniques described in the present work. In Section 11 we present more details on the design of the database. In particular, we give references to earlier work used, and indicate the list of cases we are concerned with subsequently. Furthermore, we discuss the necessary computations to determine the Fitting correspondence explicitly, where we have to take care of the fact that there might be several multiplicity-free permutation actions for a fixed group to be considered at the same time. To determine the Fitting correspondence for one of these cases, the results on Krein parameters turn out to be helpful. In Sections 12–17 we case-by-case discuss the multiplicity-free permutation actions which are not covered by earlier results. In particular, in Section 17 we deal conclusively with the permutation action of the sporadic simple Baby Monster group B on the cosets of a maximal subgroup isomorphic to the sporadic simple Fischer group Fi_{23}. For this action not even the lengths of the suborbits have been known before. Besides the character table of the corresponding endomorphism ring, we are able to find faithful permutation representations of the two-point stabilizers, which determines the isomorphism types of these subgroups. Furthermore, we deal with the exceptional case mentioned above, which is the permutation action of the double cover $2.B$ of the Baby Monster on the cosets of a subgroup isomorphic to the Fischer group Fi_{23}. This is a covering of the permutation action of the Baby Monster group B on the cosets of the Fischer group Fi_{23} considered above. Here we are able to determine the suborbit lengths and the isomorphism types of the two-point stabilizers, but the character table of the corresponding endomorphism ring (as yet) remains unknown.

Finally, we give two other applications of the techniques described earlier. In Section 18 we present an application of the new technique to enumerate long orbits to solve a problem concerning the so-called Thompson-Smith lattice, whose lattice automorphism group is a split central extension $2 \times Th$ of the sporadic simple Thompson group Th. This problem is related to the still open problem to determine the minimum of the Thompson-Smith lattice. In Section 19 we present, by way of an example, a new idea to interpret condensation results, which works for the case where the condensed module is precisely the regular representation of the condensation algebra. The example dealt with is the problem of determining the 3-modular decomposition numbers for the sporadic simple Harada-Norton group HN; we present partial results for the non-principal block of defect 2.

(0.3) We assume the reader to be familiar with the ordinary and modular representation theory of finite groups, as general references see [3, 14, 15, 16, 18, 39], and occasionally with other prerequisites as well, which are mentioned on location. The standard methods from computational representation theory, in particular MeatAxe techniques, are also assumed to be known. We use the MeatAxe implementation [69], which is referred to as *the* MeatAxe. Furthermore, the standard methods from computational group theory, in particular the techniques dealing with permutation groups, are assumed to be known. We also use the computer algebra system GAP [22]; we assume the reader to be familiar with the techniques to access the information in its libraries, such as character tables or tables of marks, and to actually apply the algorithms implemented there, in particular those dealing with permutation groups, to explicitly given examples.

As parts of the exhibition are technical in nature, we have tried to fix the notation as early as possible and to keep it fixed throughout the whole of the present work. Most of the pieces are introduced in Sections 1 and 3 as well as 5. In later sections we have tried to give suitable backward references to enhance legibility. For groups we use the notation introduced in [13], indicating the normal subgroup structure. For groups dealt with in [13] we also use the notation used there to refer to conjugacy classes or irreducible characters. For an extension of a group G by an outer automorphism of order 2, we denote the extensions of a G-invariant irreducible character χ by $\chi^{\pm}$, where for groups dealt with in [13] the character χ^{+} refers to the character actually printed there. We use the notation $\mathrm{Irr.}(\cdot)$ for the set of irreducible characters of an algebra, where the subscript indicates the ground field.

(0.4) The author gratefully acknowledges enthusiastic and helpful discussions with Robert Wilson, Max Neunhöffer, Gerhard Hiss and Thomas Breuer on the topics of the present work. Furthermore, the present work could not have been written without the possibility to make very heavy use of the computing facilities at Lehrstuhl D für Mathematik, which quite a few of the other users indeed have suffered from.

Finally, the author thanks Gerhard Hiss, Gunter Malle and Cheryl Praeger for their willingness to act as referees for the present work, and for making valuable comments and suggestions, which have been incorporated into the current version.

I Endomorphism rings and character tables

1 Endomorphisms of monomial representations

We begin by fixing the basic notation and definitions which will be in force throughout the whole of the present work. The exposition of Section 1 is inspired by [39, Ch.II.12].

(1.1) Let G be a finite group, and $H \leq G$ be a subgroup of index $n := [G\colon H]$. Let $\mathcal{I} := \{1, \ldots, r\}$, where $r \in \mathbb{N}$ is the number of H-H-double cosets in G, and let $\{g_i \in G; i \in \mathcal{I}\}$ be a set of representatives of the H-H-double cosets in G. Hence we have $G = \coprod_{i \in \mathcal{I}} Hg_iH$. Without loss of generality let $g_1 := 1_G$.

For $i \in \mathcal{I}$ let $H_i := H^{g_i} \cap H \leq H$, and $\{h_{ij} \in H; j \in \{1, \ldots, k_i\}\}$ be a set of representatives of the right cosets of H_i in H, where $k_i := [H\colon H_i]$. Hence Hg_iH decomposes into right H-cosets as $Hg_iH = \coprod_{j=1}^{k_i} Hg_ih_{ij} \subseteq G$. Without loss of generality let $h_{i1} := 1_H$.

Hence we have $G = \coprod_{i \in \mathcal{I}} \coprod_{j=1}^{k_i} Hg_ih_{ij}$. Let $\Omega := H|G$ be the set of right cosets of H in G, let $\omega_{ij} := Hg_ih_{ij}$, for $i \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$, and for short $\omega_i := \omega_{i1} = \omega_1 g_i$, as well as $\Omega_i := \{\omega_{ij} \in \Omega; j \in \{1, \ldots, k_i\}\}$. Then $\Omega = \coprod_{i \in \mathcal{I}} \Omega_i$ is the partition of Ω into H-orbits, where $H_i := \mathrm{Stab}_H(\omega_i)$ and $k_i = |\Omega_i|$. In particular we have $\Omega_1 = \{\omega_1\}$ and $k_1 = 1$.

Let $\pi_\Omega\colon G \to \mathcal{S}_\Omega$ denote the group homomorphism from G to the symmetric group $\mathcal{S}_\Omega$ on Ω defined by the transitive right action of G on Ω.

(1.2) Definition. Let $i \in \mathcal{I}$.
a) The number r is called the *rank* of H in G.
b) The number k_i is called the i-th *index parameter* of H in G. The set Ω_i is called the i-th *suborbit* of G. The suborbit Ω_1 is called the *trivial* suborbit.
c) The G-orbit $\mathcal{O}_i := (\omega_1, \omega_i) \cdot G \subseteq \Omega \times \Omega$ of $(\omega_1, \omega_i) \in \Omega \times \Omega$ is called the i-th *orbital* of G.
d) The orbital $\mathcal{O}_{i^*} := \{(\omega', \omega) \in \Omega \times \Omega; (\omega, \omega') \in \mathcal{O}_i\}$ is called the orbital *paired* to $\mathcal{O}_i$, thus defining an involution $*: \mathcal{I} \to \mathcal{I}$. If $i = i^*$, then $\mathcal{O}_i$ is called *self-paired*.

Hence $\Omega \times \Omega = \coprod_{i \in \mathcal{I}} \mathcal{O}_i$ is the partition of $\Omega \times \Omega$ into G-orbits, and we have $|\mathcal{O}_i| = n \cdot k_i$ and $\mathcal{O}_i \cap (\Omega_1 \times \Omega) = \Omega_1 \times \Omega_i$, as well as $k_{i^*} = k_i$ and $\omega_1 g_i^{-1} \in \Omega_{i^*}$.

(1.3) Let Θ be an integral domain. Let λ be a representation of ΘH, such that the underlying ΘH-module is Θ-free of degree 1. The ΘH-module endowed with the ΘH-action given by λ is denoted by Θ_λ. Let λ^G be the induced representation of ΘG obtained from λ. Its underlying ΘG-module is given as $\Theta_\lambda \otimes_{\Theta H} \Theta G = \bigoplus_{i \in \mathcal{I}} \bigoplus_{j=1}^{k_i} \Theta_\lambda \otimes g_i h_{ij} \cong \Theta_\lambda \Omega$, where $\Theta_\lambda \Omega$ is the free Θ-module with Θ-basis Ω, the subscript still indicating the underlying ΘH-action, and where the isomorphism is given by $1 \otimes g_i h_{ij} \mapsto \omega_{ij}$. Hence we may identify $\Theta_\lambda \otimes_{\Theta H} \Theta G$ and $\Theta_\lambda \Omega$ using this ΘG-isomorphism. The action of G on $\Theta_\lambda \Omega$ is described as follows.

(1.4) Definition. Let $g \in G$. For $i \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$ let $g_i h_{ij} \cdot g = h \cdot g_{i'} h_{i'j'}$, where $\omega_{i'j'} = \omega_{ij} \cdot \pi_\Omega(g)$ and $h \in H$. Let

$$\lambda_{\omega_{ij}}(g) := \lambda(h) \in \lambda(H) \subseteq \Theta.$$

Thus we have $\lambda^G(g) \colon \omega \mapsto \lambda_\omega(g) \cdot (\omega \cdot \pi_\Omega(g))$, for $\omega \in \Omega$.

(1.5) We introduce the first main actor of the present work. Let

$$E_\Theta^\lambda := \mathrm{End}_{\Theta G}(\Theta_\lambda \otimes_{\Theta H} \Theta G)$$

be the ΘG-endomorphism ring of the induced ΘG-module $\Theta_\lambda \otimes_{\Theta H} \Theta G$, where E_Θ^λ also acts from the right. Hence $\Theta_\lambda \otimes_{\Theta H} \Theta G$ is endowed with a $(\Theta G \otimes_\Theta E_\Theta^\lambda)$-right module structure.

By the Frobenius-Nakayama relations and Mackey's Theorem, see [3, Ch.3.3], we have as Θ-modules

$$\begin{aligned} E_\Theta^\lambda &\overset{(1)}{\cong} \mathrm{Hom}_{\Theta H}(\lambda, (\lambda^G)_H) \\ &\overset{(2)}{\cong} \textstyle\bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\Theta H}(\lambda, (\lambda_{H_i}^{g_i})^H) \\ &\overset{(3)}{\cong} \textstyle\bigoplus_{i \in \mathcal{I}} \mathrm{Hom}_{\Theta H_i}(\lambda_{H_i}, \lambda_{H_i}^{g_i}), \end{aligned}$$

where the representation λ^{g_i} of ΘH^{g_i} is defined as $\lambda^{g_i}(h) := \lambda(g_i h g_i^{-1})$, for $h \in H^{g_i}$. As λ is of degree 1, we have $\mathrm{Hom}_{\Theta H_i}(\lambda_{H_i}, \lambda_{H_i}^{g_i}) \neq \{0\}$ if and only if $\lambda_{H_i} = \lambda_{H_i}^{g_i}$, in which case we have $\mathrm{Hom}_{\Theta H_i}(\lambda_{H_i}, \lambda_{H_i}^{g_i}) \cong \Theta$.

(1.6) Definition. Let $\mathcal{I}_\lambda := \{i \in \mathcal{I}; \lambda_{H_i} = \lambda_{H_i}^{g_i}\}$.

We have $1 \in \mathcal{I}_\lambda$, and since $\lambda_{H_i} = \lambda_{H_i}^{g_i}$ implies $\lambda^{g_i^{-1}}_{H \cap H^{g_i^{-1}}} = \lambda_{H \cap H^{g_i^{-1}}}$, we have $i^* \in \mathcal{I}_\lambda$ whenever $i \in \mathcal{I}_\lambda$. For the case $\lambda = 1$, the trivial representation of ΘH, we have $\mathcal{I}_1 = \mathcal{I}$.

(1.7) By the explicit formulation of the Θ-isomorphisms (1), (2) and (3) in Section (1.5), we obtain an explicit basis of E_Θ^λ as follows. Let $i \in \mathcal{I}_\lambda$, and let $\alpha_i'' \in \mathrm{Hom}_{\Theta H_i}(\lambda_{H_i}, \lambda_{H_i}^{g_i})$ be defined by $\alpha_i'' \colon \Theta_\lambda \to \Theta_\lambda \otimes g_i \colon 1 \mapsto 1 \otimes g_i$, where the underlying ΘH_i-module of $\lambda_{H_i}^{g_i}$ is denoted by $\Theta_\lambda \otimes g_i$. Indeed, for $h \in H_i$ we have $\alpha_i'' \cdot \lambda^{g_i}(h) = \lambda(h) \cdot \alpha_i'' \colon 1 \mapsto \lambda(h) \otimes g_i$.

The Θ-isomorphism (3) is given by the exterior trace map, which yields $\alpha_i' \in \mathrm{Hom}_{\Theta H}(\lambda, (\lambda_{H_i}^{g_i})^H)$ given by $\alpha_i' \colon 1 \mapsto \sum_{j=1}^{k_i} \lambda(h_{ij}^{-1}) \otimes g_i h_{ij}$, where using Θ-isomorphism (2) the underlying ΘH-module of $(\lambda_{H_i}^{g_i})^H$ is $\bigoplus_{j=1}^{k_i} \Theta_\lambda \otimes g_i h_{ij} \leq \Theta_\lambda \otimes_{\Theta H} \Theta G$. Finally using Θ-isomorphism (1), which is the restriction map $\alpha \mapsto \alpha|_{\Theta_\lambda}$, this gives $\alpha_i^\lambda \in E_\Theta^\lambda$ defined by

$$\begin{aligned} \alpha_i^\lambda \colon 1 \otimes g_{i'} h_{i'j'} &\mapsto \left(\textstyle\sum_{j=1}^{k_i} \lambda(h_{ij}^{-1}) \otimes g_i h_{ij}\right) \cdot g_{i'} h_{i'j'} \\ &= \textstyle\sum_{j=1}^{k_i} \lambda(h_{ij}^{-1}) \lambda_{\omega_{ij}}(g_{i'} h_{i'j'}) \cdot (\omega_{ij} \cdot \pi_\Omega(g_{i'} h_{i'j'})), \end{aligned}$$

for $i' \in \mathcal{I}$ and $j' \in \{1, \ldots, k_{i'}\}$, where the last equality uses the identification of Section (1.3).

Let $\mathcal{A}_\lambda := \{\alpha_i^\lambda; i \in \mathcal{I}_\lambda\}$. In particular, as $\lambda_{\omega_1}(g_{i'} h_{i'j'}) = 1$, we have $\alpha_1^\lambda = \mathrm{id}_{\Theta_\lambda \Omega}$. For the case $\lambda = 1$ let $\alpha_i := \alpha_i^1$, for $i \in \mathcal{I}$, and $\mathcal{A} := \mathcal{A}_1$.

Hence we have shown the following theorem, which for the case $\lambda = 1$ first appeared in [72], see also [39, Ch.II.12], and which is *the* basic theorem of the present work.

(1.8) Theorem. E_Θ^λ is a free module over $\Theta \cdot \mathrm{id}_{\Theta_\lambda \Omega} \cong \Theta$ of Θ-rank $|\mathcal{I}_\lambda|$ and $\mathcal{A}_\lambda$ is a Θ-basis, the *Schur basis*, of E_Θ^λ.

(1.9) We collect a few facts on the Schur basis elements $\alpha_i^\lambda \in \mathcal{A}_\lambda$, for $i \in \mathcal{I}_\lambda$.

For $\alpha \in \mathrm{End}_\Theta(\Theta_\lambda \Omega)$ let $[\alpha] = [\alpha]_\Omega \in \Theta^{n \times n}$ be the representing matrix with respect to the Θ-basis Ω of $\Theta_\lambda \Omega$. The matrix entries of $[\alpha]$ are denoted by $[\alpha]_{\omega\omega'} \in \Theta$, for $\omega, \omega' \in \Omega$.

For $g \in G$ we let $\mathrm{diag}[\lambda_\omega(g); \omega \in \Omega]$ denote the diagonal matrix with entries $(\mathrm{diag}[\lambda_\omega(g); \omega \in \Omega])_{\omega', \omega''} = \delta_{\omega', \omega''} \cdot \lambda_{\omega'}(g)$, for $\omega', \omega'' \in \Omega$. Hence we obtain

$$[\lambda^G(g)] = \mathrm{diag}[\lambda_\omega(g); \omega \in \Omega] \cdot [\pi_\Omega(g)].$$

Thus we have

$$[\lambda^G(g)]^{-1} = [\pi_\Omega(g)]^T \cdot \mathrm{diag}[\lambda_\omega(g)^{-1}; \omega \in \Omega] = [\lambda^G(g)]^T \cdot \mathrm{diag}[\lambda_\omega(g)^{-2}; \omega \in \Omega].$$

(1.10) Proposition. Let $i \in \mathcal{I}_\lambda$. Then $[\alpha_i^\lambda]_{\omega\omega'} = 0$ unless $(\omega, \omega') \in \mathcal{O}_i$, in which case we have $[\alpha_i^\lambda]_{\omega\omega'} \in \lambda(H) \subseteq \Theta$. For $i' \in \mathcal{I}$ and $j' \in \{1, \ldots, k_{i'}\}$ we have

$$[\alpha_i^\lambda]_{\omega_1, \omega_{i'j'}} = \begin{cases} 0, & \text{if } i' \neq i, \\ \lambda(h_{i'j'}^{-1}), & \text{if } i' = i. \end{cases}$$

If $(\tilde{\omega}, \tilde{\omega}') = (\omega, \omega') \cdot g$ for some $g \in G$, then we have

$$[\alpha_i^\lambda]_{\tilde{\omega}\tilde{\omega}'} = [\alpha_i^\lambda]_{\omega\omega'} \cdot \frac{\lambda_{\omega'}(g)}{\lambda_\omega(g)}.$$

In particular, for the case $\lambda = 1$ we have, for $i \in \mathcal{I}$,

$$[\alpha_i]_{\omega,\omega'} := \begin{cases} 1, & \text{if } (\omega, \omega') \in \mathcal{O}_i, \\ 0, & \text{if } (\omega, \omega') \notin \mathcal{O}_i. \end{cases}$$

Proof. By Section (1.7) it only remains to prove the statement involving $[\alpha_i^\lambda]_{\tilde{\omega}\tilde{\omega}'}$. Let $\mathrm{diag}[\lambda_\omega(g)] := \mathrm{diag}[\lambda_\omega(g); \omega \in \Omega]$ for short. Then we have

$$\begin{aligned} [\alpha_i^\lambda]_{\tilde{\omega}\tilde{\omega}'} &= \left([\pi_\Omega(g)]^{-T} \cdot [\alpha_i^\lambda] \cdot [\pi_\Omega(g)]^{-1}\right)_{\omega\omega'} \\ &= \left(\mathrm{diag}[\lambda_\omega(g)] \cdot [\lambda^G(g)]^{-T} \cdot [\alpha_i^\lambda] \cdot [\lambda^G(g)]^{-1} \cdot \mathrm{diag}[\lambda_\omega(g)]\right)_{\omega\omega'} \\ &= \left(\mathrm{diag}[\lambda_\omega(g)^{-1}] \cdot [\alpha_i^\lambda] \cdot \mathrm{diag}[\lambda_\omega(g)]\right)_{\omega\omega'} \\ &= [\alpha_i^\lambda]_{\omega\omega'} \cdot \tfrac{\lambda_{\omega'}(g)}{\lambda_\omega(g)}. \end{aligned}$$

♯

We introduce a further structure on modules acted on monomially and their endomorphism rings. For technical reasons we have to adjust the base ring appropriately.

(1.11) Definition. Let $K := \mathrm{Quot}(\Theta)$ be the field of fractions of Θ and $K' \subseteq K$ be the subfield generated by $\lambda(H)$ over the prime field of K. As $\lambda(H)$ consists of roots of unity, there is an involutory field automorphism $\bar{\ }: K' \to K'$ defined by $\lambda(h) \mapsto \lambda(h)^{-1}$ for $h \in H$. Let $K'' := \mathrm{Fix}_{K'}(\bar{\ }) \subseteq K'$.

Let $\langle \cdot, \cdot \rangle_\Omega$ be the non-degenerate hermitian form on $K'_\lambda\Omega$, with respect to the field automorphism $\bar{\ }$, defined by $\langle \cdot, \cdot \rangle_\Omega : K'_\lambda\Omega \times K'_\lambda\Omega \to K' : (\omega, \omega') \mapsto \delta_{\omega,\omega'}$.

Since for $g \in G$ we have $\omega g = \lambda_\omega(g) \cdot \omega\pi_\Omega(g)$, the form $\langle \cdot, \cdot \rangle_\Omega$ is G-invariant.

(1.12) Definition. For $i \in \mathcal{I}_\lambda$ let $i^- \in \{1, \ldots, k_{i^*}\}$ and $\eta_i \in H$ such that $g_i^{-1} = \eta_i \cdot g_{i^*} \cdot h_{i^* i^-}$. Furthermore let

$$\zeta_i := \frac{\lambda(\eta_i)}{\lambda(h_{i^* i^-})} \in \lambda(H).$$

(1.13) Proposition. For $i \in \mathcal{I}_\lambda$, the adjoint map $(\alpha_i^\lambda)^\sharp \in \mathrm{End}_{K'}(K'_\lambda\Omega)$ of α_i^λ with respect to the form $\langle\cdot,\cdot\rangle_\Omega$ is given by $(\alpha_i^\lambda)^\sharp = \frac{1}{\zeta_i} \cdot \alpha_{i^*}^\lambda$. Thus we have an involutory K''-algebra antiautomorphism of $E_{K'}^\lambda$ given by

$$\sharp\colon E_{K'}^\lambda \to E_{K'}^\lambda \colon \alpha_i^\lambda \mapsto \frac{1}{\zeta_i} \cdot \alpha_{i^*}^\lambda.$$

Proof. For $i \in \mathcal{I}_\lambda$, as $[\alpha_i^\lambda] \cdot [\lambda^G(g)] = [\lambda^G(g)] \cdot [\alpha_i^\lambda]$, we have

$$[\pi_\Omega(g)]^T \cdot \mathrm{diag}[\overline{\lambda_\omega(g)}; \omega \in \Omega] \cdot \overline{[\alpha_i^\lambda]^T} = \overline{[\alpha_i^\lambda]^T} \cdot [\pi_\Omega(g)]^T \cdot \mathrm{diag}[\overline{\lambda_\omega(g)}; \omega \in \Omega].$$

Since $[\pi_\Omega(g)]^T \cdot \mathrm{diag}[\overline{\lambda_\omega(g)}; \omega \in \Omega] = [\pi_\Omega(g^{-1})]$, we conclude that $\overline{[\alpha_i^\lambda]^T}$ is a scalar multiple of $[\alpha_{i^*}^\lambda]$. Since $(\omega_1, \omega_i) \cdot g_i^{-1} = (\omega_1 g_i^{-1}, \omega_1)$ and $\lambda_{\omega_i}(g_i^{-1}) = 1$ we have

$$[\alpha_i^\lambda]_{\omega_1 g_i^{-1}, \omega_1} = [\alpha_i^\lambda]_{\omega_1, \omega_i} \cdot \frac{\lambda_{\omega_i}(g_i^{-1})}{\lambda_{\omega_1}(g_i^{-1})} = \frac{1}{\lambda_{\omega_1}(g_i^{-1})} = \frac{1}{\lambda(\eta_i)}.$$

Because of $[\alpha_{i^*}^\lambda]_{\omega_1, \omega_1 g_i^{-1}} = \lambda(h_{i^* i^-})^{-1}$, we have $\zeta_i \cdot \overline{[\alpha_i^\lambda]^T} = [\alpha_{i^*}^\lambda]$. $\sharp$

(1.14) Corollary. Let $i, j \in \mathcal{I}_\lambda$.
a) For the case $\lambda = 1$ we have $\alpha_i^\sharp = \alpha_{i^*}$ and thus $[\alpha_i]^T = [\alpha_{i^*}]$, for $i \in \mathcal{I} = \mathcal{I}_\lambda$.
b) Since $\alpha_i^\lambda = (\alpha_i^\lambda)^{\sharp\sharp} = \zeta_i \cdot (\alpha_{i^*}^\lambda)^\sharp = \frac{\zeta_i}{\zeta_{i^*}} \cdot \alpha_i^\lambda$, we have $\zeta_{i^*} = \zeta_i$.
c) If $i = i^*$, then we have $\zeta_i \cdot (\alpha_i^\lambda)^\sharp = \alpha_i^\lambda$, while if $j \neq j^*$, then we have $\zeta_j \cdot (\alpha_j^\lambda \pm \alpha_{j^*}^\lambda)^\sharp = \alpha_{j^*}^\lambda \pm \alpha_j^\lambda$. Hence α_i^λ and $\alpha_j^\lambda \pm \alpha_{j^*}^\lambda$ commute with their respective adjoint maps, and thus α_i^λ and $\alpha_j^\lambda \pm \alpha_{j^*}^\lambda$ are diagonalisable over an algebraic closure of K'.

The following notions first appeared in [28]. Their intention is to exhibit a finer structure of the suborbits $\Omega_i \subseteq \Omega$, for $i \in \mathcal{I}_\lambda$.

(1.15) Definition. **a)** For $i, j, k \in \mathcal{I}$, a triple $(\omega, \omega', \omega'') \in \Omega \times \Omega \times \Omega$ such that $(\omega, \omega') \in \mathcal{O}_i$, $(\omega', \omega'') \in \mathcal{O}_j$, and $(\omega, \omega'') \in \mathcal{O}_k$ is called *triangle of type* (i, j, k). Let $\mathcal{T}_{ijk} \subseteq \Omega \times \Omega \times \Omega$ be the set of triangles of type (i, j, k).
b) For $i, j, k \in \mathcal{I}_\lambda$, the λ-*weight* of the triangle $(\omega, \omega', \omega'') \in \mathcal{T}_{ijk}$ is defined as

$$\lambda(\omega, \omega', \omega'') := [\alpha_i^\lambda]_{\omega\omega'} \cdot [\alpha_j^\lambda]_{\omega'\omega''} \cdot ([\alpha_k^\lambda]_{\omega\omega''})^{-1} \in \lambda(H) \subseteq \Theta.$$

For $\zeta \in \lambda(H)$ let

$$\mathcal{T}_{ijk}^{\lambda,\zeta} := \{(\omega, \omega', \omega'') \in \mathcal{T}_{ijk}; \lambda(\omega, \omega', \omega'') = \zeta\}$$

be the set of triangles of type (i, j, k) and λ-weight ζ.
c) For $i, j, k \in \mathcal{I}_\lambda$ and $\zeta \in \lambda(H)$ let

$$\Omega_{ijk}^{\lambda,\zeta} := \{\omega \in \Omega; (\omega_1, \omega, \omega_k) \in \mathcal{T}_{ijk}^{\lambda,\zeta}\} \subseteq \Omega_i.$$

Let $S := \{s \in \{1, \ldots, k_i\}; \omega_{is} \in \Omega_{ijk}^{\lambda,\zeta}\}$ and $p_{ijk}^{\lambda,\zeta} := |\Omega_{ijk}^{\lambda,\zeta}| = |S| \in \mathbb{N}_0$.

(1.16) Remark. As the $\mathcal{O}_i \subseteq \Omega \times \Omega$ are invariant under diagonal G-action, the sets $\mathcal{T}_{ijk}$, for $i, j, k \in \mathcal{I}$, are invariant under diagonal action of G on $\Omega \times \Omega \times \Omega$ as well. For $i, j, k \in \mathcal{I}_\lambda$ and $(\omega, \omega', \omega'') \in \mathcal{T}_{ijk}$ as well as $g \in G$ we have

$$\begin{aligned}\lambda(\omega g, \omega' g, \omega'' g) &= [\alpha_i^\lambda]_{\omega g, \omega' g} \cdot [\alpha_j^\lambda]_{\omega' g, \omega'' g} \cdot ([\alpha_k^\lambda]_{\omega g, \omega'' g})^{-1} \\ &= [\alpha_i^\lambda]_{\omega\omega'} \cdot [\alpha_j^\lambda]_{\omega'\omega''} \cdot ([\alpha_k^\lambda]_{\omega\omega''})^{-1} \cdot \tfrac{\lambda_{\omega'}(g)}{\lambda_\omega(g)} \cdot \tfrac{\lambda_{\omega''}(g)}{\lambda_{\omega'}(g)} \cdot \tfrac{\lambda_\omega(g)}{\lambda_{\omega''}(g)} \\ &= \lambda(\omega, \omega', \omega'').\end{aligned}$$

Hence the sets $\mathcal{T}_{ijk}^{\lambda,\zeta}$ for a fixed λ-weight $\zeta \in \lambda(H)$ are unions of G-orbits as well. These are, as $\mathcal{O}_k \subseteq \Omega \times \Omega$ is a single G-orbit, in natural bijection with the set of H_k-orbits on $\Omega_{ijk}^{\lambda,\zeta}$.

As $\Omega_i = \{\omega_{is}; s \in \{1, \ldots, k_i\}\}$ is as an H-set isomorphic to the set $H_i|H$ of right cosets of H_i in H, it follows that $\Omega^{\lambda,\zeta}_{\omega_1, i, \omega_k}$ is as an H_k-set isomorphic to $\coprod_{s \in S'} (H_i^{h_{is}} \cap H_k)|H_k$, where $S' \subseteq S$ is chosen such that $\{h_{is}; s \in S'\}$ is a set of representatives of the union $\bigcup_{s \in S} H_i \cdot h_{is} \cdot H_k$ of H_i-H_k-double cosets in H. Hence we have

$$p_{ijk}^{\lambda,\zeta} = \sum_{s \in S'} [H_k \colon (H_k \cap H_i^{h_{is}})].$$

(1.17) Proposition. For $i, j, k \in \mathcal{I}_\lambda$ let $(\omega_1, \omega_{is}, \omega_k) \in \mathcal{T}_{ijk}^\lambda$, for some $s \in \{1, \ldots, k_i\}$. Let $g_i h_{is} \cdot g_k^{-1} = h_s \cdot g_{j^*} h_{j^* t}$ for some $t \in \{1, \ldots, k_{j^*}\}$ and $h_s \in H$. Then we have

$$\lambda(\omega_1, \omega_{is}, \omega_k) = \zeta_j \cdot \lambda(h_s) \cdot \frac{\lambda(h_{j^* t})}{\lambda(h_{is})}.$$

Proof. We have $[\alpha_k^\lambda]_{\omega_1, \omega_k} = 1$ and $[\alpha_i^\lambda]_{\omega_1, \omega_{is}} = \lambda(h_{is}^{-1})$, as well as

$$[\alpha_j^\lambda]_{\omega_{is}, \omega_k} = [\alpha_j^\lambda]_{\omega_{is} g_k^{-1}, \omega_1} \cdot \frac{\lambda_{\omega_1}(g_k)}{\lambda_{\omega_{is} g_k^{-1}}(g_k)} = [\alpha_j^\lambda]_{\omega_{is} g_k^{-1}, \omega_1} \cdot \lambda(h_s),$$

since $\lambda_{\omega_1}(g_k) = 1$ and $\lambda_{\omega_{is} g_k^{-1}}(g_k) = \lambda(h_s^{-1})$. Using Proposition (1.13), we have $[\alpha_j^\lambda]_{\omega_{is} g_k^{-1}, \omega_1} = \overline{[\alpha_{j^*}^\lambda]}_{\omega_1, \omega_{is} g_k^{-1}} \cdot \zeta_j$. As $[\alpha_{j^*}^\lambda]_{\omega_1, \omega_{is} g_k^{-1}} = \lambda(h_{j^* t}^{-1})$, the assertion follows. $\sharp$

The regular representation of the endomorphism ring E_Θ^λ plays a central role in the present work. The aim of the following definition is to facilitate a description of the regular representation.

(1.18) Definition.
a) For $i, j \in \mathcal{I}_\lambda$, by Theorem (1.8), we have $\alpha_i^\lambda \cdot \alpha_j^\lambda = \sum_{k \in \mathcal{I}_\lambda} p_{ijk}^\lambda \alpha_k^\lambda$, for the *structure constants* $p_{ijk}^\lambda \in \Theta$. For the case $\lambda = 1$ let $p_{ijk} := p_{ijk}^1$.

b) For $j \in \mathcal{I}_\lambda$, the representing matrix $[\alpha_j^\lambda]_{\mathcal{A}_\lambda}$ of the right regular action of α_j^λ on E_Θ^λ, with respect to the Schur basis $\mathcal{A}_\lambda$, is given by the j-th *structure constants matrix*

$$[\alpha_j^\lambda]_{\mathcal{A}_\lambda} = P_j^\lambda := [p_{ijk}^\lambda; i, k \in \mathcal{I}_\lambda] \in \Theta^{|\mathcal{I}_\lambda| \times |\mathcal{I}_\lambda|},$$

with row index i and column index k. For the case $\lambda = 1$ let $P_j := P_j^1$.

(1.19) Remark. Let $i, j, k \in \mathcal{I}_\lambda$.
a) By considering the matrix entry $[\alpha_i^\lambda \cdot \alpha_j^\lambda]_{\omega_1, \omega_k}$, where $[\alpha_i^\lambda \cdot \alpha_j^\lambda]$ still is the representing matrix of the natural action of $\alpha_i^\lambda \cdot \alpha_j^\lambda$ on $\Theta_\lambda \Omega$ with respect to the basis Ω, we obtain

$$p_{ijk}^\lambda = \sum_{\zeta \in \lambda(H)} \zeta \cdot p_{ijk}^{\lambda,\zeta}.$$

Furthermore, we have

$$p_{ijk} = \sum_{\zeta \in \lambda(H)} p_{ijk}^{\lambda,\zeta}.$$

Using the involutory K''-algebra antiautomorphism $\sharp \colon E_{K'}^\lambda \to E_{K'}^\lambda$, see Proposition (1.13), we obtain

$$p_{j^* i^* k^*}^\lambda = \frac{\zeta_i \cdot \zeta_j}{\zeta_k} \cdot \overline{p_{ijk}^\lambda}.$$

b) For the special case $j = 1$ we have $g_j = 1$ and hence $\zeta_j = 1$. Furthermore $S = \emptyset$ unless $i = k$, in which case we have $S \subseteq \{1\}$, and for $\omega_{is} = \omega_{k,1} = \omega_k$ we have $\lambda(h_s) \cdot \frac{\lambda(h_{j^*t})}{\lambda(h_{is})} = 1$. Thus $p_{i,1,k}^{\lambda,\zeta} = \delta_{i,k}\delta_{\zeta,1}$. Hence $p_{i,1,k}^\lambda = \delta_{i,k}$, as expected. Analogously, for the special case $i = 1$ we have $p_{1,j,k}^\lambda = \delta_{j,k}$.

For the special case $k = 1$ we have $g_k = 1$ and hence $S = \emptyset$ unless $j = i^*$, in which case we have $\lambda(h_s) \cdot \frac{\lambda(h_{j^*t})}{\lambda(h_{is})} = 1$. Hence $S = \emptyset$ unless $\zeta = \zeta_j$, in which case we have $S = \{1, \ldots, k_i\}$. Hence we conclude $p_{i,j,1}^{\lambda,\zeta} = \delta_{i^*,j} \cdot \delta_{\zeta,\zeta_{i^*}} \cdot k_i$ and $p_{i,j,1}^\lambda = \delta_{i^*,j} \cdot \zeta_{i^*} \cdot k_i$.
c) We have

$$\begin{aligned} p_{ijk} &= |\{\omega \in \Omega; (\omega_1, \omega) \in \mathcal{O}_i, (\omega, \omega_k) \in \mathcal{O}_j\}| \\ &= |\{\omega \in \Omega_i; (\omega_k, \omega) \in \mathcal{O}_{j^*}\}| \\ &= |\{\omega \in \Omega_i; (\omega_1, \omega g_k^{-1}) \in \mathcal{O}_{j^*}\}| \\ &= |\{\omega \in \Omega_i; \omega g_k^{-1} \in \Omega_{j^*}\}| \\ &= |\Omega_i \cap (\Omega_{j^*} g_k)| \\ &= |(\Omega_i g_k^{-1}) \cap \Omega_{j^*}|. \end{aligned}$$

Because of this the $p_{ijk} \in \mathbb{N}_0$ are also called *intersection numbers*, and the matrices P_j are also called *intersection matrices*.

For j and k fixed, the k-th column sum of P_j is

$$\sum_{i\in\mathcal{I}}[P_j]_{ik} = \sum_{i\in\mathcal{I}} p_{ijk} = \sum_{i=1}^{r} |\Omega_i \cap (\Omega_{j^*} g_k)| = |\Omega_{j^*} g_k| = k_j.$$

d) Let $K' \subseteq \overline{K}$ be an algebraic closure of K', and let $i, j \in \mathcal{I}_\lambda$, where $i = i^*$ and $j \neq j^*$. By Corollary (1.14), the maps α_i^λ and $\alpha_j^\lambda \pm \alpha_{j^*}^\lambda$ are diagonalisable over $\overline{K}$, hence have square-free minimum polynomials over $\overline{K}$. As $E_{K'}^\lambda$ acts faithfully on $K'\Omega$, the minimum polynomials of the regular action of α_i^λ and of $\alpha_j^\lambda \pm \alpha_{j^*}^\lambda$ on $E_{\overline{K}}^\lambda = E_{K'}^\lambda \otimes_{K'} \overline{K}$ also are square-free. Hence the structure constants matrices P_i^λ and $P_j^\lambda \pm P_{j^*}^\lambda$ are diagonalisable over $\overline{K}$ as well.

2 Fitting correspondence

The aim of Section 2 is to describe the connection of the representation theory of the endomorphism ring E_Θ^λ with the representation theory of the underlying group G. The exposition of Section 2 is inspired by [15, Ch.1.11.D].

(2.1) Let Θ be an integral domain such that the order $|H| \in \Theta$ of H is a unit in Θ. Let λ be a representation of ΘH of degree 1 with underlying ΘH-module Θ_λ. Let

$$\epsilon_\lambda := \frac{1}{|H|} \cdot \sum_{h\in H} \lambda(h^{-1}) \cdot h \in \Theta H \subseteq \Theta G$$

be the centrally primitive idempotent of ΘH belonging to λ.

We have an isomorphism of ΘG-modules

$$\sigma = \sigma_\lambda \colon \Theta_\lambda \Omega \to \epsilon_\lambda \Theta G \colon \omega_{ij} \mapsto \epsilon_\lambda g_i h_{ij},$$

where $\Theta_\lambda\Omega \cong \Theta_\lambda \otimes_{\Theta H} \Theta G$ is the induced ΘG-module obtained from Θ_λ, see Section (1.3), and $i \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$. The map

$$\tau = \tau_\lambda \colon \mathrm{End}_{\Theta G}(\epsilon_\lambda \Theta G) \to (\epsilon_\lambda \Theta G \epsilon_\lambda)^\circ \colon \alpha \mapsto \epsilon_\lambda \alpha$$

is an isomorphism of Θ-algebras, where $(\epsilon_\lambda \Theta G \epsilon_\lambda)^\circ$ denotes the opposed ring with multiplication given by $x \circ y := y \cdot x$, for $x, y \in \epsilon_\lambda \Theta G \epsilon_\lambda$. The inverse of τ is given by

$$\tau^{-1} = \tau_\lambda^{-1} \colon (\epsilon_\lambda \Theta G \epsilon_\lambda)^\circ \to \mathrm{End}_{\Theta G}(\epsilon_\lambda \Theta G) \colon \epsilon_\lambda g \epsilon_\lambda \mapsto (\epsilon_\lambda h \mapsto \epsilon_\lambda g \epsilon_\lambda h),$$

for $g, h \in G$.

(2.2) Proposition. We have an isomorphism of Θ-algebras

$$E_\Theta^\lambda \to (\epsilon_\lambda \Theta G \epsilon_\lambda)^\circ \colon \alpha \mapsto (\alpha^\sigma)\tau := (\sigma^{-1} \cdot \alpha \cdot \sigma)\tau,$$

and for $i \in \mathcal{I}_\lambda$ we have $((\alpha_i^\lambda)^\sigma)\tau = k_i \cdot \epsilon_\lambda g_i \epsilon_\lambda$. In particular, $\{\epsilon_\lambda g_i \epsilon_\lambda; i \in \mathcal{I}_\lambda\}$ is a Θ-basis of $(\epsilon_\lambda \Theta G \epsilon_\lambda)^\circ$, and we have $\epsilon_\lambda g_j \epsilon_\lambda = 0$ for $j \notin \mathcal{I}_\lambda$.

Proof. Using Section (1.7) we get

$$((\alpha_i^\lambda)^\sigma)\tau = \epsilon_\lambda(\alpha_i^\lambda)^\sigma = \epsilon_\lambda g_i \cdot \left(\sum_{j=1}^{k_i} \lambda(h_{ij}^{-1}) \cdot h_{ij}\right).$$

Since $\lambda_{H_i}^{g_i} = \lambda_{H_i}$, for $h \in H_i$ we have $\lambda(h^{-1}) \cdot \epsilon_\lambda g_i h = \lambda(h^{-1}) \cdot \epsilon_\lambda \cdot h^{g_i^{-1}} \cdot g_i = \lambda(h^{-1}) \cdot \lambda^{g_i}(h) \cdot \epsilon_\lambda g_i = \epsilon_\lambda g_i$. Hence we obtain $((\alpha_i^\lambda)^\sigma)\tau = \frac{|H|}{|H_i|} \cdot \epsilon_\lambda g_i \epsilon_\lambda$. The last assertion follows from the fact that for $k \in \mathcal{I}$ the support of $\epsilon_\lambda g_k \epsilon_\lambda \in \Theta G$ with respect to the Θ-basis G of ΘG is contained in the H-H-double coset Hg_kH. ♯

Proposition (2.2) exhibits E_Θ^λ as a non-unitary Θ-subalgebra of $(\Theta G)^\circ$. From this we deduce the following additional structure on E_Θ^λ.

(2.3) Proposition.
a) E_Θ^λ is a symmetric Θ-algebra with respect to the symmetrising linear form

$$t \colon E_\Theta^\lambda \to \Theta \colon \alpha_i^\lambda \mapsto \frac{1}{|H|} \cdot \delta_{i,1},$$

for $i \in \mathcal{I}_\lambda$.
b) For $i, j \in \mathcal{I}_\lambda$ we have $t(\alpha_i^\lambda \cdot \alpha_j^\lambda) = \delta_{i^*,j} \cdot \frac{\zeta_{i^*} \cdot k_i}{|H|}$.

Proof. The group algebra ΘG is a symmetric algebra with respect to the symmetrising linear form $t_G \colon \Theta G \to \Theta \colon \sum_{g \in G} c_g \cdot g \mapsto c_1$. Hence the Θ-algebra $\epsilon_\lambda \Theta G \epsilon_\lambda \subseteq \Theta G$ also is a symmetric algebra, with respect to the restriction of t_G to $\epsilon_\lambda \Theta G \epsilon_\lambda$. For $i \in \mathcal{I}_\lambda$ we have $t(k_i \cdot \epsilon_\lambda g_i \epsilon_\lambda) = \frac{1}{|H|} \cdot \delta_{i,1}$. Hence the assertion in a) follows from Proposition (2.2), and the assertion in b) follows from Remark (1.19). ♯

(2.4) Definition. For $i \in \mathcal{I}_\lambda$ let

$$\hat{\alpha}_i^\lambda := \frac{|H|}{k_i \cdot \zeta_{i^*}} \cdot \alpha_{i^*}^\lambda.$$

Then $\hat{\mathcal{A}}_\lambda := \{\hat{\alpha}_i^\lambda; i \in \mathcal{I}_\lambda\}$ is called the *dual Schur basis* of E_Θ^λ. For the case $\lambda = 1$ let $\hat{\alpha}_i := \hat{\alpha}_i^1$, for $i \in \mathcal{I}$, and $\hat{\mathcal{A}} := \hat{\mathcal{A}}_1$.

(2.5) Remark. For the moment we drop the assumption that $|H|$ is a unit in Θ, and let $\tilde{\epsilon}_\lambda := \sum_{h \in H} \lambda(h^{-1}) \cdot h \in \Theta H \subseteq \Theta G$. Then we still have an isomorphism of ΘG-modules, $\tilde{\sigma}_\lambda \colon \Theta_\lambda \to \tilde{\epsilon}_\lambda \Theta G \colon \omega_{ij} \mapsto \tilde{\epsilon}_\lambda g_i h_{ij}$, for $i \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$, analogous to Section (2.1). But in general the assertions of Proposition (2.2) and Proposition (2.3) no longer hold, even if Θ is assumed to be a field. For a treatment of this general situation see [10].

(2.6) The non-unitary embedding of Θ-algebras in Proposition (2.2) also reveals the precise relationship between the representation theory of E^λ_Θ and the representation theory of G.

Let $K = \mathrm{Quot}(\Theta)$ be a field of characteristic coprime to $|H|$, which is a splitting field for E^λ_K. For $\varphi \in \mathrm{Irr}_K(E^\lambda_K)$ let S_φ be the simple E^λ_K-module affording the character φ and $d_\varphi := \varphi(1) = \dim_K(S_\varphi) \in \mathbb{N}$. Let $e_\varphi \in E^\lambda_K$ be some primitive idempotent such that $e_\varphi E^\lambda_K/\mathrm{rad}(e_\varphi E^\lambda_K) \cong S_\varphi$ as E^λ_K-modules, and let $S^*_\varphi := \mathrm{Hom}_K(S_\varphi, K)$ be the $(E^\lambda_K)^\circ$-module dual to the E^λ_K-module S_φ. As E^λ_K is a symmetric K-algebra, we have $E^\lambda_K e_\varphi \cong (e_\varphi E^\lambda_K)^*$ as $(E^\lambda_K)^\circ$-modules, and thus $E^\lambda_K e_\varphi/\mathrm{rad}(E^\lambda_K e_\varphi) \cong S^*_\varphi$ as $(E^\lambda_K)^\circ$-modules.

Let $P_\varphi := K_\lambda\Omega \cdot e_\varphi = K_\lambda\Omega \cdot E^\lambda_K e_\varphi \leq K_\lambda\Omega$ and $m_\varphi := \dim_K(P_\varphi)$. As $K_\lambda\Omega \cong \epsilon_\lambda KG$ is a projective KG-module, P_φ is a projective indecomposable KG-module. Let $\chi_\varphi \in \mathrm{Irr}_K(G)$ be the irreducible character of KG, being afforded by the simple KG-module S_{χ_φ}, such that $P_\varphi/\mathrm{rad}(P_\varphi) \cong S_{\chi_\varphi}$ as KG-modules.

Let P and P' be projective indecomposable KG-summands of $K_\lambda\Omega$, occurring in a fixed direct sum decomposition of $K_\lambda\Omega$ into projective indecomposable KG-modules, and let $e, e' \in E^\lambda_K$ be corresponding idempotents, such that $P = K_\lambda\Omega \cdot e$ and $P' = K_\lambda\Omega \cdot e'$. Then $P \cong P'$ as KG-modules if and only if there is an isomorphism $\alpha \in E^\lambda_K$ such that $P\alpha = P' \leq K_\lambda\Omega$. Hence we have $e' = \alpha^{-1} \cdot e \cdot \alpha \in E^\lambda_K$, and thus $eE^\lambda_K/\mathrm{rad}(eE^\lambda_K) \cong e'E^\lambda_K/\mathrm{rad}(e'E^\lambda_K)$ as E^λ_K-modules. Conversely, if the latter assertion holds, then by [15, Exc.0.6.14] there is an isomorphism $\alpha \in E^\lambda_K$ such that $e' = \alpha^{-1} \cdot e \cdot \alpha \in E^\lambda_K$, and thus $P\alpha = P' \leq K_\lambda\Omega$.

Let $\mathrm{Irr}^\lambda_K(G) := \{\chi_\varphi \in \mathrm{Irr}_K(G); \varphi \in \mathrm{Irr}_K(E^\lambda_K)\}$. Let $\varphi \in \mathrm{Irr}_K(E^\lambda_K)$ and let $K \subseteq L$ be a field extension. As K is a splitting field for E^λ_K, by [18, La.I.18.8] we conclude that $e_\varphi E^\lambda_K \otimes_K L$ is an indecomposable E^λ_L-module, where $E^\lambda_L \cong E^\lambda_K \otimes_K L$. Thus $e_\varphi \in E^\lambda_K \subseteq E^\lambda_L$ is a primitive idempotent in E^λ_L, and hence $P_\varphi \otimes_K L$ is an indecomposable LG-module. Thus S_{χ_φ} is an absolutely irreducible KG-module, and hence K is a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}^\lambda_K(G)$.

Hence we have shown the following Proposition.

(2.7) Proposition. Let K be as in Section (2.6).
a) The map $\varphi \mapsto P_\varphi$ induces a bijection, the *Fitting correspondence*, between $\mathrm{Irr}_K(E^\lambda_K)$ and the set of isomorphism types of projective indecomposable summands of the KG-module $K_\lambda\Omega$. Hence it induces a bijection between $\mathrm{Irr}_K(E^\lambda_K)$ and $\mathrm{Irr}^\lambda_K(G)$.
b) As KG-modules we have

$$K_\lambda\Omega \cong \bigoplus_{\varphi \in \mathrm{Irr}_K(E^\lambda_K)} \left(\bigoplus_{i=1}^{d_\varphi} P_\varphi \right).$$

(2.8) If KG is semisimple, then we can even be a bit more specific.

Let K be of characteristic coprime to $|G|$, such that it is a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}_K^\lambda(G)$. For $\chi \in \mathrm{Irr}_K(G)$ let S_χ be the simple KG-module affording the character χ, and let $\epsilon_\chi \in KG$ be the centrally primitive idempotent corresponding to χ. Hence we have $\mathrm{Irr}_K^\lambda(G) = \{\chi \in \mathrm{Irr}_K(G); \epsilon_\lambda\epsilon_\chi \neq 0\}$, which is the set of the irreducible K-characters χ of G, such that S_χ is a constituent of λ^G, see also [15, Ch.1.11.D].

For $\chi \in \mathrm{Irr}_K^\lambda(G)$ let $\epsilon_\lambda\epsilon_\chi = \sum_{i=1}^{d_\chi} e_{\chi,i}$ be a decomposition of $\epsilon_\lambda\epsilon_\chi = \epsilon_\chi\epsilon_\lambda$ into pairwise orthogonal primitive idempotents $e_{\chi,i} \in KG$, with corresponding multiplicities $d_\chi \in \mathbb{N}$. Then we have a direct sum decomposition as KG-modules

$$\epsilon_\lambda KG \cong \bigoplus_{\chi\in\mathrm{Irr}_K^\lambda(G)} \epsilon_\lambda\epsilon_\chi KG \cong \bigoplus_{\chi\in\mathrm{Irr}_K^\lambda(G)} \left(\bigoplus_{i=1}^{d_\chi} e_{\chi,i}KG\right),$$

where $e_{\chi,i}KG \cong S_\chi$, for $i \in \{1,\ldots,d_\chi\}$. Hence in this case the Fitting correspondence is a bijection $\mathrm{Irr}_K(E_K^\lambda) \to \mathrm{Irr}_K^\lambda(G)\colon \varphi \mapsto \chi_\varphi$, and we have $S_{\chi_\varphi} = P_\varphi$ and $m_\varphi = \dim_K(S_{\chi_\varphi})$ as well as $d_{\chi_\varphi} = d_\varphi = \dim_K(S_\varphi)$. Thus we have

$$E_K^\lambda \cong \bigoplus_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} \mathrm{End}_{KG}(S_{\chi_\varphi})^{d_\varphi\times d_\varphi} \cong \bigoplus_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} K^{d_\varphi\times d_\varphi}$$

as K-algebras. In particular, E_K^λ is a semisimple K-algebra having K as a splitting field, and we have $|\mathcal{I}_\lambda| = \dim_K(E_K^\lambda) = \sum_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} d_\varphi^2$. Furthermore, for each $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, the K-algebra isomorphism τ_λ, see Section (2.1), restricts to an isomorphism, where $\chi = \chi_\varphi$,

$$K^{d_\varphi\times d_\varphi} \to (\epsilon_\lambda\epsilon_\chi KG\epsilon_\chi\epsilon_\lambda)^\circ = \bigoplus_{i=1}^{d_\varphi}\bigoplus_{j=1}^{d_\varphi} e_{\chi,j}KGe_{\chi,i}\colon E_{ij} \mapsto e_{\chi,j}f_{ji}e_{\chi,i},$$

for some $f_{ji} \in KG$, for $i,j \in \{1,\ldots,d_\chi\}$, and where $E_{ij} \in K^{d_\varphi\times d_\varphi}$ is the matrix unit given by $[E_{ij}]_{i'j'} = \delta_{i,i'}\delta_{j,j'}$, for $i',j' \in \{1,\ldots,d_\chi\}$.

Hence as $(KG \otimes_K E_K^\lambda)$-modules we have

$$K_\lambda\Omega \cong \bigoplus_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} \left(S_{\chi_\varphi} \otimes_K S_\varphi\right),$$

where the above summands are pairwise non-isomorphic absolutely irreducible $(KG \otimes_K E_K^\lambda)$-modules. We have $S_{\chi_\varphi} \otimes_K S_\varphi \cong \bigoplus_{i=1}^{d_\varphi} S_{\chi_\varphi}$ as KG-modules and $S_{\chi_\varphi} \otimes_K S_\varphi \cong \bigoplus_{i=1}^{m_\varphi} S_\varphi$ as E_K^λ-modules.

(2.9) Remark. Let K be of characteristic coprime to $|G|$ and a splitting field of E_K^λ. Then E_K^λ is commutative if and only if $d_\varphi = 1$, for all $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, which holds if and only if $|\mathrm{Irr}_K(E_K^\lambda)| = \dim_K(E_K^\lambda) = |\mathcal{I}_\lambda|$. In this case, $K_\lambda\Omega$ is called *multiplicity-free*.

(2.10) We conclude Section 2 by introducing the setting for decomposition theory, and we show how the decomposition maps of G and E^λ are related.

Let K be of characteristic 0 and a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}_K^\lambda(G)$. Hence K is a splitting field for E_K^λ as well. Without loss of generality we may assume that K is a cyclotomic field containing $\mathbb{Q}(\lambda(H))$. Let $R \subset K$ be a discrete valuation ring in K with maximal ideal $\wp \lhd R$ and finite residue class field $F := R/\wp$ of characteristic $p > 0$, where p is coprime to $|H|$. Let $\tilde{\ }: R \to F$ denote the natural epimorphism.

By Theorem (1.8), E_R^λ is an R-order in E_K^λ. As $\lambda(H) \subseteq R$, let $\tilde{\lambda} := \lambda \cdot \tilde{\ } \in \mathrm{Irr}_F(H)$. As the characteristic of F is coprime to $|H|$, we have $\lambda_{H_i} = \lambda_{H_i}^{g_i}$ if and only if $\tilde{\lambda}_{H_i} = \tilde{\lambda}_{H_i}^{g_i}$, for $i \in \mathcal{I}$. Thus $\mathcal{I}_\lambda = \mathcal{I}_{\tilde{\lambda}}$, and hence we have an R-algebra epimorphism $E_R^\lambda \to E_F^{\tilde{\lambda}}: \alpha_i^\lambda \mapsto \alpha_i^{\tilde{\lambda}}$, for $i \in \mathcal{I}_\lambda$. Without loss of generality we may assume that F is a splitting field for $E_F^{\tilde{\lambda}}$, and hence F is a splitting field for all simple FG-modules affording a character in $\mathrm{Irr}_F^{\tilde{\lambda}}(G)$ as well.

Hence we have a *decomposition map* $D_G: G(KG) \to G(FG)$, where $G(\cdot)$ denotes the corresponding Grothendieck groups, see [14, Ch.XII.82-83]. The considerations there generalise straightforwardly to the algebras E_K^λ and $E_F^{\tilde{\lambda}}$, hence we also have a decomposition map $D_E: G(E_K^\lambda) \to G(E_F^{\tilde{\lambda}})$. For $\chi \in \mathrm{Irr}_K(G)$ and $\chi' \in \mathrm{Irr}_F(G)$ let $d_{\chi\chi'}^G \in \mathbb{N}_0$ denote the corresponding *decomposition number* with respect to D_G, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $\varphi' \in \mathrm{Irr}_F(E_F^{\tilde{\lambda}})$ let $d_{\varphi\varphi'}^E \in \mathbb{N}_0$ denote the corresponding decomposition number with respect to D_E.

(2.11) Proposition. Let $\varphi' \in \mathrm{Irr}_F(E_F^{\tilde{\lambda}})$ and let $\chi' = \chi_{\varphi'} \in \mathrm{Irr}_F^{\tilde{\lambda}}(G)$ be its Fitting correspondent.
a) For $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and its Fitting correspondent $\chi = \chi_\varphi \in \mathrm{Irr}_K^\lambda(G)$ we then have $d_{\chi\chi'}^G = d_{\varphi\varphi'}^E$.
b) If $\chi \in \mathrm{Irr}_K(G) \setminus \mathrm{Irr}_K^\lambda(G)$, then $d_{\chi\chi'}^G = 0$.

Proof. By [53, Thm.3.4.1], idempotents can be lifted from FG to RG, respectively from $E_F^{\tilde{\lambda}}$ to E_R^λ. Hence the assertions follow from Brauer reciprocity, see [14, Thm.XII.83.9]. ♯

3 Characters of endomorphism rings

In Section 3 we discuss characters of endomorphism rings over fields of characteristic 0. The exposition of Section 3 is inspired by [27]. We begin by relating the character values on Schur basis elements corresponding to paired orbitals, and then use the symmetrising form to exhibit the centrally primitive idempotents of the endomorphism ring.

Let K be a cyclotomic field containing $\mathbb{Q}(\lambda(H))$ and being a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}_K^\lambda(G)$. Let $\bar{\ }: K \to K$ denote the involutory field automorphism defined by $\bar{\ }\colon \zeta \mapsto \zeta^{-1}$ for all roots of unity $\zeta \in K$, extending the field automorphism of $\mathbb{Q}(\lambda(H))$ defined in Section (1.11).

(3.1) Proposition. See also [39, Prop.II.12.12].
For $i \in \mathcal{I}_\lambda$ and $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ we have, where $\zeta_i \in K$ is as in Definition (1.12),

$$\overline{\varphi(\alpha_i^\lambda)} = \frac{1}{\zeta_i} \cdot \varphi(\alpha_{i^*}^\lambda).$$

Proof. As in Section (1.11) there is a G-invariant positive definite hermitian form $\langle\cdot,\cdot\rangle_\Omega$ on $K_\lambda\Omega$, thus the decomposition $K_\lambda\Omega \cong \bigoplus_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} \left(S_{\chi_\varphi} \otimes_K S_\varphi\right)$ as $(KG \otimes E_K^\lambda)$-modules, see Section (2.8), is an orthogonal direct sum. Thus by Proposition (1.13) we have

$$\overline{\varphi(\alpha_i^\lambda)} = \varphi((\alpha_i^\lambda)^\sharp) = \frac{1}{m_\varphi} \cdot \mathrm{tr}_\varphi((\alpha_i^\lambda)^\sharp) = \frac{1}{\zeta_i \cdot m_\varphi} \cdot \mathrm{tr}_\varphi(\alpha_{i^*}^\lambda) = \frac{1}{\zeta_i} \cdot \varphi(\alpha_{i^*}^\lambda),$$

where tr_φ denotes the K-valued trace function on $S_{\chi_\varphi} \otimes_K S_\varphi$. ♯

(3.2) Proposition.
a) The centrally primitive idempotent $\epsilon_\varphi \in E_K^\lambda$ corresponding to $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ is given as

$$\epsilon_\varphi = \frac{|H|}{c_\varphi} \cdot \sum_{i\in\mathcal{I}_\lambda} \frac{1}{k_i} \cdot \overline{\varphi(\alpha_i^\lambda)} \cdot \alpha_i^\lambda,$$

where $c_\varphi \in K$ is the corresponding *Schur element*, see also [15, Ch.1.9.B].
b) For $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ we have

$$c_\varphi = \frac{|G|}{m_\varphi} = \frac{|G|}{\chi_\varphi(1)} = c_{\chi_\varphi},$$

where $c_{\chi_\varphi} \in K$ is the Schur element belonging to $\chi_\varphi \in \mathrm{Irr}_K^\lambda(G)$ for the symmetric K-algebra KG with symmetrising form t_G, see Proposition (2.3).

Proof. Using the symmetrising form t we have

$$\epsilon_\varphi = \frac{1}{c_\varphi} \cdot \sum_{i\in\mathcal{I}_\lambda} \varphi(\hat{\alpha}_i^\lambda) \cdot \alpha_i^\lambda = \frac{|H|}{c_\varphi} \cdot \sum_{i\in\mathcal{I}_\lambda} \frac{1}{k_i \cdot \zeta_{i^*}} \cdot \varphi(\alpha_{i^*}^\lambda) \cdot \alpha_i^\lambda.$$

Hence the assertion in a) follows from Proposition (3.1) and Corollary (1.14).

For $i \in \mathcal{I}_\lambda$, the trace of the action of α_i^λ on $K_\lambda\Omega$ is given as $\mathrm{tr}_{K_\lambda\Omega}(\alpha_i^\lambda) = \delta_{1,i} \cdot n$. Hence using a) we have $\mathrm{tr}_{K_\lambda\Omega}(\epsilon_\varphi) = \frac{|H|}{c_\varphi} \cdot d_\varphi \cdot n = \frac{|G|\cdot d_\varphi}{c_\varphi}$. Furthermore, the idempotent $\epsilon_\varphi \in E_K^\lambda$ acts as the identity on $S_{\chi_\varphi} \otimes_K S_\varphi$ and annihilates the other summands $S_{\chi_{\varphi'}} \otimes_K S_{\varphi'}$, for $\varphi \neq \varphi' \in \mathrm{Irr}_K(E_K^\lambda)$. Hence we have $\mathrm{tr}_{K_\lambda\Omega}(\epsilon_\varphi) = d_\varphi \cdot m_\varphi$. ♯

We address the question of semisimplicity of the endomorphism ring $E_F^{\tilde{\lambda}}$ over a field F of positive characteristic.

(3.3) Remark. Let $\tilde{}: R \to F$ and $\tilde{\lambda}$ be as in Section (2.10), where in particular the characteristic of F is coprime to $|H|$.
a) For $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ let $D_\varphi: E_K^\lambda \to \mathrm{End}_K(S_\varphi)$ denote the corresponding representation. Then the Schur element $c_\varphi \in K$ is defined by the Frobenius-Schur relations $\sum_{i \in \mathcal{I}_\lambda} D_\varphi(\hat{\alpha}_i^\lambda) \cdot M \cdot D_\varphi(\alpha_i^\lambda) = c_\varphi \cdot \mathrm{tr}(M) \cdot \mathrm{id}_{S_\varphi}$, for $M \in \mathrm{End}_K(S_\varphi)$, where tr is the K-valued trace function on $\mathrm{End}_K(S_\varphi)$. Hence we have $c_\varphi \in R$, and $\widetilde{c_\varphi}$ is well-defined.
b) As $E_F^{\tilde{\lambda}}$ is a symmetric algebra, $\tilde{\varphi} \in \mathrm{Irr}_F(E_F^{\tilde{\lambda}})$ is afforded by a projective simple $E_F^{\tilde{\lambda}}$-module if and only if $\tilde{\varphi}$ occurs with multiplicity $d_{\tilde{\varphi}}$ as a constituent of the regular $E_F^{\tilde{\lambda}}$-module $E_F^{\tilde{\lambda}}$, while for the non-projective simple $E_F^{\tilde{\lambda}}$-modules this multiplicity is at least $2 \cdot d_{\tilde{\varphi}}$.

(3.4) Proposition. See also Tits' Deformation Theorem [16, Thm.8.68.17], [23, Thm.1.3.8] and [19].

We keep the notation of Section (2.10), where in particular the characteristic of F is coprime to $|H|$. Then the decomposition map D_E induces a bijection

$$\{\varphi \in \mathrm{Irr}_K(E_K^\lambda); \widetilde{c_\varphi} \neq 0 \in F\} \to \{\tilde{\varphi} \in \mathrm{Irr}_F(E_F^{\tilde{\lambda}}); \tilde{\varphi} \text{ projective}\}.$$

In particular, $E_F^{\tilde{\lambda}}$ is semisimple if and only if $\widetilde{c_\varphi} \neq 0 \in F$ for all $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$.

Proof. If $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ such that $d^E_{\varphi\tilde{\varphi}} \neq 0$, then $\tilde{\varphi}$ occurs in the regular $E_F^{\tilde{\lambda}}$-module $E_F^{\tilde{\lambda}} \cong \widetilde{E_K^\lambda}$ at least with multiplicity $d_\varphi \cdot d^E_{\varphi\tilde{\varphi}}$. If $\tilde{\varphi}$ is projective, then by Remark (3.3) we conclude from $d_\varphi \geq d_{\tilde{\varphi}}$ that $d_\varphi = d_{\tilde{\varphi}}$ and $d^E_{\varphi\tilde{\varphi}} = 1$. Hence $d^E_{\varphi\tilde{\varphi}'} = 0$, for $\tilde{\varphi} \neq \tilde{\varphi}' \in \mathrm{Irr}_F(E_F^{\tilde{\lambda}})$, and $d^E_{\varphi'\tilde{\varphi}} = 0$, for $\varphi \neq \varphi' \in \mathrm{Irr}_K(E_K^\lambda)$. Furthermore we have $\widetilde{c_\varphi} = c_{\tilde{\varphi}}$, and by the Gaschütz-Ikeda Theorem, see [14, Thm.IX.62.11], $S_{\tilde{\varphi}}$ is a projective $E_F^{\tilde{\lambda}}$-module if and only if $c_{\tilde{\varphi}} \neq 0 \in F$.

Conversely, if $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ such that $\widetilde{c_\varphi} \neq 0 \in F$, then $c_\varphi \in R$ is a unit. Let

$$\epsilon_{\varphi,j} := \frac{1}{c_\varphi} \cdot \sum_{i \in \mathcal{I}_\lambda} [D_\varphi(\hat{\alpha}_i^\lambda)]_{jj} \cdot \alpha_i^\lambda \in E_R^\lambda \subseteq E_K^\lambda,$$

for $j \in \{1, \ldots, d_\varphi\}$. Then $\epsilon_\varphi = \sum_{j=1}^{d_\varphi} \epsilon_{\varphi,j} \in E_R^\lambda$ is a decomposition of ϵ_φ into pairwise orthogonal primitive idempotents. Hence $\epsilon_\varphi E_R^\lambda \epsilon_\varphi \cong R^{d_\varphi \times d_\varphi}$ as R-algebras, thus $\widetilde{\epsilon_\varphi E_R^\lambda \epsilon_\varphi} \cong F^{d_\varphi \times d_\varphi}$ as F-algebras. Hence $\widetilde{S_\varphi}$ is an irreducible $E_F^{\tilde{\lambda}}$-module, with corresponding Schur element $\widetilde{c_\varphi} \neq 0 \in F$. ♯

(3.5) Remark. As $c_{\varphi_\chi} = c_\chi$, for $\chi \in \mathrm{Irr}_K^\lambda(G)$, we conclude that $E_F^{\tilde{\lambda}}$ is semisimple if and only if all KG-constituents of λ^G are of p-defect 0. Hence for $\lambda = 1$, where the trivial KG-character is an element of $\mathrm{Irr}_K^\lambda(G)$, the F-algebra E_F is semisimple if and only if p does not divide the group order $|G|$. But for $\lambda \neq 1$ the F-algebra $E_F^{\tilde{\lambda}}$ might be semisimple even if p divides $|G|$, as the following examples show.

(3.6) Example.
a) Let $F := \mathbb{F}_4$ be the finite field of order 4, let $G := \mathcal{S}_3$ be the symmetric group on 3 letters and $H := \mathcal{A}_3$ be the alternating group on 3 letters, and let $1 \neq \tilde{\lambda} \in \mathrm{Irr}_F(\mathcal{A}_3)$ be a non-trivial F-representation. Then $\tilde{\lambda}^G$ is an irreducible F-representation of $\mathcal{S}_3$ of degree 2, and we have $\tilde{\lambda}^G \cong \widetilde{\rho}$ as $F\mathcal{S}_3$-modules, where ρ is the reflection K-representation of $\mathcal{S}_3$, whose Schur element is $c_\rho = 3 \in K$.
b) Let $G := SL_2(\mathbb{F}_q)$ be the special linear group of degree 2 over $\mathbb{F}_q$, where q is a prime power $q \geq 4$, and let $H := U \ltimes T < G$ be a split Borel subgroup with torus $T \cong C_{q-1}$. We have $|H| = q(q-1)$ and $|G| = q(q-1)(q+1)$. Let F be a finite field of characteristic coprime to $q(q-1)$ containing primitive $(q-1)$-st roots of unity. Hence B has exactly $q-1$ different F-representations $\tilde{\lambda} \in \mathrm{Irr}_F(H)$ of degree 1, all of which are inflated from T. If $1 \neq \tilde{\lambda}^2$, then $\tilde{\lambda}^G$ is irreducible of degree $q+1$, and hence has Schur element $0 \neq q(q-1) \in F$. If $\tilde{\lambda} \neq 1$, but $\tilde{\lambda}^2 = 1$, then $\tilde{\lambda}^G$ has two non-isomorphic constituents of degree $\frac{q+1}{2}$, whose Schur elements hence are $0 \neq 2q(q-1) \in F$.

We introduce the second main actor of the present work.

(3.7) Definition. The matrix

$$\Phi_\lambda := [\varphi(\alpha_i^\lambda); \varphi \in \mathrm{Irr}_K(E_K^\lambda), i \in \mathcal{I}_\lambda] \in K^{|\mathrm{Irr}_K(E_K^\lambda)| \times |\mathcal{I}_\lambda|},$$

with row index φ and column index i, is called the *character table* of E_K^λ. For $\lambda = 1$ let $\Phi := \Phi_1$.

Explicit examples of character tables are shown in Examples (3.12) and (4.10) as well as (5.17), and of course in Part III. In all the explicitly given tables we also indicate the Fitting correspondence $\mathrm{Irr}_K(E_K^\lambda) \to \mathrm{Irr}_K^\lambda(G)$, see Proposition (2.7). We proceed to prove the most important structural feature of the character table of an endomorphism ring, the orthogonality relations.

(3.8) Proposition. Orthogonality relations.
a) We have the *first orthogonality relations*

$$\overline{\Phi_\lambda} \cdot \mathrm{diag}[k_i^{-1}; i \in \mathcal{I}_\lambda] \cdot \Phi_\lambda^T = n \cdot \mathrm{diag}[\frac{d_\varphi}{m_\varphi}; \varphi \in \mathrm{Irr}_K(E_K^\lambda)].$$

b) If E_K^λ is commutative, then we have the *second orthogonality relations*

$$\Phi_\lambda^T \cdot \mathrm{diag}[m_\varphi; \varphi \in \mathrm{Irr}_K(E_K^\lambda)] \cdot \overline{\Phi_\lambda} = n \cdot \mathrm{diag}[k_i; i \in \mathcal{I}_\lambda].$$

Proof. Because of $\varphi(\epsilon_{\varphi'}) = \delta_{\varphi,\varphi'} \cdot d_\varphi$, for $\varphi, \varphi' \in \mathrm{Irr}_K(E_K^\lambda)$, by Proposition (3.2) we have $\sum_{i \in \mathcal{I}_\lambda} \frac{1}{k_i} \cdot \overline{\varphi(\alpha_i^\lambda)} \cdot \varphi'(\alpha_i^\lambda) = \delta_{\varphi,\varphi'} \cdot \frac{|G| \cdot d_\varphi}{|H| \cdot m_\varphi} = \delta_{\varphi,\varphi'} \cdot \frac{n \cdot d_\varphi}{m_\varphi}$, which in terms of matrices is just the assertion in a).

If E_K^λ is commutative, then by Remark (2.9) we have $d_\varphi = 1$ for all $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $|\mathrm{Irr}_K(E_K^\lambda)| = r$, hence Φ_λ is a square matrix. Because of the first orthogonality relations Φ_λ is invertible, and we have

$$\Phi_\lambda^{-T} \cdot \mathrm{diag}[k_i; i \in \mathcal{I}_\lambda] \cdot \overline{\Phi_\lambda^{-1}} = \frac{1}{n} \cdot \mathrm{diag}[m_\varphi; \varphi \in \mathrm{Irr}_K(E_K^\lambda)].$$

From this the assertion in b) follows. ♯

(3.9) Remark. In particular, by the first orthogonality relations we obtain

$$\frac{1}{m_\varphi} = \frac{1}{d_\varphi \cdot n} \cdot \sum_{i \in \mathcal{I}_\lambda} \frac{1}{k_i} \cdot \overline{\varphi(\alpha_i^\lambda)} \cdot \varphi(\alpha_i^\lambda),$$

for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$. As $d_\varphi = \varphi(\alpha_1^\lambda)$ is known from Φ_λ the degree $\chi_\varphi(1) = m_\varphi$ of the Fitting correspondent $\chi_\varphi \in \mathrm{Irr}_K^\lambda(G)$ of $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ can be read off from Φ_λ as soon as the k_i, for $i \in \mathcal{I}_\lambda$, are known; see also Remark (3.21).

As a direct consequence of the orthogonality relations we obtain the following notion, which for the case $\lambda = 1$ first appeared in [20]. Part of the statements in c) of Proposition (3.10) have been proved in [21], see also [80, Thm.V.30.1].

(3.10) Proposition.
a) For $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $i \in \mathcal{I}_\lambda$, the character value $\varphi(\alpha_i^\lambda) \in K$ is an algebraic integer. If E_K^λ is commutative, then $\det \Phi_\lambda \in K$ and $\det \overline{\Phi_\lambda} \in K$ are algebraic integers, and we have $(\det \Phi_\lambda)^2 \in \mathbb{Q}(\lambda(H))$ and $\det \Phi_\lambda \cdot \det \overline{\Phi_\lambda} \in \mathbb{Q}$.
b) Let E_K^λ be commutative. Then the *generalised Frame number*

$$N_\lambda := n^{|\mathcal{I}_\lambda|} \cdot (\prod_{i \in \mathcal{I}_\lambda} k_i) \cdot (\prod_{\varphi \in \mathrm{Irr}_K(E_K^\lambda)} \frac{1}{m_\varphi})$$

is a rational integer.
c) Let $\lambda = 1$ and E_K be commutative. Then the *Frame number* $N_1 \in \mathbb{Z}$ is divisible by n^2. Furthermore, $N_1 \in \mathbb{Z}$ is a square in $\mathbb{Z}$, if and only if either
i) $|\mathcal{I}| - |\{i \in \mathcal{I}; i^* = i\}| \equiv 0 \bmod 4$ and $\det \Phi \in \mathbb{Z}$, or
ii) $|\mathcal{I}| - |\{i \in \mathcal{I}; i^* = i\}| \equiv 2 \bmod 4$ and $\det \Phi \in i\mathbb{Z}$.
In particular, N_1 is a square in $\mathbb{Z}$, if all characters in $\mathrm{Irr}_K^1(G)$ are rational-valued.

Proof. Let $\mathcal{R}$ be the set of all discrete valuation rings in K, without any restriction to the characteristic of the residue class field of R. As the representation of E_K^λ affording φ can be realized over all rings $R \in \mathcal{R}$, see Section (2.10), we conclude that $\varphi(\alpha_i^\lambda) \in \bigcap_{R \in \mathcal{R}} R$, which by [15, Ch.I.4.C] is the ring of algebraic integers in K.

If E_K^λ is commutative, then from the second orthogonality relations, see Proposition (3.8), we obtain by taking determinants

$$\det \Phi_\lambda \cdot \det \overline{\Phi_\lambda} \cdot (\prod_{\varphi \in \mathrm{Irr}_K(E_K^\lambda)} m_\varphi) = n^{|\mathcal{I}_\lambda|} \cdot (\prod_{i \in \mathcal{I}_\lambda} k_i).$$

Thus $\det \Phi_\lambda \cdot \det \overline{\Phi_\lambda} = N_\lambda \in \mathbb{Q}$ is an algebraic integer. By Proposition (3.1), we have $\overline{\Phi_\lambda} = \Phi_\lambda \cdot Q_\lambda \cdot \mathrm{diag}[\zeta_i^{-1}; i \in \mathcal{I}_\lambda]$, where $Q_\lambda \in \mathbb{Z}^{|\mathcal{I}_\lambda| \times |\mathcal{I}_\lambda|}$ is the permutation matrix describing the permutation of the columns of Φ_λ induced by the pairing involution $*: \mathcal{I}_\lambda \to \mathcal{I}_\lambda$. Hence we have $\det Q_\lambda = (-1)^{\frac{|\mathcal{I}_\lambda| - |\{i \in \mathcal{I}_\lambda; i^* = i\}|}{2}}$. Thus we obtain

$$N_\lambda = (-1)^{\frac{|\mathcal{I}_\lambda| - |\{i \in \mathcal{I}_\lambda; i^* = i\}|}{2}} \cdot (\det \Phi_\lambda)^2 \cdot \prod_{i \in \mathcal{I}_\lambda} \frac{1}{\zeta_i}.$$

Hence we have $(\det \Phi_\lambda)^2 \in \mathbb{Q}(\lambda(H))$. This proves the assertions in a) and b).

For $\lambda = 1$, we have $(\det \Phi)^2 \in \mathbb{Q}$. Hence $\det \Phi \in \mathbb{R}$ or $\det \Phi \in i\mathbb{R}$. From this the characterisation of $N_1 \in \mathbb{Z}$ being square in $\mathbb{Z}$ follows. As is shown in Remark (3.21), we have $k_i = \varphi_1(\alpha_i)$ for $i \in \mathcal{I}$, where $\varphi_1 \in \mathrm{Irr}_K(E_K)$ denotes the Fitting correspondent of the trivial KG-character. Hence using the first orthogonality relations, see Proposition (3.8), we obtain $\overline{\Phi} \cdot [1, \ldots, 1]^T = n \cdot [1, 0, \ldots, 0]^T$. Hence $\det \Phi$ is divisible by n in the ring of algebraic integers in K. If all characters in $\mathrm{Irr}_K^1(G)$ are rational-valued, by Remark (3.21) below, we have $\varphi(\alpha_i) \in \mathbb{Q}$, for $\varphi \in \mathrm{Irr}_K(E_K)$ and $i \in \mathcal{I}$, hence $\det \Phi \in \mathbb{Z}$, and by Proposition (3.1) we have $i^* = i$ for all $i \in \mathcal{I}$. This proves the assertions in c). ♯

(3.11) Remark. In general, it is not true that $N_1 \in \mathbb{Z}$ is a square, if only $i^* = i$ holds for all $i \in \mathcal{I}$, but no further assumption on $\det \Phi$ is made, as the following example shows, thus disproving a conjecture in [20].

(3.12) Example. Let $G := J_1$ and $H := L_2(11) < G$ as well as $\lambda = 1$. The character table Φ of the endomorphism ring E_K is contained in the database, see Section (11.1), and is given as follows, where $r_5 := \sqrt{5} \in \mathbb{R}$. According to Definition (3.7), the rows and columns of Φ are indexed by $\varphi \in \mathrm{Irr}_K(E_K)$ and $i \in \mathcal{I} = \{1, \ldots, 5\}$, respectively, and the entry of Φ in row φ and column i is the character value $\varphi(\alpha_i) \in K$, for the Schur basis element $\alpha_i \in \mathcal{A}$. Furthermore we indicate the Fitting correspondence $\mathrm{Irr}_K(E_K) \to \mathrm{Irr}_K^1(G): \varphi \mapsto \chi_\varphi$, see Proposition (2.7).

φ	χ_φ	1	2	3	4	5
1	$1a$	1	11	12	110	132
2	$56a$	1	$\frac{-7-r_5}{2}$	$\frac{-3+3r_5}{2}$	$\frac{5+7r_5}{2}$	$\frac{3-9r_5}{2}$
3	$56b$	1	$\frac{-7+r_5}{2}$	$\frac{-3-3r_5}{2}$	$\frac{5-7r_5}{2}$	$\frac{3+9r_5}{2}$
4	$76a$	1	4	-2	5	-8
5	$77a$	1	1	4	-10	4

As is shown in Remark (3.21), we have $k_i = \varphi_1(\alpha_i)$ for $i \in \mathcal{I} = \{1, \ldots, 5\}$. Hence the index parameters are pairwise different, and thus we have $i^* = i$ for all $i \in \mathcal{I}$. But we have $\det \Phi = -2 \cdot 3 \cdot 7 \cdot 11 \cdot 19^2 \cdot r_5 \in \mathbb{Q}(r_5)$ and hence $N_1 = (\det \Phi)^2 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 19^4 \in \mathbb{Z}$, which is not a square in $\mathbb{Z}$.

(3.13) Remark. Let E_K^λ be commutative. Let $e \in \mathbb{N}$ and let $\zeta_e \in \mathbb{Q}(\lambda(H))$ be a primitive e-th root of unity such that $\mathbb{Q}(\lambda(H)) = \mathbb{Q}(\zeta_e)$. Let $\zeta_{2e} \in \mathbb{C}$ is a primitive $2e$-th root of unity. It follows from the proof of Proposition (3.10) that N_λ is a square in the ring of integers of $\mathbb{Q}(\zeta_{2e})$, which by [50, Cor.2.2] coincides with $\mathbb{Z}[\zeta_{2e}]$. But for the case $\lambda \neq 1$, no characterisation for the generalised Frame number $N_\lambda \in \mathbb{Q}(\lambda(H))$ to be a square in the ring of integers $\mathbb{Z}[\lambda(H)] = \mathbb{Z}[\zeta_e]$ of $\mathbb{Q}(\lambda(H))$ is known to the author.

This question is related to the question whether $\prod_{i\in\mathcal{I}_\lambda} \zeta_i \in \mathbb{Q}(\lambda(H))$ is a square in $\mathbb{Z}[\lambda(H)]$. As by Corollary (1.14) we have $\zeta_i = \zeta_{i^*}$, for $i \in \mathcal{I}_\lambda$, we only have to consider $\prod_{i\in\mathcal{I}_\lambda, i=i^*} \zeta_i \in \mathbb{Q}(\lambda(H))$. Hence let $i \in \mathcal{I}_\lambda$ such that $i = i^*$. By Definition (1.12) we have $\zeta_i = \frac{\lambda(\eta_i)}{\lambda(h_{ii^-})}$. Let $h \in H_i$ and $\eta' \in H$ such that

$$\eta_i \cdot g_i \cdot h_{ii^-} = g_i^{-1} = \eta' \cdot g_i \cdot h \cdot h_{ii^-} = \eta' \cdot g_i h g_i^{-1} \cdot g_i \cdot h_{ii^-}$$

Hence we have $\eta' \cdot g_i h g_i^{-1} = \eta_i$ and thus

$$\frac{\lambda(\eta')}{\lambda(h \cdot h_{ii^-})} = \frac{\lambda(\eta_i)}{\lambda(g_i h g_i^{-1}) \cdot \lambda(h) \cdot \lambda(h_{ii^-})} = \frac{1}{\lambda(h)^2} \cdot \frac{\lambda(\eta_i)}{\lambda(h_{ii^-})}.$$

Hence without loss of generality we may change the set of representatives of the right cosets $H_i|H$ of H_i in H. Let $h, h' \in H$. Then we have

$$(hg_ih')^{-1} = h'^{-1} \cdot \eta_i g_i h_{ii^-} \cdot h^{-1} = h'^{-1}\eta_i h^{-1} \cdot hg_ih' \cdot h'^{-1} h_{ii^-} h^{-1},$$

and thus $\frac{\lambda(h'^{-1}\cdot\eta_i\cdot h^{-1})}{\lambda(h'^{-1}\cdot h_{ii^-}\cdot h^{-1})} = \frac{\lambda(\eta_i)}{\lambda(h_{ii^-})} = \zeta_i$. Hence without loss of generality we may as well change the set of representatives of the H-H-double cosets in G.

Thus we may choose $g_i \in G$ such that for $\omega_i := \omega_1 g_i \in \Omega$ we have $\omega_i g_i = \omega_1$, Thus we have $g_i^2 \in H_i$ and $g_i^{-1} = \eta_i \cdot g_i = g_i \cdot \eta_i$, where $\eta_i \in H_i \leq H$. Having made these choices, we have reduced the question whether $\prod_{i\in\mathcal{I}_\lambda} \zeta_i \in \mathbb{Q}(\lambda(H))$ is a square in $\mathbb{Z}[\lambda(H)]$ to the question whether $\prod_{i\in\mathcal{I}_\lambda, i=i^*} \lambda(\eta_i) \in \mathbb{Q}(\lambda(H))$ is a square in $\mathbb{Z}[\lambda(H)]$. For this, again, no general statement is known to the author.

We discuss the relationship between the character table of a commutative endomorphism ring E_K^λ, the set of its centrally primitive idempotents, and its structure constants matrices.

(3.14) Proposition. Let E_K^λ be commutative, let

$$\mathcal{E}_\lambda := \{\epsilon_\varphi \in E_K^\lambda; \varphi \in \mathrm{Irr}_K(E_K^\lambda)\}$$

be the set of centrally primitive idempotents of E_K^λ, and let $[\mathcal{E}_\lambda]_{\mathcal{A}_\lambda} \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$ be the matrix describing the centrally primitive idempotents in terms of the Schur basis $\mathcal{A}_\lambda$, see Proposition (3.2).

a) Then $\mathcal{E}_\lambda$ is a K-basis of E_K^λ and $\{\epsilon_\varphi E_K^\lambda \leq E_K^\lambda; \varphi \in \mathrm{Irr}_K(E_K^\lambda)\}$ is the set of all 1-dimensional E_K^λ-submodules of E_K^λ. For $j \in \mathcal{I}_\lambda$ we have

$$[\mathcal{E}_\lambda]_{\mathcal{A}_\lambda} \cdot P_j^\lambda = \mathrm{diag}[\varphi(\alpha_j^\lambda); \varphi \in \mathrm{Irr}_K(E_K^\lambda)] \cdot [\mathcal{E}_\lambda]_{\mathcal{A}_\lambda},$$

where P_j^λ is the j-th structure constants matrix, see Definition (1.18).
b) We have $[\mathcal{E}_\lambda]_{\mathcal{A}_\lambda} = \Phi_\lambda^{-T}$ as well as

$$n \cdot \mathrm{diag}[m_\varphi^{-1}; \varphi \in \mathrm{Irr}_K(E_K^\lambda)] \cdot [\mathcal{E}_\lambda]_{\mathcal{A}_\lambda} = \overline{\Phi_\lambda} \cdot \mathrm{diag}[k_i^{-1}; i \in \mathcal{I}_\lambda].$$

Proof. The regular E_K^λ-module E_K^λ decomposes as $E_K^\lambda \cong \bigoplus_{\varphi \in \mathrm{Irr}_K(E_K^\lambda)} \epsilon_\varphi E_K^\lambda$, where $\dim_K(\epsilon_\varphi E_K^\lambda) = 1$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$. We have $\epsilon_\varphi \cdot \alpha_j^\lambda = \varphi(\alpha_j^\lambda) \cdot \epsilon_\varphi$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$. From that and the uniqueness of the centrally primitive idempotents the assertions in a) follow. The assertions in b) follow from Proposition (3.2) and the second orthogonality relations, see Proposition (3.8). ♯

(3.15) Corollary. Let $\mathcal{J} \subseteq \mathcal{I}_\lambda$ and $\mathcal{C} := \langle \alpha_j^\lambda; j \in \mathcal{J} \rangle_{K-\mathrm{algebra}} \leq E_K^\lambda$. Then we have $\mathcal{C} = E_K^\lambda$ if and only if $E_K^\lambda \cong \bigoplus_{i \in \{1,\ldots,|\mathcal{I}_\lambda|\}} S_i$ as $\mathcal{C}$-modules, where the $S_i \leq E_K^\lambda$ are pairwise non-isomorphic $\mathcal{C}$-modules such that $\dim_K(S_i) = 1$.

(3.16) Definition.
a) For $i, j \in \mathcal{I}_\lambda$ let $\hat{\alpha}_i^\lambda \cdot \alpha_j^\lambda = \sum_{k \in \mathcal{I}_\lambda} p_{\hat{i}j\hat{k}}^\lambda \cdot \hat{\alpha}_k^\lambda$, for the *dual structure constants* $p_{\hat{i}j\hat{k}}^\lambda \in \mathbb{Q}(\lambda(H))$, where $\hat{\mathcal{A}}_\lambda = \{\hat{\alpha}_k^\lambda; k \in \mathcal{I}_\lambda\}$ is the dual Schur basis, see Definition (2.4). For the case $\lambda = 1$ let $p_{\hat{i}j\hat{k}} := p_{\hat{i}j\hat{k}}^1$.
b) For $j \in \mathcal{I}_\lambda$, the representing matrix $[\alpha_j^\lambda]_{\hat{\mathcal{A}}_\lambda}$ of the right regular action of α_j^λ on E_K^λ, with respect to the dual Schur basis $\hat{\mathcal{A}}_\lambda$, is given by the j-th *dual structure constants matrix*

$$[\alpha_j^\lambda]_{\hat{\mathcal{A}}_\lambda} = \hat{P}_j^\lambda := [p_{\hat{i}j\hat{k}}^\lambda; i, k \in \mathcal{I}_\lambda] \in \mathbb{Q}(\lambda(H))^{|\mathcal{I}_\lambda| \times |\mathcal{I}_\lambda|},$$

with row index i and column index k. For the case $\lambda = 1$ let $\hat{P}_j := \hat{P}_j^1$.

(3.17) Proposition. Let $i, j, l \in \mathcal{I}_\lambda$.
a) For the structure constants matrix $P_{i^*}^\lambda$ we have

$$P_{i^*}^\lambda = \zeta_i \cdot \mathrm{diag}[k_j; j \in \mathcal{I}_\lambda] \cdot \overline{(P_i^\lambda)^T} \cdot \mathrm{diag}[k_l^{-1}; l \in \mathcal{I}_\lambda].$$

b) For the dual structure constants $p_{\hat{i}j\hat{l}}^\lambda$ we have $p_{\hat{i}j\hat{l}}^\lambda = \frac{k_l \cdot \zeta_{l^*}}{k_i \cdot \zeta_{i^*}} \cdot p_{i^*jl^*}^\lambda$.

c) If E_K^λ is commutative, then we have $\hat{P}_j^\lambda = (P_j^\lambda)^T$.

Proof. Using Corollary (1.14), Remark (1.19), the symmetrising form t, see Proposition (2.3), and Definition (2.4) we obtain

$$\begin{aligned}
p^\lambda_{ji^*l} &= \tfrac{\zeta_i\cdot\zeta_{j^*}}{\zeta_{l^*}}\cdot\overline{p^\lambda_{ij^*l^*}}\\
&= \tfrac{\zeta_i\cdot\zeta_{j^*}}{\zeta_{l^*}}\cdot\overline{t(\alpha^\lambda_i\alpha^\lambda_{j^*}\hat{\alpha}^\lambda_{l^*})}\\
&= \tfrac{\zeta_i\cdot\zeta_{j^*}}{\zeta_{l^*}}\cdot\tfrac{|H|\cdot\zeta_l}{k_l}\cdot\overline{t(\alpha^\lambda_i\alpha^\lambda_{j^*}\alpha^\lambda_l)}\\
&= \tfrac{\zeta_i\cdot\zeta_{j^*}}{\zeta_{l^*}}\cdot\tfrac{|H|\cdot\zeta_l}{k_l}\cdot\tfrac{k_j}{|H|\cdot\zeta_{j^*}}\cdot\overline{t(\alpha^\lambda_l\alpha^\lambda_i\hat{\alpha}^\lambda_j)}\\
&= \tfrac{k_j\cdot\zeta_i}{k_l}\cdot\overline{p^\lambda_{lij}}.
\end{aligned}$$

This shows the assertion in a). Furthermore, we have

$$p^\lambda_{\hat{i}j\hat{l}} = t(\hat{\alpha}^\lambda_i\alpha^\lambda_j\alpha^\lambda_l) = \frac{k_l\cdot\zeta_{l^*}}{k_i\cdot\zeta_{i^*}}\cdot t(\alpha^\lambda_{i^*}\alpha^\lambda_j\hat{\alpha}^\lambda_{l^*}) = \frac{k_l\cdot\zeta_{l^*}}{k_i\cdot\zeta_{i^*}}\cdot p^\lambda_{i^*jl^*}.$$

This shows the assertion in b), while the assertion in c) follows from

$$p^\lambda_{\hat{i}j\hat{l}} = t(\hat{\alpha}^\lambda_i\alpha^\lambda_j\alpha^\lambda_l) = t(\hat{\alpha}^\lambda_i\alpha^\lambda_l\alpha^\lambda_j) = t(\alpha^\lambda_l\alpha^\lambda_j\hat{\alpha}^\lambda_i) = p^\lambda_{lji}.$$ ♯

(3.18) Proposition. Let E^λ_K be commutative and $j\in\mathcal{I}_\lambda$.
a) Let $[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda}\in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$ be the matrix describing the centrally primitive idempotents of E^λ_K in terms of the dual Schur basis $\hat{\mathcal{A}}_\lambda$. Then we have

$$[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda}\cdot(P^\lambda_j)^T = \mathrm{diag}[\varphi(\alpha^\lambda_j);\varphi\in\mathrm{Irr}_K(E^\lambda_K)]\cdot[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda}$$

and

$$[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda} = \frac{1}{|G|}\cdot\mathrm{diag}[m_\varphi;\varphi\in\mathrm{Irr}_K(E^\lambda_K)]\cdot\Phi_\lambda.$$

b) We have $P^\lambda_j = \Phi^T_\lambda\cdot\mathrm{diag}[\varphi(\alpha^\lambda_j);\varphi\in\mathrm{Irr}_K(E^\lambda_K)]\cdot\Phi^{-T}_\lambda$.

Proof. The first assertion in a) follows from Proposition (3.14) and Proposition (3.17). By Proposition (3.2) and its proof we have

$$[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda} = [\frac{1}{c_\varphi}\cdot\varphi(\alpha^\lambda_i);\varphi\in\mathrm{Irr}_K(E^\lambda_K), i\in\mathcal{I}_\lambda],$$

with row index φ and column index i. Hence the second assertion in a) follows. The assertion in b) follows from those in a). ♯

Finally, we discuss the relationship between the character table Φ_λ of E^λ_K, see Definition (3.7), and the ordinary character table of the underlying group G.

(3.19) Definition. Let $\mathcal{C}l(G)$ denote the set of conjugacy classes of G and let
$$\mathcal{X}_\lambda := [\chi(C); \chi \in \mathrm{Irr}_K^\lambda(G), C \in \mathcal{C}l(G)] \in K^{|\mathrm{Irr}_K^\lambda(G)| \times |\mathcal{C}l(G)|}$$
denote the character table of $\mathrm{Irr}_K^\lambda(G)$, with row index χ and column index C. For $C \in \mathcal{C}l(G)$ and $i \in \mathcal{I}$ let
$$\gamma_i^\lambda(C) := \sum_{h \in H} \delta_C(hg_i) \cdot \lambda(h^{-1}) \in \mathbb{Q}(\lambda(H)),$$
where $\delta_C\colon G \to \{0,1\}$ is defined by $\delta_C(g) = 1$ if and only if $g \in C$. Note that $\gamma_i^\lambda(C)$ does not depend on the particular choice of the representative of the double coset $Hg_iH \subseteq G$. Let
$$\Gamma_\lambda := [\gamma_i^\lambda(C); i \in \mathcal{I}_\lambda, C \in \mathcal{C}l(G)] \in \mathbb{Q}(\lambda(H))^{|\mathcal{I}_\lambda| \times |\mathcal{C}l(G)|}$$
with row index $i \in \mathcal{I}_\lambda$ and column index C. The $\gamma_j^\lambda(C)$ for $j \notin \mathcal{I}_\lambda$ are dealt with in Proposition (3.22).

For $\lambda = 1$ let $\gamma_i(C) := \gamma_i^1(C) = |C \cap Hg_i| \in \mathbb{Z}$ and $\Gamma := \Gamma_1 \in \mathbb{Q}^{r \times |\mathcal{C}l(G)|}$, for $i \in \mathcal{I}$ and $C \in \mathcal{C}l(G)$.

(3.20) Proposition. For the character table Φ_λ of E_K^λ we have
$$\Phi_\lambda = \frac{1}{|H|} \cdot \mathcal{X}_\lambda \cdot \Gamma_\lambda^T \cdot \mathrm{diag}[k_i; i \in \mathcal{I}_\lambda].$$

Proof. Let $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $\chi \in \mathrm{Irr}_K^\lambda(G)$ be its Fitting correspondent, see Proposition (2.7) and Section (2.8). For $\alpha \in E_K^\lambda$ and $\sigma = \sigma_\lambda$ as in Section (2.1), let α_χ^σ, denote the image of $\alpha^\sigma \in \mathrm{End}_{KG}(\epsilon_\lambda KG)$ under the projection onto the direct summand $K^{d_\varphi \times d_\varphi}$ belonging to χ. Hence we have $\varphi(\alpha) = \mathrm{tr}(\alpha_\chi^\sigma)$. Furthermore we have $K^{d_\varphi \times d_\varphi} \cdot \tau = \epsilon_\lambda \epsilon_\chi KG \epsilon_\chi \epsilon_\lambda \subseteq \epsilon_\chi KG \epsilon_\chi \cong K^{m_\varphi \times m_\varphi}$, where $m_\varphi = \chi(1)$ and $\tau = \tau_\lambda$ is as in Section (2.1). Hence the Pierce decomposition of $\epsilon_\chi KG \epsilon_\chi$ with respect to $\epsilon_\lambda \epsilon_\chi$ shows that we have $\mathrm{tr}(\alpha_\chi^\sigma) = \chi((\alpha_\chi^\sigma)\tau)$. Since for $\chi \neq \chi' \in \mathrm{Irr}_K^\lambda(G)$ we have $\chi((\alpha_{\chi'}^\sigma)\tau) = 0$, we conclude that $\varphi(\alpha) = \chi((\alpha_\chi^\sigma)\tau) = \chi((\alpha^\sigma)\tau)$. For $i \in \mathcal{I}_\lambda$ we hence have
$$\begin{aligned}
\varphi(\alpha_i^\lambda) &= \chi(((\alpha_i^\lambda)^\sigma)\tau) \\
&= k_i \cdot \chi(\epsilon_\lambda g_i \epsilon_\lambda) \\
&= \tfrac{k_i}{|H|^2} \cdot \textstyle\sum_{h,h' \in H} \lambda(h^{-1}) \cdot \lambda((h')^{-1}) \cdot \chi(hg_ih') \\
&= \tfrac{k_i}{|H|^2} \cdot \textstyle\sum_{h,h' \in H} \lambda((h'h)^{-1}) \cdot \chi(h'hg_i) \\
&= \tfrac{k_i}{|H|} \cdot \textstyle\sum_{h \in H} \lambda(h^{-1}) \cdot \chi(hg_i) \\
&= \tfrac{k_i}{|H|} \cdot \textstyle\sum_{C \in \mathcal{C}l(G)} \gamma_i^\lambda(C)\chi(C).
\end{aligned}$$
♯

(3.21) Remark.
a) In particular, for $\lambda = 1$ let $\varphi_1 \in \mathrm{Irr}_K(E_K)$ be the Fitting correspondent of the trivial KG-character. Then we have, for $i \in \mathcal{I}$,

$$\varphi_1(\alpha_i) = \frac{k_i}{|H|} \cdot \sum_{C \in \mathcal{C}l(G)} |C \cap Hg_i| = k_i \in \mathbb{N}.$$

Hence the index parameters k_i, for $i \in \mathcal{I}$, can be read off from the character table Φ of E_K. Note that the values of φ_1 on $\mathcal{A}$ are positive integers, and that by the orthogonality relations, see Proposition (3.8), φ_1 is uniquely determined by this condition.

b) Let for the moment E_K^λ be commutative, and let

$$K' := \mathbb{Q}(\lambda(H))[\chi(C); \chi \in \mathrm{Irr}_K^\lambda(G), C \in \mathcal{C}l(G)].$$

As $d_\varphi = 1$ for all $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, by [18, La.IV.9.1] the Schur indices over $\mathbb{Q}(\lambda(H))$ of all $\chi \in \mathrm{Irr}_K^\lambda(G)$ are equal to 1. Thus K' is a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}_K^\lambda(G)$, and hence for E_K^λ as well.

c) Let without loss of generality K be a splitting field for KG, and let $\rho \in \mathrm{Gal}(K/\mathbb{Q}(\lambda(H)))$. As $\lambda^\rho = \lambda$, we conclude that $\mathrm{Irr}_K^\lambda(G)$ is $\mathrm{Gal}(K/\mathbb{Q}(\lambda(H)))$-invariant. As $(\Gamma_\lambda)^\rho = \Gamma_\lambda$, the set $\mathrm{Irr}_K(E_K^\lambda)$ also is $\mathrm{Gal}(K/\mathbb{Q}(\lambda(H)))$-invariant and we have $\chi_{\varphi^\rho} = (\chi_\varphi)^\rho \in \mathrm{Irr}_K^\lambda(G)$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$. In particular, if $\lambda(H) \subseteq \mathbb{R}$, we have $\chi_{\overline{\varphi}} = \overline{\chi_\varphi} \in \mathrm{Irr}_K^\lambda(G)$.

(3.22) Proposition.
a) For $j \notin \mathcal{I}_\lambda$ we have $\gamma_j^\lambda(C) = 0$ for all $C \in \mathcal{C}l(G)$.
b) For $\chi \notin \mathrm{Irr}_K^\lambda(G)$ we have $\sum_{C \in \mathcal{C}l(G)} \gamma_i^\lambda(C)\chi(C) = 0$ for all $i \in \mathcal{I}_\lambda$.

Proof. Without loss of generality we assume that K is a splitting field for KG. Let $\mathcal{X} = [\chi(C); \chi \in \mathrm{Irr}_K(G), C \in \mathcal{C}l(G)] \in K^{|\mathcal{C}l(G)| \times |\mathcal{C}l(G)|}$ denote the full K-character table of G. Hence as in the proof of Proposition (3.20) we have $\chi(\epsilon_\lambda g_k \epsilon_\lambda) = \frac{1}{|H|} \cdot \sum_{C \in \mathcal{C}l(G)} \gamma_k^\lambda(C)\chi(C)$, for $k \in \mathcal{I}$.

By Proposition (2.2), we have $\epsilon_\lambda g_j \epsilon_\lambda = 0 \in KG$, for $j \notin \mathcal{I}_\lambda$. Hence for all $\chi \in \mathrm{Irr}_K(G)$ we have $\sum_{C \in \mathcal{C}l(G)} \gamma_j^\lambda(C)\chi(C) = 0$. Thus $\mathcal{X} \cdot [\gamma_j^\lambda(C); C \in \mathcal{C}l(G)]^T = 0 \in K^{|\mathcal{C}l(G)| \times 1}$. As $\mathcal{X}$ is invertible, the assertion in a) follows. For $\chi \notin \mathrm{Irr}_K^\lambda(G)$ we have $\epsilon_\lambda \epsilon_\chi = 0 \in KG$, hence $\chi(\epsilon_\lambda KG \epsilon_\lambda) = 0$. From that the assertion in b) follows. ♯

(3.23) Proposition. See also Ree's Formula [15, Thm.1.11.28].
We have

$$\mathcal{X}_\lambda = \mathrm{diag}[\frac{m_\varphi}{d_\varphi}; \varphi \in \mathrm{Irr}(E_K^\lambda)] \cdot \Phi_\lambda \cdot \overline{\Gamma_\lambda} \cdot \mathrm{diag}[|C|^{-1}; C \in \mathcal{C}l(G)].$$

Proof. For $C \in \mathcal{C}l(G)$ let $C^+ := \sum_{g \in C} g \in KG$ be the corresponding conjugacy class sum. Since $C^+ \in Z(KG)$, we have $\lambda^G(C^+) \in E_K^\lambda$. Thus we have $\lambda^G(C^+) = \sum_{i \in \mathcal{I}_\lambda} \gamma_i \cdot \alpha_i^\lambda \in E_K^\lambda$, for $\gamma_i \in K$. By the definition of the α_i^λ, see Section (1.7), we have, for $i \in \mathcal{I}_\lambda$ and fixed $j \in \{1, \ldots, k_i\}$,

$$\gamma_i := \sum_{h \in H} \delta_C(hg_ih_{ij}) \cdot \lambda_{\omega_1}(hg_ih_{ij}) \cdot \lambda(h_{ij}) = \sum_{h \in H} \delta_C(hg_i) \cdot \lambda(h) = \overline{\gamma_i^\lambda(C)}.$$

For $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ let $\chi \in \mathrm{Irr}_K^\lambda(G)$ be its Fitting correspondent, see Section (2.8), and let tr_φ denote the K-valued trace function on the $KG \otimes_K E_K^\lambda$-module $S_{\chi_\varphi} \otimes_K S_\varphi$. For $i \in \mathcal{I}_\lambda$ we then have $\mathrm{tr}_\varphi(\alpha_i^\lambda) = m_\varphi \cdot \varphi(\alpha_i^\lambda)$ and $\mathrm{tr}_\varphi(C^+) = d_\varphi \cdot |C| \cdot \chi(C)$. Thus $\chi(C) = \frac{m_\varphi}{d_\varphi \cdot |C|} \cdot \sum_{i \in \mathcal{I}_\lambda} \overline{\gamma_i^\lambda(C)} \cdot \varphi(\alpha_i^\lambda)$. ♯

(3.24) Remark. Proposition (3.20) describes Φ_λ in terms of $\mathcal{X}_\lambda$ and Γ_λ, while Proposition (3.23) describes $\mathcal{X}_\lambda$ in terms of Γ_λ and Φ_λ. We briefly discuss the remaining case of describing Γ_λ in terms of Φ_λ and $\mathcal{X}_\lambda$.

Let E_K^λ be commutative. Then from Proposition (3.23) and the orthogonality relations, see Proposition (3.8), we obtain

$$\Gamma_\lambda = \frac{1}{n} \cdot \mathrm{diag}[k_i^{-1}; i \in \mathcal{I}_\lambda] \cdot \Phi_\lambda^T \cdot \overline{\mathcal{X}_\lambda} \cdot \mathrm{diag}[|C|; C \in \mathcal{C}l(G)].$$

Hence we have

$$\begin{aligned} \mathcal{Y}_\lambda &:= \langle [\gamma_i^\lambda(C); C \in \mathcal{C}l(G)]; i \in \mathcal{I}_\lambda \rangle_K \\ &= \langle [\chi_\varphi(C) \cdot |C|; C \in \mathcal{C}l(G)]; \varphi \in \mathrm{Irr}_K(E_K^\lambda) \rangle_K \\ &\leq K^{1 \times |\mathcal{C}l(G)|}. \end{aligned}$$

By Proposition (3.20) and the second orthogonality relations, see Proposition (3.8), we have

$$\Gamma_\lambda \cdot \left(\mathcal{X}_\lambda^T \cdot \mathrm{diag}[m_\varphi; \varphi \in \mathrm{Irr}(E_K^\lambda)] \cdot \overline{\mathcal{X}_\lambda} \right) \cdot \overline{\Gamma_\lambda^T} = |G| \cdot |H| \cdot \mathrm{diag}[k_i^{-1}; i \in \mathcal{I}_\lambda].$$

Hence $\{[\gamma_i^\lambda(C); C \in \mathcal{C}l(G)]; i \in \mathcal{I}_\lambda\}$ is an orthogonal K-basis of $\mathcal{Y}_\lambda$, with respect to the hermitian form defined by the bracketed term. The latter hence is positive definite on $\mathcal{Y}_\lambda$. Furthermore, because of the orthogonality relations for $\mathcal{X}_\lambda$ we have

$$\overline{\mathcal{X}'_\lambda} \cdot \left(\mathcal{X}_\lambda^T \cdot \mathrm{diag}[m_\varphi; \varphi \in \mathrm{Irr}(E_K^\lambda)] \cdot \overline{\mathcal{X}_\lambda} \right) \cdot (\mathcal{X}'_\lambda)^T = |G|^2 \cdot \mathrm{diag}[m_\varphi; \varphi \in \mathrm{Irr}(E_K^\lambda)],$$

where for short $\mathcal{X}'_\lambda := \mathcal{X}_\lambda \cdot \mathrm{diag}[|C|; C \in \mathcal{C}l(G)]$. From that we conclude that $\{[\chi_\varphi(C) \cdot |C|; C \in \mathcal{C}l(G)]; \varphi \in \mathrm{Irr}_K(E_K^\lambda)\}$ also is an orthogonal K-basis of $\mathcal{Y}_\lambda$.

4 Krein parameters

In Section 4 we restrict ourselves to the case $\lambda = 1$, and discuss another algebraic structure on E_K, which has a connection to the tensor product structure on $\mathrm{Irr}_K^1(G)$. As general references see for example [8, Ch.2.3] and [2, Ch.II.3].

Let K be a cyclotomic field being a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}^1_K(G)$.

(4.1) Definition.
a) For $A = [a_{ij}; i,j \in \{1,\ldots,n\}] \in K^{n\times n}$ and $B = [b_{ij}; i,j \in \{1,\ldots,n\}] \in K^{n\times n}$, both with row index i and column index j, let the *Hadamard product* be defined by

$$A \star B := [a_{ij}b_{ij}; i,j \in \{1,\ldots,n\}] \in K^{n\times n}.$$

b) As $E_K \to K^{n\times n}\colon \alpha_i \mapsto [\alpha_i]_\Omega$, for $i \in \mathcal{I}$, is a faithful K-representation, E_K becomes a commutative K-algebra, denoted by $E_K^\star$, by the *Hadamard product* $\alpha_i \star \alpha_j := \delta_{i,j} \cdot \alpha_i$, for $i,j \in \mathcal{I}$.

(4.2) Remark.
a) Hence $\mathcal{A}$ is the set of centrally primitive idempotents of $E_K^\star$.
b) For λ arbitrary and $i,j \in \mathcal{I}_\lambda \subseteq \mathcal{I}_{\lambda^2}$, by Proposition (1.10) we have

$$[\alpha_i^\lambda]_\Omega \star [\alpha_j^\lambda]_\Omega = \delta_{i,j} \cdot [\alpha_i^{\lambda^2}]_\Omega.$$

Hence there is a generalised Hadamard product $\star\colon E_K^\lambda \times E_K^\lambda \to E_K^{\lambda^2}$.

(4.3) We give an interpretation of the Hadamard product on E_K in terms of the permutation module $K\Omega$.

Let $\Delta\colon \Omega \to \Omega\times\Omega\colon \omega \mapsto (\omega,\omega)$ be the diagonal map, and $\Delta\Omega^\perp := (\Omega\times\Omega)\setminus\Delta\Omega \subseteq \Omega\times\Omega$. Thus $K\Omega \otimes_K K\Omega$ is endowed with the structure of a $((KG \otimes_K E_K) \otimes_K (KG \otimes_K E_K))$-module, and it decomposes as KG-module as

$$K\Omega \otimes_K K\Omega \cong K(\Omega\times\Omega) \cong K(\Delta\Omega) \oplus K(\Delta\Omega^\perp) \cong K\Omega \oplus K(\Delta\Omega^\perp).$$

Let $\iota\colon K\Omega \to K\Omega \otimes_K K\Omega$ and $\pi\colon K\Omega \otimes_K K\Omega \to K\Omega$ be the KG-injection and the KG-projection corresponding to the above direct sum decomposition.

(4.4) Proposition. Keeping the notation of Section (4.3), let $\alpha, \alpha' \in E_K$. Then we have

$$\iota \cdot (\alpha \otimes \alpha') \cdot \pi = \alpha \star \alpha' \in E_K.$$

Proof. With respect to the K-bases Ω of $K\Omega$ and $\Omega\otimes\Omega$ of $K(\Omega\otimes\Omega)$ we have $[\iota]_{\omega,(\omega'\otimes\omega'')} = \delta_{\omega,\omega'}\delta_{\omega',\omega''}$ and $[\pi]_{(\omega'\otimes\omega''),\omega} = \delta_{\omega,\omega'}\delta_{\omega',\omega''}$. Furthermore $[\alpha\otimes\alpha']_{(\omega'\otimes\omega''),(\tilde\omega'\otimes\tilde\omega'')} = [\alpha]_{\omega',\tilde\omega'} \cdot [\alpha']_{\omega'',\tilde\omega''}$. Hence for $\omega, \tilde\omega \in \Omega$ we get

$$\begin{aligned}
& [\iota \cdot (\alpha\otimes\alpha') \cdot \pi]_{\omega,\tilde\omega} \\
= \;& \textstyle\sum_{\omega',\omega'',\tilde\omega',\tilde\omega''\in\Omega} \left([\iota]_{\omega,(\omega'\otimes\omega'')} \cdot [\alpha\otimes\alpha']_{(\omega'\otimes\omega''),(\tilde\omega'\otimes\tilde\omega'')} \cdot [\pi]_{(\tilde\omega'\otimes\tilde\omega''),\tilde\omega}\right) \\
= \;& [\alpha\otimes\alpha']_{(\omega\otimes\omega),(\tilde\omega\otimes\tilde\omega)} \\
= \;& [\alpha]_{\omega,\tilde\omega} \cdot [\alpha']_{\omega,\tilde\omega}.
\end{aligned}$$

♯

We introduce suitable structure constants of $E_K^\star$ and show their relationship to the character values of Schur basis elements.

(4.5) Definition. Let E_K be commutative. Let $\mathrm{Irr}_K(E_K) = \{\varphi_i; i \in \mathcal{I}\}$, and for $i \in \mathcal{I}$ let $\epsilon_i \in E_K$ be the centrally primitive idempotent corresponding to φ_i. For $i, j \in \mathcal{I}$ we have $\epsilon_i \star \epsilon_j = \sum_{k \in \mathcal{I}} q_{ijk} \cdot \epsilon_k$, for the *Krein parameters* $q_{ijk} \in K$ of $E_K^\star$.

(4.6) Proposition. Let E_K be commutative. Then for $i, j, k \in \mathcal{I}$ we have

$$q_{ijk} = \frac{m_i \cdot m_j}{n^2} \cdot \sum_{l \in \mathcal{I}} \frac{1}{k_l^2} \cdot \overline{\varphi_i(\alpha_l)} \cdot \overline{\varphi_j(\alpha_l)} \cdot \varphi_k(\alpha_l).$$

Proof. By Proposition (3.2), for $i \in \mathcal{I}$ we have $\epsilon_i = \sum_{j \in \mathcal{I}} \frac{m_i}{n \cdot k_j} \cdot \overline{\varphi_i(\alpha_j)} \cdot \alpha_j$, where $m_i := m_{\varphi_i}$. Hence for $i, j \in \mathcal{I}$ we obtain

$$\epsilon_i \star \epsilon_j = \frac{m_i \cdot m_j}{n^2} \cdot \sum_{l \in \mathcal{I}} \frac{1}{k_l^2} \cdot \overline{\varphi_i(\alpha_l)} \cdot \overline{\varphi_j(\alpha_l)} \cdot \alpha_l.$$

Let $\mathcal{E} := \{\epsilon_i \in E_K; i \in \mathcal{I}\}$ be the K-basis of E_K consisting of the centrally primitive idempotents, and let $\hat{\mathcal{E}} = \{\hat{\epsilon}_i; i \in \mathcal{I}\}$ the corresponding dual K-basis with respect to the symmetrising form t, see Proposition (2.3). By Proposition (3.14) we have $[\mathcal{E}]_{\mathcal{A}} = \Phi^{-T}$, hence we conclude $[\hat{\mathcal{E}}]_{\hat{\mathcal{A}}} = \Phi$, where $\Phi = [\varphi_i(\alpha_j); i, j \in \mathcal{I}] \in K^{r \times r}$ denotes the character table of E_K. Hence for $i \in \mathcal{I}$ we have $\hat{\epsilon}_i = \sum_{j \in \mathcal{I}} \varphi_i(\alpha_j) \cdot \hat{\alpha}_j$. Thus we obtain

$$\begin{aligned} q_{ijk} &= t((\epsilon_i \star \epsilon_j) \cdot \hat{\epsilon}_k) \\ &= \tfrac{m_i \cdot m_j}{n^2} \cdot \textstyle\sum_{l \in \mathcal{I}} \sum_{s \in \mathcal{I}} \frac{1}{k_l^2} \cdot \overline{\varphi_i(\alpha_l)} \cdot \overline{\varphi_j(\alpha_l)} \cdot \varphi_k(\alpha_s) \cdot t(\alpha_l, \hat{\alpha}_s). \end{aligned}$$

♯

(4.7) Remark. Let E_K be commutative and let φ_1 be the Fitting correspondent of the trivial KG-character. As by Remark (3.21) we have $\varphi_1(\alpha_i) = k_i$, for all $i \in \mathcal{I}$, and $m_1 = 1$, by the first orthogonality relations, see Proposition (3.8), we obtain

$$q_{1jk} = \frac{m_j}{n^2} \cdot \sum_{l \in \mathcal{I}} \frac{1}{k_l} \cdot \overline{\varphi_j(\alpha_l)} \cdot \varphi_k(\alpha_l) = \delta_{j,k} \cdot \frac{m_j}{n^2} \cdot \frac{n}{m_k} = \delta_{j,k} \cdot \frac{1}{n},$$

for $j, k \in \mathcal{I}$. Furthermore, for $i, j \in \mathcal{I}$, we have

$$q_{ij1} = \frac{m_i \cdot m_j}{n^2} \cdot \sum_{l \in \mathcal{I}} \frac{1}{k_l} \cdot \varphi_{\bar{i}}(\alpha_l) \cdot \overline{\varphi_j(\alpha_l)} = \delta_{\bar{i},j} \cdot \frac{m_i}{n},$$

where by Remark (3.21) we let $\bar{i} \in \mathcal{I}$ such that $\varphi_{\bar{i}} = \overline{\varphi_i}$.

As was promised at the beginning of Section 4, we prove the relationship between the Hadamard product on E_K and the tensor product structure on $\mathrm{Irr}_K^1(G)$. An application of Proposition (4.8) is given in Section (11.5).

(4.8) Proposition. See also [73].
Let E_K be commutative and $i, j, k \in \mathcal{I}$, such that $q_{ijk} \neq 0$. Then the character $\chi_{\varphi_k} \in \mathrm{Irr}^1_K(G)$ is a constituent of the product $\chi_{\varphi_i} \cdot \chi_{\varphi_j} \in \mathbb{Z}\mathrm{Irr}_K(G)$.

Proof. For $i \in \mathcal{I}$ let $S_i := S_{\chi_{\varphi_i}}$ denote the simple KG-module affording the character χ_i. Hence we have $K\Omega \cong \bigoplus_{i\in\mathcal{I}} S_i$. Let $\iota_i : S_i \to K\Omega$ be the KG-injections and $\pi_i : K\Omega \to S_i$ be the KG-projections corresponding to the above direct sum decomposition. Hence we have $\pi_i \cdot \iota_i = \epsilon_i \in E_K$, for $i \in \mathcal{I}$. By assumption we have $\epsilon_k \cdot (\epsilon_i \star \epsilon_j) = q_{ijk} \cdot \epsilon_k \neq 0$. By Proposition (4.4) we have

$$\begin{aligned} \epsilon_k \cdot (\epsilon_i \star \epsilon_j) &= \pi_k \cdot \iota_k \cdot \iota \cdot ((\pi_i \cdot \iota_i) \otimes (\pi_j \cdot \iota_j)) \cdot \pi \\ &= \pi_k \cdot \iota_k \cdot \iota \cdot (\pi_i \otimes \pi_j) \cdot (\iota_i \otimes \iota_j) \cdot \pi, \end{aligned}$$

where the natural tensor product maps $\pi_i \otimes \pi_j : K\Omega \otimes_K K\Omega \to S_i \otimes_K S_j$ and $\iota_i \otimes \iota_j : S_i \otimes_K S_j \to K\Omega \otimes_K K\Omega$ are KG-homomorphisms with respect to the diagonal KG-action. It follows that $0 \neq \iota_k \cdot \iota \cdot (\pi_i \otimes \pi_j) : S_k \to (S_i \otimes_K S_j)$. As S_k is a simple KG-module the assertion follows. ♯

(4.9) Remark. Using Remark (4.7), as $\chi_{\varphi_1} \in \mathrm{Irr}^1_K(G)$ is the trivial KG-character, Proposition (4.8) for $i, j, k \in \mathcal{I}$ implies the trivial statements that for $j = k$ the character χ_{φ_k} is a constituent of $\chi_{\varphi_1} \cdot \chi_{\varphi_j} = \chi_{\varphi_j}$, and that for $\bar{i} = j$ the trivial character χ_{φ_1} is a constituent of $\chi_{\varphi_i} \cdot \chi_{\varphi_j} = \chi_{\varphi_i} \cdot \overline{\chi_{\varphi_i}}$.

If at least one of i, j, k equals 1, by Remark (4.7), the converse of Proposition (4.8) holds as well. But this is not true in general, as the following example shows.

(4.10) Example. Let $G := M_{11}$ and $H := A_6 < A_6.2_3 < G$. The character table of the endomorphism ring E_K, see Definition (3.7), is contained in the database, see Section (11.1), and is given as follows, where we also indicate the Fitting correspondence $\mathrm{Irr}_K(E_K) \to \mathrm{Irr}^1_K(G)$, see Proposition (2.7).

φ	χ_φ	1	2	3
1	$1a$	1	1	20
2	$10a$	1	1	-2
3	$11a$	1	-1	.

Let $i = j = k = 3$. Using GAP, see Section (8.1) and Table 6, by Proposition (4.6) we get $q_{333} = 0$. But using the ordinary character table of G, also available in GAP, we find that χ_{φ_3} indeed is a constituent of the tensor product $\chi_{\varphi_3} \cdot \chi_{\varphi_3}$.

(4.11) Remark. We conclude Section 4 by discussing briefly a rationality property of the Krein parameters.

Let E_K be commutative and $i, j, k \in \mathcal{I}$. It follows from Propositions (4.6) and (3.1) that we have $q_{i,j,k} \in \mathbb{R}$, and by [2, Thm.II.3.8] we even have the *Krein* *condition* $q_{i,j,k} \geq 0$. According to [2, p.70] it is an open question, when the Krein parameter $q_{i,j,k} \in \mathbb{R}$ is rational. In [2, p.71] an example is given, where the $q_{i,j,k} \in \mathbb{R}$ are at most quadratic irrationalities.

The database, see Section (11.1), contains quite a few examples where some of the Krein parameters are irrational, many of these are quadratic irrationalities. But there also occur irrationalities of higher degree. Using GAP, see Section (8.1) and Table 6, by Proposition (4.6) we find the following examples.

a) Let $G := M_{12}.2$ and $H := M_8.(A_4 \times 2) < M_8.(S_4 \times 2) < G$. The character table of the endomorphism ring E_K has entries from both the quadratic number fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$, and there are Krein parameters being irrationalities of degree 4.

b) Let $G := J_1$ and $H := 2^3.7.3 < G$. The character table of the endomorphism ring E_K has entries both from the quadratic number field $\mathbb{Q}(\sqrt{5})$ and from the cubic number field contained in the 19-th cyclotomic field, and there are both Krein parameters being irrationalities of degree 3 and of degree 6, respectively.

5 Coverings

In Section 5 we examine the situation where we have given transitive G-sets Ω and Ω' such that there exists an epimorphism $\Omega' \to \Omega$ of G-sets. In particular, we discuss how the character tables of the endomorphism ring corresponding to Ω and of the endomorphism ring corresponding to Ω' are related, provided a disjointness condition on the KG-constituents of λ^G and of $(\lambda'^H - \lambda)^G$ holds, see Section (5.3).

(5.1) We begin by fixing some more notation, which will be in force for the remaining parts of the present work. Let $H' \leq H \leq G$ be another subgroup.

As in Section (1.1) let $\mathcal{I}' := \{1, \ldots, r'\}$, where $r' \in \mathbb{N}$ is the number of H'-H'-double cosets in G, and let $\{g_i' \in G; i \in \mathcal{I}'\}$ be a set of representatives of the H'-H'-double cosets in G, where $g_1' := 1_G$. For $i \in \mathcal{I}'$ let $H_i' := (H')^{g_i'} \cap H' \leq H'$, and $\{h_{ij}' \in H'; j \in \{1, \ldots, k_i'\}\}$ be a set of representatives of the right cosets of H_i' in H', where $k_i' = [H' \colon H_i']$ and $h_{i1}' := 1_{H'}$. Let $\Omega' := H'|G$ be the set of right cosets of H' in G, and $n' := [G \colon H']$. Let $\omega_{ij}' := H' g_i' h_{ij}'$, for $i \in \mathcal{I}'$ and $j \in \{1, \ldots, k_i'\}$, and for short $\omega_i' := \omega_{i1}'$, as well as $\Omega_i' := \{\omega_{ij}' \in \Omega'; j \in \{1, \ldots, k_i'\}\}$.

(5.2) Let Θ and λ be as in Section (1.3), and let $\lambda' := \lambda_{H'}$. We describe the relationship between λ^G and λ'^G.

Let $\mathcal{I}_{\lambda'} := \{i \in \mathcal{I}'; \lambda'_{H_i'} = \lambda'^{g_i'}_{H_i'}\}$. As in Section (1.5) and Section (1.7) we have

$$\begin{aligned}
\mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G) &\overset{(1)}{\cong} \mathrm{Hom}_{\Theta H'}(\lambda', (\lambda^G)_{H'}) \\
&\overset{(2)}{\cong} \bigoplus_{g \in H|G|H'} \mathrm{Hom}_{\Theta H'}(\lambda', (\lambda^g_{H^g \cap H'})^{H'}) \\
&\overset{(3)}{\cong} \bigoplus_{g \in H|G|H'} \mathrm{Hom}_{\Theta(H^g \cap H')}(\lambda'_{H^g \cap H'}, \lambda^g_{H^g \cap H'}),
\end{aligned}$$

where the sums run over a set of representatives of the H-H'-double cosets in G. Again we have $\mathrm{Hom}_{\Theta(H^g\cap H')}(\lambda'_{H^g\cap H'}, \lambda^g_{H^g\cap H'}) \neq \{0\}$ if and only if $\lambda'_{H^g\cap H'} = \lambda^g_{H^g\cap H'}$, in which case we have $\mathrm{Hom}_{\Theta(H^g\cap H')}(\lambda'_{H^g\cap H'}, \lambda^g_{H^g\cap H'}) \cong \Theta$. Furthermore, the Θ-isomorphism (1) still is given by $\alpha \mapsto \alpha' := \alpha|_{\Theta_{\lambda'}}$.

(5.3) We fix an appropriate setting to be able to describe the relationship between the character tables of E_K^λ and $E_K^{\lambda'}$. In particular the disjointness condition formulated below will be in force throughout Section 5. We encounter several examples for this situation in Part III. After some preparation, the precise relationship between the character values on the Schur basis elements of E_K^λ and $E_K^{\lambda'}$, respectively, is given in Corollary (5.13).

Let K, R and F be as in Section (2.10), where in particular the characteristic of F is coprime to $|H|$. Let K be a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}_K^{\lambda'}(G)$. We have $\lambda'^G = \lambda^G + (\lambda'^H - \lambda)^G$, thus K is a splitting field for all simple KG-modules affording a character in $\mathrm{Irr}_K^\lambda(G)$ as well. We furthermore assume that λ^G and $(\lambda'^H - \lambda)^G$ have no KG-constituents in common. In particular this holds if $E_K^{\lambda'}$ is commutative, since then $d_\varphi = 1$ for all $\varphi \in \mathrm{Irr}(E_K^{\lambda'})$.

We remark that for the case $\lambda = 1$, and hence also $\lambda' = 1$, the condition of 1_H^G and $(1_{H'}^H - 1_H)^G$ having no KG-constituents in common is related to the notion of *generalised normal subgroups* introduced in [75, Ch.I.5], see [75, Thm.III.19.15].

Hence $\mathrm{Irr}_K^\lambda(G) \subseteq \mathrm{Irr}_K^{\lambda'}(G)$ is the set of constituents of λ^G and $\mathrm{Irr}_K^{\lambda'}(G) \setminus \mathrm{Irr}_K^\lambda(G)$ is the set of constituents of $(\lambda'^H - \lambda)^G$. Thus as KG-modules we have

$$K_{\lambda'}\Omega' \cong K_\lambda\Omega \oplus \sum_{\chi\in\mathrm{Irr}_K^{\lambda'}(G)\setminus\mathrm{Irr}_K^\lambda(G)} K_{\lambda'}\Omega'\epsilon_\chi,$$

where $\epsilon_\chi \in KG$ is the centrally primitive idempotent belonging to $\chi \in \mathrm{Irr}_K(G)$. Let $\alpha_{\lambda'\lambda} \in E_K^{\lambda'}$ denote the corresponding KG-projection onto $K_\lambda\Omega$. Hence we have $E_K^\lambda \cong \alpha_{\lambda'\lambda}E_K^{\lambda'}\alpha_{\lambda'\lambda}$ and $E_K^{\lambda'} \cong \alpha_{\lambda'\lambda}E_K^{\lambda'}\alpha_{\lambda'\lambda} \oplus (1-\alpha_{\lambda'\lambda})E_K^{\lambda'}(1-\alpha_{\lambda'\lambda})$ as K-algebras. Thus in this sense we can consider E_K^λ is a subset of $E_K^{\lambda'}$, and $\mathrm{Irr}_K(E_K^\lambda)$ as a subset of $\mathrm{Irr}_K(E_K^{\lambda'})$.

(5.4) Proposition. Let λ and λ' be as in Section (5.3), where in particular λ^G and $(\lambda'^H - \lambda)^G$ have no KG-constituents in common.
a) $\mathrm{Hom}_{RG}(\lambda'^G, \lambda^G)$ has an R-basis $\mathcal{A}_{\lambda'\lambda} := \{\alpha_i^{\lambda'\lambda}; i \in \mathcal{I}_\lambda\}$, defined using the R-isomorphism (1) in Section (5.2) by

$$(\alpha_i^{\lambda'\lambda})' = (\alpha_i^\lambda)' \in \mathrm{Hom}_{RH}(\lambda, (\lambda^G)_H) \leq \mathrm{Hom}_{RH'}(\lambda', (\lambda^G)_{H'}),$$

where $\mathcal{A}_\lambda = \{\alpha_i^\lambda; i \in \mathcal{I}_\lambda\}$ is the Schur basis of E_R^λ.
b) Using the natural embedding $\mathrm{Hom}_{RG}(\lambda'^G, \lambda^G) \to \mathrm{Hom}_{KG}(\lambda'^G, \lambda^G)$ of R-modules, the set $\mathcal{A}_{\lambda'\lambda}$ also is a K-basis of $\mathrm{Hom}_{KG}(\lambda'^G, \lambda^G)$.

c) $\mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G)$ has an F-basis $\mathcal{A}_{\widetilde{\lambda'}\tilde{\lambda}} := \{\alpha_i^{\widetilde{\lambda'}\tilde{\lambda}}; i \in \mathcal{I}_\lambda\}$, defined by

$$(\alpha_i^{\widetilde{\lambda'}\tilde{\lambda}})' = (\alpha_i^{\tilde{\lambda}})' \in \mathrm{Hom}_{FH}(\tilde{\lambda}, (\tilde{\lambda}^G)_H) \leq \mathrm{Hom}_{FH'}(\widetilde{\lambda'}, (\tilde{\lambda}^G)_{H'}),$$

where $\mathcal{A}_{\tilde{\lambda}} = \{\alpha_i^{\tilde{\lambda}}; i \in \mathcal{I}_\lambda\}$ is the Schur basis of $E_F^{\tilde{\lambda}}$.

Proof. By Section (2.10), we have $\mathcal{I}_{\tilde{\lambda}} = \mathcal{I}_\lambda$. Furthermore, we have

$$\dim_K \mathrm{Hom}_{KG}(\lambda'^G, \lambda^G) = \mathrm{rk}_R \mathrm{Hom}_{RG}(\lambda'^G, \lambda^G) = \dim_F \mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G)$$

and $\dim_K E_K^\lambda = \mathrm{rk}_R E_R^\lambda = \dim_F E_F^{\tilde{\lambda}}$. As λ^G and $(\lambda'^H - \lambda)^G$ have no KG-constituents in common, we have $\mathrm{Hom}_{KG}(\lambda'^G, \lambda^G) \cong \mathrm{Hom}_{KG}(\lambda^G, \lambda^G) = E_K^\lambda$. ♯

(5.5) Corollary. For $i \in \mathcal{I}_\lambda$ we have $Hg_iH = Hg_iH' \subseteq G$, and thus H' acts transitively on Ω_i. In particular, we have $[H' : (H' \cap H^{g_i})] = k_i = [H : H_i]$.

(5.6) Definition.
a) For $i' \in \mathcal{I}'$ and $j' \in \{1, \ldots, k'_i\}$ let $i \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$ as well as $h''_{i'j'} \in H$ be defined by $g'_{i'}h'_{i'j'} = h''_{i'j'} \cdot g_ih_{ij} \in G$. For $j' = 1$ let for short $h''_{i'} \in H$ be defined by $g'_{i'} = h''_{i'} \cdot g_ih_{ij} \in G$, and let

$$\zeta'_{i'} := \lambda(h''_{i'}) \cdot \lambda(h_{ij}) \in \lambda(H).$$

Furthermore, as $i \in \mathcal{I}$ depends on $i' \in \mathcal{I}'$ but not on $j' \in \{1, \ldots, k'_i\}$, this defines a surjective map $\alpha_{H',H} : \mathcal{I}' \to \mathcal{I}$.
b) The map of G-sets $\Omega' \to \Omega : \omega'_{i'j'} \mapsto \omega_{ij}$, for $i' \in \mathcal{I}'$ and $j' \in \{1, \ldots, k_{i'}\}$, where $i = \alpha_{H',H}(i') \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$, by Corollary (5.5) induces surjective maps $\Omega'_{i'} \to \Omega_i$, for $i \in \mathcal{I}_\lambda$ and $i' \in \alpha^{-1}_{H',H}(i)$. The suborbit Ω_i is said to *split* into the suborbits $\{\Omega'_{i'}; i' \in \alpha^{-1}_{H',H}(i)\}$. If $|\alpha^{-1}_{H',H}(i)| = 1$, then Ω_i is said to be a *non-split* suborbit.

(5.7) Remark.
a) For $i \in \mathcal{I}$ we have $\sum_{i' \in \alpha^{-1}_{H',H}(i)} k'_{i'} = [H : H'] \cdot k_i$.
b) By Proposition (5.4), for $1 \in \mathcal{I}_\lambda$ we obtain $\alpha_1^{\lambda'\lambda} \in \mathrm{Hom}_{RG}(\lambda'^G, \lambda^G)$ as

$$\alpha_1^{\lambda'\lambda} : R_{\lambda'}\Omega' \to R_\lambda\Omega : \omega'_{i'j'} \mapsto \lambda(h''_{i'j'}) \cdot \omega_{ij},$$

for $i' \in \mathcal{I}'$ and $j' \in \{1, \ldots, k'_i\}$, where $i = \alpha_{H',H}(i') \in \mathcal{I}$ and $j \in \{1, \ldots, k_i\}$, and $h''_{i'j'} \in H$ is as in Definition (5.6). Furthermore, an analogous statement holds for $\alpha_1^{\widetilde{\lambda'}\tilde{\lambda}} \in \mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G)$.

(5.8) Lemma.
a) For $i \in \mathcal{I}$ we have

$$Hg_iH = \coprod_{x \in H'|H|(H \cap H^{g_i^{-1}})} \left(\coprod_{y \in (H \cap H'^{xg_i})|H|H'} H' \cdot x \cdot g_i \cdot y \cdot H' \right),$$

where x and y run through sets of representatives of the double cosets indicated.
b) For $i \in \mathcal{I}_\lambda$ we have

$$Hg_iH = \coprod_{y \in (H \cap H'^{g_i})|H_i|(H' \cap H^{g_i})} H' \cdot g_i \cdot y \cdot H',$$

where y runs through a set of representatives of the double cosets indicated.

Proof. The group $H^\circ \times H$ acts transitively on Hg_iH by $(h, \tilde{h}) \colon x \mapsto hx\tilde{h}$, for $h, \tilde{h} \in H$ and $x \in Hg_iH$, where H° denotes the opposed group. Hence $\mathrm{Stab}_{H^\circ \times H}(g_i) = \{(h^{g_i^{-1}}, h^{-1}) \in H \times H; h \in H_i\}$. As the H'-H'-double cosets contained in Hg_iH are exactly the $H'^\circ \times H'$-orbits under this action, we have to find representatives $(x, y) \in H \times H$ of the $\mathrm{Stab}_{H^\circ \times H}(g_i)$-$(H'^\circ \times H')$-double cosets in $H^\circ \times H$, hence representatives of the orbits of $\mathrm{Stab}_{H^\circ \times H}(g_i)^\circ \times (H'^\circ \times H')$ on $H^\circ \times H$ with respect to the action $((a, b), (c, d)) \colon (h, \tilde{h}) \mapsto (cha, b\tilde{h}d)$, for $(a, b) \in \mathrm{Stab}_{H^\circ \times H}(g_i)$, as well as $c, d \in H'$ and $h, \tilde{h} \in H$.

Without loss of generality we let the first component $x \in H$ run through a fixed set of representatives of the H'-$(H \cap H^{g_i^{-1}})$-double cosets in H. For the action of $(H \cap H^{g_i^{-1}}) \times H'^\circ$ on H we get

$$\mathrm{Stab}_{(H \cap H^{g_i^{-1}}) \times H'^\circ}(x) = \{(h^x, h^{-1}) \in (H \cap H^{g_i^{-1}}) \times H'; h \in (H \cap H^{g_i^{-1}})^{x^{-1}} \cap H'\}.$$

Hence, for fixed $x \in H$, the second component $y \in H$ is to be chosen from a set of representatives of the orbits of

$$\{((a, b), (c, d)) \in \mathrm{Stab}_{H^\circ \times H}(g_i)^\circ \times (H'^\circ \times H'); (a, c) \in \mathrm{Stab}_{(H \cap H^{g_i^{-1}}) \times H'^\circ}(x)\}$$

on $\{x\} \times H$. This proves the assertion in a).

Since we have $i^* \in \mathcal{I}_\lambda$, it follows from by Corollary (5.5) that we have

$$[H \colon H'] = [(H \cap H^{g_i^{-1}}) \colon (H' \cap H^{g_i^{-1}})].$$

Hence we have $H' \cdot (H \cap H^{g_i^{-1}}) = H$. Furthermore, because of $[H_i \colon (H' \cap H^{g_i})] = [H \colon H']$, we have a bijection

$$\begin{array}{rcl} (H \cap H'^{g_i})|H_i|(H' \cap H^{g_i}) & \to & (H \cap H'^{g_i})|H|H' \colon \\ (H \cap H'^{g_i}) \cdot y \cdot (H' \cap H^{g_i}) & \mapsto & (H \cap H'^{g_i}) \cdot y \cdot H'. \end{array}$$

Thus the assertion in b) follows from a). $\sharp$

(5.9) Remark. Let $i \in \mathcal{I}_\lambda$. Hence there is a bijection between the set $\{\Omega'_{i'}; i' \in \alpha^{-1}_{H',H}(i)\}$ and the set of of representatives y of the double cosets indicated in Lemma (5.8). Using this bijection we may write the index parameters $k'_{i'}$, for $i' \in \alpha^{-1}_{H',H}(i)$, also as $k'_{i,y}$.

Because of $H' \cap H'^{g_i y} \leq H' \cap H^{g_i} \leq H'$ we have

$$\frac{k'_{i,y}}{k_i} = [(H' \cap H^{g_i}) \colon (H' \cap H'^{g_i y})] \in \mathbb{N}.$$

Because of $H' \cap H'^{g_i y} = (H' \cap H^{g_i}) \cap (H \cap H'^{g_i})^y$, the quotients $\frac{k'_{i,y}}{k_i}$ are the lengths of the orbits of the subgroup $H' \cap H^{g_i} \leq H_i$ with respect to the action of H_i on the set of right cosets $(H \cap H'^{g_i})|H_i$. As $H \cap H'^{g_i}$ and $H' \cap H^{g_i}$ are not necessarily conjugate in H_i, we might in particular have $\frac{k'_{i,y}}{k_i} > 1$ for all double coset representatives y.

If y runs through the set of representatives of the double cosets indicated in Lemma (5.8), then $y^{-1} \cdot g_i^{-1}$ runs through a set of representatives of the H'-H'-double cosets of G contained in $Hg_i^{-1}H = Hg_{i^*}H$. As $H' \cap H'^{y^{-1}g_i^{-1}} \leq H' \cap H^{g_i^{-1}}$, we conclude that $\{k'_{j'}; j' \in \alpha^{-1}_{H',H}(i^*)\} = \{k'_{i'}; i' \in \alpha^{-1}_{H',H}(i)\}$, with multiplicities. As $k_i = k_{i^*}$, the same holds for $\{\frac{k'_{j'}}{k_{i^*}}; j' \in \alpha^{-1}_{H',H}(i^*)\}$.

The following Proposition gives a description of the cardinality $|\alpha^{-1}_{H',H}(i) \cap \mathcal{I}_{\lambda'}|$, for $i \in \mathcal{I}$, in terms of irreducible characters of H.

(5.10) Proposition. Let $E_K^{\lambda'}$ be commutative. Then for $i \in \mathcal{I}$ we have

$$|\alpha^{-1}_{H',H}(i) \cap \mathcal{I}_{\lambda'}| = \sum_{\chi \in \mathrm{Irr}_K^{\lambda'}(H)} \langle \chi_{H_i}, \chi^{g_i}_{H_i} \rangle_{H_i},$$

where $\langle \cdot, \cdot \rangle_{H_i}$ denotes the hermitian product on $\mathrm{Irr}_K(H_i)$.

Proof. Let for short $\langle \cdot, \cdot \rangle$ denote the hermitian product on ordinary characters, where the group in question always will be clear from the context. As $g_i^{-1} = \eta_i g_{i^*} h_{i^* i^-}$, where $\eta_i, h_{i^* i^-} \in H$, see Definition (1.12), for $\chi \in \mathrm{Irr}_K^{\lambda'}(H)$ we have

$$\langle \chi_{H_i}, \chi^{g_i}_{H_i} \rangle = \langle \chi^{g_i^{-1}}_{H \cap H^{g_i^{-1}}}, \chi_{H \cap H^{g_i^{-1}}} \rangle = \langle \chi^{g_{i^*}}_{H_{i^*}}, \chi_{H_{i^*}} \rangle.$$

Hence it is enough to show that the right hand side of the asserted equation equals $|\alpha^{-1}_{H',H}(i^*) \cap \mathcal{I}_{\lambda'}|$. As $d_\varphi = 1$ for all $\varphi \in \mathrm{Irr}_K(E_K^{\lambda'})$, we have $0 = \langle \chi^G, \tilde{\chi}^G \rangle = \sum_{i \in \mathcal{I}} \langle \chi_{H_i}, \tilde{\chi}^{g_i}_{H_i} \rangle$, for $\chi \neq \tilde{\chi} \in \mathrm{Irr}_K^{\lambda'}(H)$, hence $\langle \chi_{H_i}, \tilde{\chi}^{g_i}_{H_i} \rangle = 0$

for all $i \in \mathcal{I}$. As all $\chi \in \mathrm{Irr}_K^{\lambda'}(H)$ occur in λ'^H with multiplicity 1, we have

$$\begin{aligned}
& \textstyle\sum_{\chi \in \mathrm{Irr}_K^{\lambda'}(H)} \langle \chi_{H_i}, \chi_{H_i}^{g_i} \rangle \\
= \;& \langle (\lambda'^H)_{H_i}, (\lambda'^H)_{H_i}^{g_i} \rangle \\
= \;& \textstyle\sum_{t \in H'|H|H_i} \langle (\lambda'^t_{H'^t \cap H_i})^{H_i}, (\lambda'^H)_{H_i}^{g_i} \rangle \\
= \;& \textstyle\sum_{t \in H'|H|H_i} \langle \lambda'^t_{H'^t \cap H_i}, ((\lambda'^{g_i})^{H^{g_i}})_{H'^t \cap H_i} \rangle \\
= \;& \textstyle\sum_{t \in H'|H|H_i} \sum_{s \in H'^{g_i}|H^{g_i}|(H'^t \cap H_i)} \langle \lambda'^t_{H'^t \cap H_i}, (\lambda'^{g_i s}_{H'^{g_i s} \cap H'^t \cap H_i})^{H'^t \cap H_i} \rangle \\
= \;& \textstyle\sum_{t \in H'|H|H_i} \sum_{s \in H'^{g_i}|H^{g_i}|(H'^t \cap H_i)} \langle \lambda'^t_{H'^{g_i s} \cap H'^t \cap H_i}, \lambda'^{g_i s}_{H'^{g_i s} \cap H'^t \cap H_i} \rangle \\
\overset{(1)}{=} \;& \textstyle\sum_{t \in H'|H|H_i} \sum_{s \in (H'^{t g_i^{-1}} \cap H)|H|H'} \langle \lambda'^{t g_i^{-1} s}_{H' \cap H'^{t g_i^{-1} s}}, \lambda'_{H' \cap H'^{t g_i^{-1} s}} \rangle,
\end{aligned}$$

where the sums run over sets of representatives of the double cosets indicated, and where equation (1) because of $t \in H$ and $s^{g_i^{-1}} \in H$ follows from

$$\begin{aligned}
\langle \lambda'^t_{H'^{g_i s} \cap H'^t \cap H_i}, \lambda'^{g_i s}_{H'^{g_i s} \cap H'^t \cap H_i} \rangle &= \langle \lambda'^t_{H'^{g_i s} \cap H'^t \cap H^{g_i}}, (\lambda'^{g_i s g_i^{-1}}_{H'^{g_i s g_i^{-1}} \cap H'^{t g_i^{-1}} \cap H})^{g_i} \rangle \\
&= \langle \lambda'^{t g_i^{-1}}_{H'^{g_i s g_i^{-1}} \cap H'^{t g_i^{-1}}}, \lambda'^{g_i s g_i^{-1}}_{H'^{g_i s g_i^{-1}} \cap H'^{t g_i^{-1}}} \rangle \\
&= \langle \lambda'^{t g_i^{-1} \cdot g_i s^{-1} g_i^{-1}}_{H' \cap H'^{t s^{-1} g_i^{-1}}}, \lambda'_{H' \cap H'^{t s^{-1} g_i^{-1}}} \rangle.
\end{aligned}$$

If s and t run through sets of representatives of the double cosets indicated on the right hand side of equation (1), then by Lemma (5.8) the elements $t \cdot g_i^{-1} s$ run through a set of representatives of the H'-H'-double cosets in G contained in $Hg_i^{-1}H$. ♯

(5.11) Corollary. For $s, t \in H$ as on the right hand side of equation (1) in the proof of Proposition (5.10), we have $H'^{s^{-1} g_i} \cap H'^t \leq H_i$. Hence because of $\lambda_{H'} = \lambda'$ we conclude

$$1 \geq \langle \lambda'^{t g_i^{-1} s}_{H' \cap H'^{t g_i^{-1} s}}, \lambda'_{H' \cap H'^{t g_i^{-1} s}} \rangle \geq \langle \lambda_{H_i}, \lambda_{H_i}^{g_i} \rangle \geq 0.$$

Thus for $i \in \mathcal{I}_\lambda$ we obtain $\alpha_{H',H}^{-1}(i) \subseteq \mathcal{I}_{\lambda'}$.

(5.12) Proposition. For $i \in \mathcal{I}_\lambda$ and $i' \in \alpha_{H',H}^{-1}(i) \subseteq \mathcal{I}_{\lambda'}$, see Corollary (5.11), using the identification from Section (5.3), we have

$$\alpha_{\lambda'\lambda} \cdot \alpha_{i'}^{\lambda'} \cdot \alpha_{\lambda'\lambda} = \frac{k'_{i'} \cdot \zeta'_{i'}}{k_i} \cdot \alpha_i^\lambda,$$

where $\zeta'_{i'}$ is as in Definition (5.6), and

$$\alpha_i^\lambda = \frac{1}{[H\colon H']} \cdot \sum_{i' \in \alpha_{H',H}^{-1}(i)} \frac{1}{\zeta'_{i'}} \cdot \alpha_{i'}^{\lambda'}.$$

Proof. By Section (5.3) we have $E_K^\lambda \cong \alpha_{\lambda'\lambda} E_K^{\lambda'} \alpha_{\lambda'\lambda} \subseteq E_K^{\lambda'}$. Because of $((\alpha_{\lambda'\lambda})^{\sigma_{\lambda'}})\tau_{\lambda'} = \epsilon_\lambda$ and $\epsilon_\lambda \epsilon_{\lambda'} = \epsilon_\lambda = \epsilon_{\lambda'}\epsilon_\lambda$, the K-algebra isomorphism in Proposition (2.2) translates the non-unitary embedding $\alpha_{\lambda'\lambda} E_K^{\lambda'} \alpha_{\lambda'\lambda} \subseteq E_K^{\lambda'}$ of K-algebras into the embedding $\epsilon_\lambda KG\epsilon_\lambda \subseteq \epsilon_{\lambda'} KG\epsilon_{\lambda'}$. Hence

$$\epsilon_\lambda \cdot (\epsilon_{\lambda'} g'_{i'} \epsilon_{\lambda'}) \cdot \epsilon_\lambda = \epsilon_\lambda \cdot h''_{i'} g_i h_{ij} \cdot \epsilon_\lambda = \zeta'_{i'} \cdot (\epsilon_\lambda g_i \epsilon_\lambda).$$

Let $\mathcal{H} \subseteq H' \times H'$ be chosen such that $H' g'_{i'} H' = \{h' g'_{i'} h'' \in G; (h', h'') \in \mathcal{H}\}$. Hence we have

$$\begin{aligned} k'_{i'} \cdot \epsilon_{\lambda'} g'_{i'} \epsilon_{\lambda'} &= \tfrac{1}{|H'|} \cdot \textstyle\sum_{(h',h'')\in\mathcal{H}} \lambda'((h'h'')^{-1}) \cdot h' g'_{i'} h'' \\ &= \tfrac{[H:H']}{|H|} \cdot \zeta'_{i'} \cdot \textstyle\sum_{(h',h'')\in\mathcal{H}} \lambda((h' h''_{i'} h_{ij} h'')^{-1}) \cdot h' h''_{i'} g_i h_{ij} h''. \end{aligned}$$

Rewriting $k_i \cdot \epsilon_\lambda g_i \epsilon_\lambda$ analogously, the assertion follows. ♯

(5.13) Corollary. Let $i \in \mathcal{I}_\lambda$.
a) For $\varphi \in \mathrm{Irr}_K(E_K^\lambda) \subseteq \mathrm{Irr}_K(E_K^{\lambda'})$ and $i' \in \alpha_{H',H}^{-1}(i)$ we have

$$\varphi(\alpha_{i'}^{\lambda'}) = \frac{k'_{i'} \cdot \zeta'_{i'}}{k_i} \cdot \varphi(\alpha_i^\lambda).$$

b) For $\varphi \in \mathrm{Irr}_K(E_K^{\lambda'})$ we have

$$\sum_{i' \in \alpha_{H',H}^{-1}(i)} \frac{1}{\zeta'_{i'}} \cdot \varphi(\alpha_{i'}^{\lambda'}) = \begin{cases} [H\colon H'] \cdot \varphi(\alpha_i^\lambda), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^\lambda) \subseteq \mathrm{Irr}_K(E_K^{\lambda'}), \\ 0, & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{\lambda'}) \setminus \mathrm{Irr}_K(E_K^\lambda). \end{cases}$$

In particular, if Ω_i is a non-split suborbit and thus $\alpha_{H',H}^{-1}(i) = \{i'\}$, then for $\varphi \in \mathrm{Irr}_K(E_K^{\lambda'}) \setminus \mathrm{Irr}_K(E_K^\lambda)$ we have $\varphi(\alpha_{i'}^{\lambda'}) = 0$.

(5.14) Example.
a) Let $G := J_4$ and $H := 2^{11}\colon M_{24}$ as well as $H' := 2^{11}\colon M_{23}$, where $\lambda = 1$ and $\lambda' = 1$. Hence we have $r = 7$ and $r' = 11$ as well as $\mathcal{I}_\lambda = \mathcal{I}$ and $\mathcal{I}_{\lambda'} = \mathcal{I}'$. The character tables of the endomorphism rings E_K and $E_K^{1_{H'}}$ are given in Sections (16.1) and (16.2), see Table 21 and Table 22, respectively. The splitting of the suborbits Ω_i is given as follows, where $i' \in \alpha_{H',H}(i)^{-1}$ and $i \in \mathcal{I}$.

i	$\frac{k'_{i'}}{k_i}$
1	$1, 23$
2	$8, 16$
3	24
4	24
5	$4, 20$
6	24
7	$1, 23$

b) Let $G := HN.2$ and $H := S_{12}$ as well as $H' := S_{11}$, where $\lambda = 1$ and $\lambda' = 1$. Hence we have $r = 10$ and $r' = 17$. The character table of the endomorphism ring $E_K^{1_{H'}}$ is given in Section (13.1), see Table 13. The splitting of the Ω_i is given as follows, where $i' \in \alpha_{H',H}(i)^{-1}$ and $i \in \mathcal{I}$. Note that, even since $\langle 1_{H'}^H, 1_{H'}^H \rangle_H = 2$, the suborbit Ω_5 of Ω splits into three suborbits of Ω'.

i	$\frac{k'_{i'}}{k_i}$
1	$1, 11$
2	$6, 6$
3	12
4	$2, 10$
5	$1, 5, 6$
6	12
7	12
8	$4, 8$
9	12
10	$1, 11$

We conclude Section 5 by discussing three particular cases of the above general situation, which are of importance later on.

(5.15) Remark. Let $[H\colon H'] = 2$ and $\lambda' = 1$. Hence we have $\mathrm{Irr}_K^1(H) = \{1, 1^-\}$, where $1^- \in \mathrm{Irr}_K(H)$ denotes the inflation of the non-trivial irreducible character of H/H' to H. Hence both elements of $\mathrm{Irr}_K^1(H)$ can be chosen as λ as above. By Remark (5.7) and Remark (5.9), for $i \in \mathcal{I}$, we distinguish two cases.

a) We have $\alpha_{H',H}^{-1}(i) = \{i'\}$, thus $k'_{i'} = 2 \cdot k_i$. Hence we have $Hg_iH = H'g_iH'$ and $[(H' \cap H^{g_i})\colon (H' \cap H'^{g_i})] = 2$. By Proposition (5.10) we have $1^-_{H_i} \neq (1^-)^{g_i}_{H_i}$, hence $i \notin \mathcal{I}_{1^-}$. Thus, by Corollary (5.13), still using the identification from Section (5.3), for $\varphi \in \mathrm{Irr}_K(E_K^{1_{H'}})$ we have

$$\varphi(\alpha_{i'}^{1_{H'}}) = \begin{cases} 2 \cdot \varphi(\alpha_i^{1_H}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{1_H}), \\ 0, & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{1_H^-}). \end{cases}$$

b) We have $\alpha_{H',H}^{-1}(i) = \{i', i''\}$, thus $k'_{i'} = k'_{i''} = k_i$. Hence we have $Hg_iH = H'g_iH' \;\dot\cup\; H'g_iyH'$, where $\{g_i, g_iy\}$ is a set of representatives of the double cosets indicated in Lemma (5.8). We have $H' \cap H'^{g_i} = H' \cap H'^{g_iy} = H' \cap H^{g_i}$. By Proposition (5.10) we have $1^-_{H_i} = (1^-)^{g_i}_{H_i}$, hence $i \in \mathcal{I}_{1^-}$. Without loss of generality let $g'_{i'} := g_i$ and $g'_{i''} := g_iy$. As $y \in H_i \setminus (H \cap H'^{g_i})$, by Definition (5.6), applied to $\lambda = 1^-$, we obtain $\zeta'_{i'} = 1$ and $\zeta'_{i''} = \lambda(y) = -1$. Thus, by Corollary (5.13), for $\varphi \in \mathrm{Irr}_K(E_K^{1_{H'}})$ we have

$$\varphi(\alpha_{i'}^{1_{H'}}) = \begin{cases} \varphi(\alpha_{i''}^{1_{H'}}) = \varphi(\alpha_i^{1_H}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{1_H}), \\ -\varphi(\alpha_{i''}^{1_{H'}}) = \varphi(\alpha_i^{1_H^-}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{1_H^-}). \end{cases}$$

An example for this situation is given in Section (17.1).

(5.16) Remark. Let $G' \leq G$ such that $[G{:}G'] = 2$. Let $H \not\leq G'$ and $H' := G' \cap H$, hence we have $[H{:}H'] = 2$, and we may identify $H'|G'$ with $\Omega := H|G$ via $H'g \mapsto Hg$, for $g \in G'$. Hence without loss of generality we may in particular choose the double coset representatives $g_i \in G' < G$, for $i \in \mathcal{I}$.

Let $\lambda' = 1$. As in Remark (5.15) we have $\mathrm{Irr}^1_K(H) = \{1, 1^-\}$, where now $1^- \in \mathrm{Irr}_K(H)$ is extendible to $1^- \in \mathrm{Irr}_K(G)$, where $1^- \in \mathrm{Irr}_K(G)$ is the inflation of the non-trivial irreducible character of G/G' to G. Hence the condition that 1_H^G and $(1^-)_H^G$ have no KG-constituents in common is equivalent to $\chi \neq \chi \cdot 1^- \in \mathrm{Irr}_K(G)$ and $\chi \cdot 1^- \not\in \mathrm{Irr}^1_K(G)$, for $\chi \in \mathrm{Irr}^1_K(G)$.

a) Let $E_K^{1_{H'}^{G'}} := \mathrm{End}_{KG'}(K\Omega)$ and as usual $E_K = E_K^{1_H^G} := \mathrm{End}_{KG}(K\Omega)$. From Clifford theory and the condition on the KG-constituents of 1_H^G we conclude that $\dim_K(E_K^{1_{H'}^{G'}}) = \dim_K(E_K)$ holds. Hence the H'-orbits and the H-orbits on Ω coincide, and thus for the corresponding Schur K-bases $\{\alpha_i^{1_{H'}^{G'}}; i \in \mathcal{I}\}$ and $\mathcal{A} = \{\alpha_i; i \in \mathcal{I}\}$ of $E_K^{1_{H'}^{G'}}$ and E_K, respectively, we have $\alpha_i^{1_{H'}^{G'}} = \alpha_i \in \mathrm{End}_K(K\Omega)$, for $i \in \mathcal{I}$. Hence $E_K^{1_{H'}^{G'}}$ and E_K are isomorphic K-algebras, and the sets $\mathrm{Irr}_K(E_K^{1_{H'}^{G'}})$ and $\mathrm{Irr}_K(E_K)$ can be identified via $\varphi' \mapsto \varphi{:}\, (\alpha_i \mapsto \varphi'(\alpha_i^{1_{H'}^{G'}}))$, for $i \in \mathcal{I}$. Thus $E_K^{1_{H'}^{G'}}$ and E_K have the same character table.

Furthermore, by Proposition (3.2), we have $\mathcal{E}^{1_{H'}^{G'}} = \mathcal{E}$, where $\mathcal{E}^{1_{H'}^{G'}}$ and $\mathcal{E}$ are the centrally primitive idempotents of $E_K^{1_{H'}^{G'}}$ and E_K, respectively. Thus for the Fitting correspondents of a pair of characters $\mathrm{Irr}_K(E_K^{1_{H'}^{G'}}) \ni \varphi' \mapsto \varphi \in \mathrm{Irr}_K(E_K)$ being identified as above we have $\chi_{\varphi'} = (\chi_\varphi)_{G'} \in \mathrm{Irr}_K^{1_{H'}}(G')$.

b) Assume that, for $i \in \mathcal{I}$, we have $|\alpha^{-1}_{H',H}(i)| = 1$. Then, by Remark (5.15), we have $Hg_iH = H'g_iH' \subseteq G'$, a contradiction. Hence we have $\alpha^{-1}_{H',H}(i) = \{i', i''\}$, and $k'_{i'} = k'_{i''} = k_i$. Let $z \in H \setminus H'$. Since $H'g_izH' \subseteq G \setminus G'$ and $H'g_iH' \,\dot\cup\, H'g_izH' \subseteq Hg_iH$, we have equality here. Hence without loss of generality let $g'_{i'} := g_i$ and $g'_{i''} := g_iz$.

Let $\epsilon := \epsilon_{1_{H'}} \in KH' \subseteq KG' \subseteq KG$. Then for $E_K^{1_{H'}} = E_K^{1_{H'}^G}$ we have $(E_K^{1_{H'}})^\circ \cong \epsilon KG\epsilon$ as K-algebras. The latter has $\{\epsilon g_i\epsilon; i \in \mathcal{I}\} \,\dot\cup\, \{\epsilon g_iz\epsilon; i \in \mathcal{I}\}$ as a K-basis, see Proposition (2.2). For $i, j \in \mathcal{I}$ we have $\epsilon g_i\epsilon \cdot \epsilon g_j\epsilon \in KG'$ and $\epsilon g_iz\epsilon \cdot \epsilon g_jz\epsilon \in KG'$ as well as $\epsilon g_i\epsilon \cdot \epsilon g_jz\epsilon \in K(G \setminus G')$ and $\epsilon g_iz\epsilon \cdot \epsilon g_j\epsilon \in K(G \setminus G')$. Hence we have

$$\epsilon KG\epsilon = \epsilon KG'\epsilon \oplus \epsilon K(G \setminus G')\epsilon$$

as K-vector spaces and $\epsilon KG\epsilon$ is a $\mathbb{Z}/2\mathbb{Z}$-graded K-algebra, also called a K-*superalgebra*. Furthermore, we have $\epsilon KG'\epsilon \cong (E_K^{1_{H'}^{G'}})^\circ$ as K-algebras.

Let $(k_i \cdot \epsilon g_i\epsilon) \cdot (k_j \cdot \epsilon g_j\epsilon) = \sum_{l \in \mathcal{I}} p'_{jil} \cdot (k_l \cdot \epsilon g_l\epsilon)$, where the $p'_{ijl} \in K$, for $i, j, l \in \mathcal{I}$, denote the structure constants of $E_K^{1_{H'}^{G'}}$, see Definition (1.18). As $z^2 \in H'$, we obtain the structure constants matrices of $E_K^{1_{H'}}$ as

$$P_{j'}^{1_{H'}} = \left[\begin{array}{c|c} P'_j & \cdot \\ \hline \cdot & P'_j \end{array}\right] \in K^{2r\times 2r} \quad \text{and} \quad P_{j''}^{1_{H'}} = \left[\begin{array}{c|c} \cdot & P'_j \\ \hline P'_j & \cdot \end{array}\right] \in K^{2r\times 2r},$$

where $P'_j \in K^{r\times r}$ is the corresponding structure constants matrix of $E_K^{1_{H'}^{G'}}$.

Using the above form of the structure constants matrices of $E_K^{1_{H'}}$, we conclude that $E_K^{1_{H'}}$ also is commutative, and for a pair of characters $\mathrm{Irr}_K(E_K^{1_{H'}^{G'}}) \ni \varphi' \mapsto \varphi \in \mathrm{Irr}_K(E_K)$ being identified as above we obtain $\varphi_+, \varphi_- \in \mathrm{Irr}_K(E_K^{1_{H'}})$ such that, for $i \in \mathcal{I}$,

$$\begin{array}{rcccccl} \varphi_+(\alpha_{i'}) & = & \varphi_+(\alpha_{i''}) & = & \varphi'(\alpha_i^{1_{H'}^{G'}}) & = & \varphi(\alpha_i), \\ \varphi_-(\alpha_{i'}) & = & -\varphi_-(\alpha_{i''}) & = & \varphi'(\alpha_i^{1_{H'}^{G'}}) & = & \varphi(\alpha_i), \end{array}$$

where $\{\alpha_{i'}; i \in \mathcal{I}\} \,\dot\cup\, \{\alpha_{i''}; i \in \mathcal{I}\}$ is the Schur K-basis of $E_K^{1_{H'}}$.

By Remark (5.15), the set $\{\varphi_+; \varphi \in \mathrm{Irr}_K(E_K)\}$ is in Fitting correspondence to $\mathrm{Irr}_K^1(G)$, while $\{\varphi_-; \varphi \in \mathrm{Irr}_K(E_K)\}$ corresponds to $\mathrm{Irr}_K^{1_H^-}(G) = 1^- \cdot \mathrm{Irr}_K^1(G)$. As $H'g_i \subseteq G'$ and $H'g_i z \subseteq G \setminus G'$, for $i \in \mathcal{I}$, we conclude from Proposition (3.20) that, for $\chi \in \mathrm{Irr}_K^1(G)$, we have $\chi_{\varphi^+} \cdot 1^- = \chi_{\varphi^-}$.

(5.17) Example. Let $G := J_2.2$ and $H := U_3(3).2$, as well as $G' := J_2$ and $H' := U_3(3)$. The character table of the endomorphism ring E_K, see Definition (3.7), which equals the character table of $E_K^{1_{H'}^{G'}}$, and the character table of $E_K^{1_{H'}}$ are both contained in the database, see Section (11.1). They are given as follows, where for all three cases the Fitting correspondence, see Proposition (2.7), is indicated as well.

φ	$\chi_{\varphi'}$	χ_φ	1	2	3
1	$1a$	$1a^+$	1	36	63
2	$36a$	$36a^+$	1	6	-7
3	$63a$	$63a^+$	1	-4	3

φ	χ_φ	$1'$	$1''$	$2'$	$2''$	$3'$	$3''$
1	$1a^+$	1	1	36	36	63	63
2	$1a^-$	1	-1	36	-36	63	-63
3	$36a^+$	1	1	6	6	-7	-7
4	$36a^-$	1	-1	6	-6	-7	7
5	$63a^+$	1	1	-4	-4	3	3
6	$63a^-$	1	-1	-4	4	3	-3

(5.18) Remark. Let $H' \trianglelefteq H$ such that $[H\colon H'] = 3$, and $\lambda' = 1$. Hence we have $\mathrm{Irr}_K^1(H) = \{1, \lambda_3, \lambda_3^{-1}\}$, where $\lambda_3, \in \mathrm{Irr}_K(H)$ denotes the inflation of one of the non-trivial irreducible characters of H/H' to H. Hence all the elements of $\mathrm{Irr}_K^1(H)$ can be chosen as λ as above, and we have $\mathcal{I}_{\lambda_3} = \mathcal{I}_{\lambda_3^{-1}}$. For $i \in \mathcal{I}$ we have $\langle (\lambda_3)_{H_i}, (\lambda_3)_{H_i}^{g_i} \rangle_{H_i} = 1$ if and only if $\langle (\lambda_3^{-1})_{H_i}, (\lambda_3^{-1})_{H_i}^{g_i} \rangle_{H_i} = 1$. Hence by Proposition (5.10) we distinguish two cases.

a) We have $\alpha_{H',H}^{-1}(i) = \{i'\}$, thus $k'_{i'} = 3 \cdot k_i$. Hence we have $i \notin \mathcal{I}_{\lambda_3}$. Thus, by Corollary (5.13), still using the identification from Section (5.3), for $\varphi \in \mathrm{Irr}_K(E_K^{1_{H'}})$ we have

$$\varphi(\alpha_{i'}^{1_{H'}}) = \begin{cases} 3 \cdot \varphi(\alpha_i^{1_H}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{1_H}), \\ 0, & \text{if } \varphi \notin \mathrm{Irr}_K(E_K^{1_H}). \end{cases}$$

b) We have $\alpha_{H',H}^{-1}(i) = \{i', i'', i'''\}$, thus $k'_{i'} = k'_{i''} = k'_{i'''} = k_i$. Hence we have $i \in \mathcal{I}_{\lambda_3}$, and $Hg_iH = H'g_iH' \,\dot\cup\, H'g_iyH' \,\dot\cup\, H'g_iy'H'$, where $\{g_i, g_iy, g_iy'\}$ is a set of representatives of the double cosets indicated in Lemma (5.8). As $H' \cap H^{g_i} \trianglelefteq H_i$ and $H \cap H'^{g_i} \trianglelefteq H_i$, we may choose $y' = y^{-1}$. Without loss of generality let $g'_{i'} := g_i$ as well as $g'_{i''} := g_iy$ and $g'_{i'''} := g_iy^{-1}$. As $y \in H_i \setminus (H \cap H'^{g_i})$, by Definition (5.6), applied to λ_3, we obtain $\zeta'_{i'} = 1$, as well as $\zeta'_{i''} = \lambda_3(y) = \zeta_3$ and $\zeta'_{i'''} = \lambda_3(y^{-1}) = \frac{1}{\zeta_3}$, where $\zeta_3 \in K$ is a primitive third root of unity. Thus, by Corollary (5.13), for $\varphi \in \mathrm{Irr}_K(E_K^{1_{H'}})$ we have

$$\varphi(\alpha_{i'}^{1_{H'}}) = \begin{cases} \varphi(\alpha_{i''}^{1_{H'}}) = \varphi(\alpha_{i'''}^{1_{H'}}) = \varphi(\alpha_i^{1_H}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{1_H}), \\ \frac{1}{\zeta_3} \cdot \varphi(\alpha_{i''}^{1_{H'}}) = \zeta_3 \cdot \varphi(\alpha_{i'''}^{1_{H'}}) = \varphi(\alpha_i^{\lambda_3}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{\lambda_3}), \\ \zeta_3 \cdot \varphi(\alpha_{i''}^{1_{H'}}) = \frac{1}{\zeta_3} \cdot \varphi(\alpha_{i'''}^{1_{H'}}) = \varphi(\alpha_i^{\lambda_3^{-1}}), & \text{if } \varphi \in \mathrm{Irr}_K(E_K^{\lambda_3^{-1}}). \end{cases}$$

Let $\overline{}\colon K \to K$ denote the involutory field automorphism as in Section 3. As the set $\mathrm{Irr}_K^{1_{H'}}(G)$ is invariant under $\overline{}$, by Remark (3.21) we conclude that $\overline{\varphi} \in \mathrm{Irr}_K(E_K^{\lambda_3^{-1}})$ if and only if $\varphi \in \mathrm{Irr}_K(E_K^{\lambda_3})$. Hence to determine the character table of $E_K^{1_{H'}}$ it is sufficient to know, for which $i \in \mathcal{I}$ we have $|\alpha_{H',H}^{-1}(i)| > 1$, and to determine the character tables of $E_K^{1_H}$ and $E_K^{\lambda_3}$.

An example for this situation is given in Section (12.2).

6 Condensation functors

In Section 6 we occupy a much more general point of view, which encompasses the cases of the endomorphism rings E_K^{λ} as special cases. It turns out that this is the right setting to formulate and understand some of the most powerful techniques of computational representation theory. We begin in a fairly general setting, thereby correcting an impreciseness in [57].

Let Θ be a principal ideal domain. Let A be a Θ-algebra, which is a finitely generated Θ-free Θ-module. Let **mod**-A be the abelian category of finitely generated right A-modules. For the necessary notions from category theory see [1, Ch.II.1] and [36, Ch.I].

(6.1) Definition.
a) Let V be a finitely generated Θ-free Θ-module, and let $U \leq V$ be a Θ-submodule. Then the Θ-pure Θ-submodule

$$U^V := \bigcap\{X; X \leq V \text{ is a } \Theta\text{-pure } \Theta\text{-submodule}, U \leq X\} \leq V$$

is called the *pure closure* of U in V. For the notion of Θ-purity see [39, Ch.I.17].
b) Let $\mathbf{mod}_\Theta$-A be the full additive subcategory of **mod**-A consisting of its Θ-free objects. In particular, if Θ is a field we have $\mathbf{mod}_\Theta$-A = **mod**-A.

(6.2) Proposition. Let $V, W \in \mathbf{mod}_\Theta$-$A$ and $\alpha \in \mathrm{Hom}_A(V, W)$.
a) Then a kernel $\ker\alpha$ and a cokernel $\mathrm{cok}\,\alpha$ exist in $\mathbf{mod}_\Theta$-A.
b) The natural map, induced by α,

$$\mathrm{coim}\,\alpha := \mathrm{cok}\,(\ker\alpha) \to \ker(\mathrm{cok}\,\alpha) =: \mathrm{im}\,\alpha$$

from the coimage of α to the image of α is an isomorphism if and only if $V\alpha \leq W$ is a Θ-pure submodule. In particular, if Θ is not a field then $\mathbf{mod}_R$-A fails to be an exact category.

Proof. The set theoretic kernel $\ker\alpha \in$ **mod**-A of α again is a Θ-free module, and hence, together with its natural embedding into V, it is a categorical kernel of α in $\mathbf{mod}_\Theta$-A.

As $(V\alpha)^W \leq W$ is a Θ-pure submodule, we have $W/(V\alpha)^W \in \mathbf{mod}_\Theta$-$A$. Let $\beta: W \to W/(V\alpha)^W$ denote the natural epimorphism. Let $X \in \mathbf{mod}_\Theta$-$A$ and $\gamma \in \mathrm{Hom}_A(W, X)$, such that $\alpha\gamma = 0$. Then for $w \in (V\alpha)^W$ there is $\theta \in \Theta$ such that $\theta w \in V\alpha$, hence we have $\theta w \cdot \gamma = 0$, and since X is a Θ-free module we conclude $w\gamma = 0$. Hence γ factors through β, and $\mathrm{cok}\,\alpha := W/(V\alpha)^W$ together with β is a categorical cokernel of α in $\mathbf{mod}_\Theta$-A. This shows the assertions in a).

As $\ker\alpha \leq V$ is a Θ-pure submodule, we have $\mathrm{cok}\,(\ker\alpha) \cong V/\ker\alpha$. As $(V\alpha)^W \leq W$ is a Θ-pure submodule, we have $\ker(\mathrm{cok}\,\alpha) \cong (V\alpha)^W$. From that the assertion in b) follows. ♯

(6.3) Definition. Let $V, W, U \in \mathbf{mod}_\Theta$-$A$ as well as $\alpha \in \mathrm{Hom}_A(V, W)$ and $\beta \in \mathrm{Hom}_A(W, U)$. The sequence $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ is called *exact*, if $\mathrm{im}\,\alpha = \ker\beta$ in the category $\mathbf{mod}_\Theta$-A.

We introduce the objects of interest in Section 6, condensation functors and functors related to them, and discuss a few of their properties. The intention is to show their usefulness as a tool to analyse a given module category in practice.

(6.4) Definition. See also [25, Ch.6.2].
Let $e \in A$ be an idempotent.
a) The additive exact functor

$$C_e\colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}eAe\colon V \mapsto Ve,$$

mapping $\alpha \in \mathrm{Hom}_A(V,W)$ to its restriction $\alpha|_{Ve} \in \mathrm{Hom}_{eAe}(Ve,We)$ to Ve, is called the *condensation functor* or *Schur functor* with respect to e. For $V \in \mathbf{mod}\text{-}A$ the eAe-module $Ve \in \mathbf{mod}\text{-}eAe$ is called the *condensed module* of V.
b) The *uncondensation functor* with respect to e is the additive functor

$$U_e := ? \otimes_{eAe} eA\colon \mathbf{mod}\text{-}eAe \to \mathbf{mod}\text{-}A.$$

For $W \in \mathbf{mod}\text{-}eAe$, the A-module $W \otimes_{eAe} eA \in \mathbf{mod}\text{-}A$ is called the *uncondensed module* of W.

(6.5) Remark. C_e is equivalent to the tensor functor $? \otimes_A Ae\colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}eAe$, using the equivalence $\sigma_e\colon C_e \to ? \otimes_A Ae$ of functors from $\mathbf{mod}\text{-}A$ to $\mathbf{mod}\text{-}eAe$ given by $\sigma_e(V)\colon Ve \to V \otimes_A Ae\colon ve \mapsto v \otimes e$.

Furthermore, there is an equivalence $\tau_e\colon \mathrm{Hom}_A(eA, ?) \to ? \otimes_A Ae$ of functors from $\mathbf{mod}\text{-}A$ to $\mathbf{mod}\text{-}eAe$, given by $\tau_e(V)\colon \mathrm{Hom}_A(eA,V) \to Ve\colon \alpha \mapsto e\alpha$, with inverse given by $\tau_e^{-1}(V)\colon Ve \to \mathrm{Hom}_A(eA,V)\colon v \mapsto (ea \mapsto v \cdot a)$.

$C_e \circ U_e$ is equivalent to the identity functor on $\mathbf{mod}\text{-}eAe$ using the equivalence given by $V \otimes_{eAe} eA \cdot e \to V\colon v \otimes ea \cdot e \mapsto veae$, for $V \in \mathbf{mod}\text{-}eAe$.

(6.6) Proposition.
a) C_e induces an additive functor from $\mathbf{mod}_\Theta\text{-}A$ to $\mathbf{mod}_\Theta\text{-}eAe$.
b) Let $V, W, U \in \mathbf{mod}_\Theta\text{-}A$ and let $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ be an exact sequence in $\mathbf{mod}_\Theta\text{-}A$, see Definition (6.3). Then $Ve \xrightarrow{\alpha|_{Ve}} We \xrightarrow{\beta|_{We}} Ue$ is an exact sequence in $\mathbf{mod}_\Theta\text{-}eAe$.

Proof. If $V \in \mathbf{mod}\text{-}A$ is a Θ-free module, then $Ve \in \mathbf{mod}\text{-}eAe$ also is a Θ-free module. This shows the assertion in a).

Both $(V\alpha)^W \cdot e \leq (V\alpha)^W$ and $(V\alpha)^W \leq W$ are Θ-pure submodules. Hence $(V\alpha)^W \cdot e \leq W$ is a Θ-pure submodule, thus this holds for $(V\alpha)^W \cdot e \leq We$ as well. Hence we have $(V\alpha \cdot e)^{We} \leq (V\alpha)^W \cdot e$. Furthermore, for $w \in (V\alpha)^W \cdot e = (V\alpha)^W \cap We$ there is $\theta \in \Theta$ such that $\theta w \in V\alpha \cap We = V\alpha \cdot e$. Hence we also have $(V\alpha)^W \cdot e \leq (V\alpha \cdot e)^{We}$, and thus equality holds. Using the exactness of C_e as a functor from $\mathbf{mod}\text{-}A$ to $\mathbf{mod}\text{-}eAe$, the assertion in b) follows. ♯

The most important case, as far as computational applications are concerned, is where the base ring Θ is a field.

(6.7) Proposition. See also [57, La.3.2].
Let Θ be a field.
a) Let $S \in \mathbf{mod}\text{-}A$ be a simple A-module. Then we have $Se \neq \{0\}$, if and only if S is a constituent of $eA/\mathrm{rad}(eA) \in \mathbf{mod}\text{-}A$. If $Se \neq \{0\}$, then $Se \in \mathbf{mod}\text{-}eAe$ is a simple eAe-module.
b) Let $S, S' \in \mathbf{mod}\text{-}A$ be simple A-modules, such that $Se \neq \{0\}$. Then we have $S \cong S'$ in $\mathbf{mod}\text{-}A$ if and only if $Se \cong S'e$ in $\mathbf{mod}\text{-}eAe$.
c) Let $T \in \mathbf{mod}\text{-}eAe$ be a simple eAe-module. Then there is a simple A-module $S \in \mathbf{mod}\text{-}A$ such that $T \cong Se$ as eAe-modules.

Proof. By Remark (6.5) we have $Se \cong \mathrm{Hom}_A(eA, S) \cong \mathrm{Hom}_A(eA/\mathrm{rad}(eA), S)$ as Θ-vector spaces. From this the first assertion in a) follows. Let $0 \neq v \in Se$. Since S is a simple A-module, we have $v \cdot eAe = vA \cdot e = Se$. From this the second assertion in a) follows.

Let $Se \cong S'e$ in $\mathbf{mod}\text{-}eAe$. Choose a decomposition of $e \in A$ as a sum of pairwise orthogonal primitive idempotents in A. We have $\mathrm{Hom}_A(eA, S) \cong Se \neq \{0\}$ as Θ-vector spaces, if and only if there is a summand $e_S \in eAe \subseteq A$ such that $e_S A$ is a projective indecomposable module with $e_S A/\mathrm{rad}(e_S A) \cong S$ in $\mathbf{mod}\text{-}A$. Applying the condensation functor $C_{e_S}\colon \mathbf{mod}\text{-}eAe \rightarrow \mathbf{mod}\text{-}e_S A e_S$, we obtain $Se_S \cong S'e_S$ in $\mathbf{mod}\text{-}e_S A e_S$. Hence we have $\{0\} \neq S'e_S \cong \mathrm{Hom}_A(e_S A, S')$ as Θ-vector spaces, thus $S' \cong S$ in $\mathbf{mod}\text{-}A$. This proves the assertion in b).

By Remark (6.5) we have $C_e \circ U_e(T) \cong T \neq \{0\}$ in $\mathbf{mod}\text{-}eAe$, hence $U_e(T) \neq \{0\}$. Thus there is a simple A-module $S \in \mathbf{mod}\text{-}A$ such that $\mathrm{Hom}_A(U_e(T), S) \neq \{0\}$. By the Adjointness Theorem [15, Thm.0.2.19] we have as Θ-vector spaces

$$\mathrm{Hom}_A(T \otimes_{eAe} eA, S) \cong \mathrm{Hom}_{eAe}(T, \mathrm{Hom}_A(eA, S)) \cong \mathrm{Hom}_{eAe}(T, Se) \neq \{0\}.$$

Thus we conclude that $\{0\} \neq Se \in \mathbf{mod}\text{-}eAe$ is a simple eAe-module, hence $Se \cong T$ in $\mathbf{mod}\text{-}eAe$. $\sharp$

Given an idempotent $e \in A$, this leads to some further structural features of the category of A-modules. Their usefulness becomes clearer below.

(6.8) Definition. Let Θ be a field and let $e \in A$ be an idempotent.
a) Let $\Sigma_e \subseteq \mathbf{mod}\text{-}A$ be a set of representatives of the isomorphism types of simple A-modules $S \in \mathbf{mod}\text{-}A$ such that $Se \neq \{0\}$. In particular, Σ_1 is a set of representatives of the isomorphism types of all simple A-modules.
b) Let $\mathbf{mod}_e\text{-}A$ be the full subcategory of $\mathbf{mod}\text{-}A$ consisting of all A-modules all of whose constituents are isomorphic to an element of Σ_e. The natural embedding induces the fully faithful exact functor $I_e\colon \mathbf{mod}_e\text{-}A \rightarrow \mathbf{mod}\text{-}A$. Let

$$C_e^{\Sigma} := C_e \circ I_e\colon \mathbf{mod}_e\text{-}A \rightarrow \mathbf{mod}\text{-}eAe.$$

c) For $V \in \mathbf{mod}\text{-}A$ let $\mathcal{P}(V) \xrightarrow{\rho} V$ denote its projective cover, and let $\Omega(V) := \ker \rho \in \mathbf{mod}\text{-}A$ be the *Heller module* of V. Let $\mathbf{mod}_{\Omega,e}\text{-}A$ be the full subcategory of $\mathbf{mod}\text{-}A$ consisting of all A-modules V such that both $V/\mathrm{rad}(V) \in$

$\mathbf{mod}_e$-A and $\Omega(V)/\mathrm{rad}(\Omega(V)) \in \mathbf{mod}_e$-$A$. The natural embedding induces the fully faithful exact functor $I_{\Omega,e}\colon \mathbf{mod}_{\Omega,e}$-$A \to \mathbf{mod}$-$A$. Let

$$C_e^{\Omega} := C_e \circ I_{\Omega,e}\colon \mathbf{mod}_{\Omega,e}\text{-}A \to \mathbf{mod}\text{-}eAe.$$

(6.9) Remark. Let Θ be a field and let $e \in A$ be an idempotent.
a) By Proposition (6.7), the set $\{Se; S \in \Sigma_e\} \subseteq \mathbf{mod}$-$eAe$ is a set of representatives of the isomorphism types of all simple eAe-modules.
b) If $\Sigma_e = \Sigma_1$, then the projective A-module $eA \in \mathbf{mod}$-A is a progenerator of $\mathbf{mod}$-A. Hence in this case, by [15, Thm.0.3.54], C_e induces an equivalence between $\mathbf{mod}$-A and $\mathbf{mod}$-eAe. Thus C_e is fully faithful and essentially surjective.

We discuss properties of the condensation functor C_e in the general case, where we do not assume that C_e induces an equivalence. Proposition (6.10) shows that C_e^{Σ} is a suitable functor to examine the submodule structure of A-modules. Proposition (6.11) and Example (6.14) show that C_e^{Σ} is fully faithful, but in general is not essentially surjective. Proposition (6.15) then shows how this failure to be an equivalence can be remedied by using the functor C_e^{Ω}.

(6.10) Proposition. Let Θ be a field, $e \in A$ be an idempotent and let $V \in \mathbf{mod}_e$-A. Then C_e^{Σ} induces a lattice isomorphism between the submodule lattices of V and $C_e^{\Sigma}(V)$.

Proof. Clearly C_e^{Σ} preserves inclusion of submodules and commutes with forming sums and intersections of submodules. Hence C_e^{Σ} induces a lattice homomorphism from the submodule lattice of V to the submodule lattice of $C_e^{\Sigma}(V)$. Since $V \in \mathbf{mod}_e$-A this homomorphism is injective. It remains to prove that it is also surjective.

Let $\alpha\colon W \to Ve$ be an injective homomorphism of eAe-modules. Applying C_e to $\mathrm{Hom}_A(U_e(W), V)$ and using the equivalences of Remark (6.9) yields a Θ-linear map

$$(C_e)_{U_e(W),V}\colon \begin{cases} \mathrm{Hom}_A(W \otimes_{eAe} eA, V) & \to \quad \mathrm{Hom}_{eAe}(W, \mathrm{Hom}_A(eA, V))\colon \\ \qquad\qquad\qquad \beta & \mapsto \quad \big(w \mapsto (ea \mapsto (w \otimes e)^{\beta} \cdot a)\big)\,. \end{cases}$$

This coincides with the adjointness Θ-homomorphism given by [15, Thm.0.2.19], and hence is a Θ-isomorphism. Let $\beta := (C_e)^{-1}_{U_e(W),V}(\alpha) \in \mathrm{Hom}_A(U_e(W), V)$. Then we have $U_e(W)\beta \leq V$ and $C_e(U_e(W)\beta) = (C_e \circ U_e(W))\alpha = W\alpha$. ♯

(6.11) Proposition. Let Θ be a field and let $e \in A$ be an idempotent. Then the functor $C_e^{\Sigma}\colon \mathbf{mod}_e$-$A \to \mathbf{mod}$-$eAe$ is fully faithful.

Proof. If $\Sigma_e = \Sigma_1$, then we have $C_e^\Sigma = C_e$, and by Remark (6.9) the functor C_e is an equivalence of categories, in particular C_e is fully faithful. Hence we may assume $\Sigma_e \neq \Sigma_1$. Let $e' \in A$ be an idempotent orthogonal to e, such that $Se' \neq \{0\}$ if and only if $S \in \textbf{mod}\text{-}A$ is a simple A-module isomorphic to an element of $\Sigma_1 \setminus \Sigma_e$, and let $f := e + e' \in A$. Hence $\Sigma_f = \Sigma_1$ and thus the functor $C_f \colon \textbf{mod}\text{-}A \to \textbf{mod}\text{-}fAf$ is an equivalence of categories, in particular C_f is fully faithful. Note that, since there might be a simple A-module $S \in \textbf{mod}\text{-}A$ isomorphic to an element of Σ_e such that $S(1-e) \neq \{0\}$, in general we cannot simply let $f = 1 \in A$.

We have the Pierce decomposition $fAf = eAe \oplus eAe' \oplus e'Ae \oplus e'Ae'$ of fAf as a Θ-vector space. Hence, for $V \in \textbf{mod}\text{-}eAe$ and $v \in V$ as well as $a \in A$, let $v \cdot eae' = v \cdot e'ae = v \cdot e'ae := 0$. It is straightforward to check that this defines an fAf-module structure on V. Thus we obtain a functor $I_e^f \colon \textbf{mod}\text{-}eAe \to \textbf{mod}\text{-}fAf$. As, for $V, W \in \textbf{mod}\text{-}eAe$, we have $\mathrm{Hom}_{fAf}(I_e^f(V), I_e^f(W)) = \mathrm{Hom}_{eAe}(V, W)$, the functor I_e^f is fully faithful. By the choice of $e' \in A$ we furthermore conclude $I_e^f \circ C_e \circ I_e = C_f \circ I_e$ as functors from $\textbf{mod}_e\text{-}A$ to $\textbf{mod}\text{-}fAf$. As both I_e and I_e^f, as well as C_f, are fully faithful, the assertion follows. ♯

(6.12) Corollary. Let Θ be a field and let $e \in A$ be an idempotent.
a) For $V \in \textbf{mod}_e\text{-}A$ we then have $\mathrm{End}_A(V) \cong \mathrm{End}_{eAe}(Ve)$.
b) In particular, if $S \in \textbf{mod}_e\text{-}A$ is a simple A-module, then S is absolutely simple if and only if $Se \in \textbf{mod}\text{-}eAe$ is.

(6.13) Remark. Let Θ be a field and let $e \in A$ be an idempotent.
a) Let $V \in \textbf{mod}_e\text{-}A$ and let $\mathcal{C} \subseteq eAe$ be a Θ-subalgebra. Then we have $\mathrm{End}_{eAe}(Ve) \subseteq \mathrm{End}_{\mathcal{C}}(Ve)$, and by Corollary (6.12) we have equality if and only if $\dim_\Theta \mathrm{End}_{\mathcal{C}}(Ve) = \dim_\Theta \mathrm{End}_A(V)$
b) The functor $C_e^\Sigma \colon \textbf{mod}_e\text{-}A \to \textbf{mod}\text{-}eAe$ is not necessarily essentially surjective, hence not necessarily an equivalence of categories, as the following example shows.

(6.14) Example. Let Θ be a field of characteristic 2, let $G := \mathcal{A}_5$ be the alternating group on 5 letters, and $A := \Theta G$, where we assume Θ to be a splitting field for A. The 2-modular Brauer characters of G can be found in [37]. Let $H \leq G$ be a cyclic subgroup of order 5, let $\lambda = 1$ be the trivial representation of ΘH and $\epsilon = \epsilon_1 \in \Theta H \subseteq A$, where the notation is as in Section (2.1).

As $\epsilon A \cong 1_H^G$ as A-modules, we have $\mathrm{Hom}(\epsilon A, S_1) \neq \{0\}$, where S_1 denotes the trivial A-module. Furthermore, ϵA is a projective A-module, and since $\dim_\Theta(\mathcal{P}(S_1)) = 12$, where $\mathcal{P}(S_1)$ denotes the projective cover of S_1, we conclude $\epsilon A \cong \mathcal{P}(S_1)$ as A-modules, hence $\epsilon \in A$ is a primitive idempotent, and thus $\mathrm{Hom}_A(\epsilon A, S) = \{0\}$ for all simple A-modules $S \not\cong S_1$. Hence we have $\Sigma_\epsilon = \{S_1\}$ and $S_1\epsilon$ is the only simple $\epsilon A\epsilon$-module, up to isomorphism.

As ϵA is a non-simple, projective indecomposable module for the symmetric algebra A, its endomorphism ring $\mathrm{End}_A(\epsilon A) \cong (\epsilon A\epsilon)^\circ$ as Θ-algebras, see Section

(2.1), is a local Θ-algebra containing non-zero nilpotent elements. Hence $\epsilon A\epsilon$ is not semisimple and in particular we have $\mathrm{Ext}^1_{\epsilon A\epsilon}(S_1\epsilon, S_1\epsilon) \neq \{0\}$. As G is a perfect group, we have $\mathrm{Ext}^1_A(S_1, S_1) = \{0\}$. Hence all modules in $\mathbf{mod}_\epsilon$-A are semisimple. Thus C^Σ_ϵ is not essentially surjective.

(6.15) Proposition. See also [1, Prop.II.2.5].
Let Θ be a field and $e \in A$ be an idempotent. Then the functor $C^\Omega_e\colon \mathbf{mod}_{\Omega,e}\text{-}A \to \mathbf{mod}\text{-}eAe$ is an equivalence of categories.

Proof. Let $V \in \mathbf{mod}\text{-}eAe$ and $S \in \Sigma_1$. By the Adjointness Theorem [15, Thm.0.2.19] we have $\mathrm{Hom}_A(U_e(V), S) \cong \mathrm{Hom}_{eAe}(V, \mathrm{Hom}_A(eA, S))$ as Θ-vector spaces. As $\mathrm{Hom}_A(eA, S) = \{0\}$ if $S \notin \Sigma_e$, we have $U_e(V)/\mathrm{rad}(U_e(V)) \in \mathbf{mod}_e\text{-}A$. By [3, Cor.2.5.4] we have $\mathrm{Hom}_A(\Omega(U_e(V)), S) \cong \mathrm{Ext}^1_A(V \otimes_{eAe} eA, S)$ as Θ-vector spaces. If $P \in \mathbf{mod}\text{-}eAe$ is a projective eAe-module, and hence a direct summand of a free eAe-module, then $P \otimes_{eAe} eA \in \mathbf{mod}\text{-}A$ is a projective A-module. Thus by the Eckmann-Shapiro Lemma [3, Cor.2.8.4] we conclude $\mathrm{Ext}^1_A(V \otimes_{eAe} eA, S) \cong \mathrm{Ext}^1_{eAe}(V, \mathrm{Hom}_A(eA, S))$ as Θ-vector spaces. Hence we also have $\Omega(U_e(V))/\mathrm{rad}(\Omega(U_e(V))) \in \mathbf{mod}_e\text{-}A$.

Thus U_e restricts to a functor $U_e\colon \mathbf{mod}\text{-}eAe \to \mathbf{mod}_{\Omega,e}\text{-}A$. By Remark (6.5) $C^\Omega_e \circ U_e$ is equivalent to the identity functor on $\mathbf{mod}\text{-}eAe$. Conversely, for $V \in \mathbf{mod}_{\Omega,e}\text{-}A$ we have $U_e \circ C_e(V) \cong \mathrm{Hom}_A(eA, V) \otimes_{\mathrm{End}_A(eA)^\circ} eA \in \mathbf{mod}_{\Omega,e}\text{-}A$. Hence it is sufficient to show that the natural evaluation map

$$\nu\colon \mathrm{Hom}_A(eA, V) \otimes_{\mathrm{End}_A(eA)^\circ} eA \to V\colon \alpha \otimes ea \mapsto (ea)\alpha$$

is an isomorphism of A-modules.

Assume that ν is not surjective. Then there is $S \in \Sigma_e$ and $0 \neq \beta \in \mathrm{Hom}_A(V, S)$ such that $\mathrm{im}\,\nu \leq \ker\beta \leq V$. As β is surjective, $eA \in \mathbf{mod}\text{-}A$ is a projective A-module, and $\mathrm{Hom}_A(eA, S) \neq \{0\}$, there is $\alpha \in \mathrm{Hom}_A(eA, V)$ such that $\alpha\beta \neq 0$. Hence $\mathrm{im}\,\alpha \not\leq \ker\beta \leq V$, which is a contradiction. Hence ν is surjective, and we thus have an exact sequence

$$\{0\} \to \ker\nu \to \mathrm{Hom}_A(eA, V) \otimes_{\mathrm{End}_A(eA)^\circ} eA \xrightarrow{\nu} V \to \{0\}$$

of A-modules. Since $C_e \circ U_e$ is equivalent to the identity functor on $\mathbf{mod}\text{-}eAe$, applying C_e yields the exact sequence $\{0\} \to (\ker\nu)e \to Ve \xrightarrow{\mathrm{id}} Ve \to \{0\}$ of eAe-modules. Hence we conclude $(\ker\nu)e = \{0\}$.

As ν is surjective, the projective cover $\mathcal{P}(V) \xrightarrow{\rho} V$ yields the existence of $\mu \in \mathrm{Hom}_A(\mathcal{P}(V), \mathrm{Hom}_A(eA, V) \otimes_{\mathrm{End}_A(eA)^\circ} eA)$ such that $\mu\nu = \rho$. As $(\Omega(V)\mu)\nu = (\ker\rho)\mu\nu = \{0\}$, there is $\kappa \in \mathrm{Hom}_A(\Omega(V), \ker\nu)$ such that $\Omega(V)\mu = \Omega(V)\kappa \leq \ker\nu$. From $(\ker\nu)e = \{0\}$ and $\Omega(V)/\mathrm{rad}(\Omega(V)) \in \mathbf{mod}_e\text{-}A$ we conclude that $\Omega(V)\mu = \{0\}$. Hence there is $\upsilon \in \mathrm{Hom}_A(V, \mathrm{Hom}_A(eA, V) \otimes_{\mathrm{End}_A(eA)^\circ} eA)$ such that $\rho\upsilon = \mu$. Thus we have $\rho\upsilon\nu = \rho$. As ρ is surjective, we conclude $\upsilon\nu = \mathrm{id}_V$. Hence $\ker\nu$ is a direct summand of $\mathrm{Hom}_A(eA, V) \otimes_{\mathrm{End}_A(eA)^\circ} eA \in \mathbf{mod}_{\Omega,e}\text{-}A$, and hence $\ker\nu/\mathrm{rad}(\ker\nu) \in \mathbf{mod}_e\text{-}A$. As $(\ker\nu)e = \{0\}$ we conclude $\ker\nu = \{0\}$, and thus ν is injective as well. $\sharp$

(6.16) Remark. Let $V \in \mathbf{mod}\text{-}A$ and $e \in A$ be an idempotent. The natural evaluation map $\nu\colon \mathrm{Hom}_A(eA, V) \otimes_{eAe} eA \to V$ used in the proof of Proposition (6.15) is the preimage of $\mathrm{id}_{\mathrm{Hom}_A(eA,V)}$ under the adjointness Θ-isomorphism, see [15, Thm.0.2.19],

$$\mathrm{Hom}_A(\mathrm{Hom}_A(eA, V) \otimes_{eAe} eA, V) \cong \mathrm{Hom}_{eAe}(\mathrm{Hom}_A(eA, V), \mathrm{Hom}_A(eA, V)).$$

This leads to the definition of relative uncondensation functors, which are of practical importance, see Section (6.22).

(6.17) Definition. Let $V \in \mathbf{mod}\text{-}A$ and $e \in A$ be an idempotent. Let $\alpha\colon W \to Ve$ be an injective homomorphism of eAe-modules. Then we have a homomorphism of A-modules

$$(\alpha \otimes \mathrm{id}) \cdot \nu\colon\ W \otimes_{eAe} eA \xrightarrow{\alpha\otimes\mathrm{id}} Ve \otimes_{eAe} eA \xrightarrow{\nu} V,$$

where $\nu\colon \mathrm{Hom}_A(eA, V) \otimes_{eAe} eA \to V$ is the natural evaluation map as in Remark (6.16). The A-module $\mathrm{im}\,((\alpha \otimes \mathrm{id}) \cdot \nu) \leq V$ is called the *uncondensed module of W relative to α and V.*

(6.18) We consider the question how condensation functors relate to modular reduction.

Let K be an algebraic number field, and let $R \subset K$ be a discrete valuation ring in K with maximal ideal $\wp \lhd R$ and finite residue class field $F := R/\wp$ of characteristic $p > 0$. Let $\tilde{\ }\colon R \to F$ denote the natural epimorphism.

Let A be an R-algebra, which is a finitely generated R-free R-module, let $A_K := A \otimes_R K$ and $A_F := A \otimes_R F$, and let $\tilde{\ }\colon A \to A_F$ denote the natural epimorphism. Let $e \in A \subseteq A_K$ be an idempotent. We have the Pierce decomposition of R-modules $A = eAe \oplus (1-e)Ae \oplus eA(1-e) \oplus (1-e)A(1-e)$. As A is an R-free R-module, this also holds for $eAe \leq A$, and we have $eAe \otimes_R K \cong eA_Ke$ as K-algebras and $eAe \otimes_R F \cong \tilde{e}A_F\tilde{e}$ as F-algebras.

If $V \xrightarrow{\alpha} W \xrightarrow{\beta} U$ is an exact sequence in $\mathbf{mod}_R\text{-}A$, see Definition (6.3), then it follows from the proof of Proposition (6.2) that the induced sequence of eA_Ke-modules $V \otimes_R K \xrightarrow{\alpha\otimes\mathrm{id}} W \otimes_R K \xrightarrow{\beta\otimes\mathrm{id}} U \otimes_R K$ is an exact sequence in $\mathbf{mod}\text{-}eA_Ke$. Note that this does not necessarily hold for the induced sequence of $\tilde{e}A_F\tilde{e}$-modules $V \otimes_R F \xrightarrow{\alpha\otimes\mathrm{id}} W \otimes_R F \xrightarrow{\beta\otimes\mathrm{id}} U \otimes_R F$ in $\mathbf{mod}\text{-}\tilde{e}A_F\tilde{e}$.

As in the group algebra case, see [14, Ch.XII.82-83], which straightforwardly generalises to the general case considered here, we define *decomposition maps* $D\colon G(A_K) \to G(A_F)$ and $D_e\colon G(eA_Ke) \to G(\tilde{e}A_F\tilde{e})$, where $G(\cdot)$ denotes the corresponding Grothendieck groups, as follows. Let $S \in \mathbf{mod}\text{-}A_K$ is a simple A_K-module, and let $\hat{S} \in \mathbf{mod}_R\text{-}A$, such that $\hat{S} \otimes_R K \cong S$ as A_K-modules. Let $T \in \mathbf{mod}\text{-}A_F$ be a simple A_K-module. Then the *decomposition number* $d_{S,T} \in \mathbb{N}_0$ is defined as the multiplicity of the constituent T in an A_F-module composition series of $\widetilde{\hat{S}} := \hat{S} \otimes_R F \in \mathbf{mod}\text{-}A_F$. The decomposition numbers $d^e_{S,T} \in \mathbb{N}_0$ for simple modules $S \in \mathbf{mod}\text{-}eA_Ke$ and $T \in \mathbf{mod}\text{-}\tilde{e}A_F\tilde{e}$ are defined analogously.

(6.19) Proposition. Let A be as in Section (6.18) and let $e \in A \subseteq A_K$ be an idempotent.
a) The additive functors $\mathrm{Hom}_A(eA, ?) \otimes_R K$ and $\mathrm{Hom}_{A_K}(eA_K, ? \otimes_R K)$ from $\mathbf{mod}_R\text{-}A$ to $\mathbf{mod}\text{-}eA_Ke$ are equivalent.
b) The additive functors $\mathrm{Hom}_A(eA, ?) \otimes_R F$ and $\mathrm{Hom}_{A_F}(\tilde{e}A_F, ? \otimes_R F)$ from $\mathbf{mod}_R\text{-}A$ to $\mathbf{mod}\text{-}\tilde{e}A_F\tilde{e}$ are equivalent.

Proof. As A is an R-free R-module, this also holds for $eA \leq A$. For $V \in \mathbf{mod}_R\text{-}A$, hence $\mathrm{Hom}_A(eA, V) \leq \mathrm{Hom}_R(eA, V)$ also is an R-free R-module. From that the assertions follow. ♯

(6.20) Proposition. Let A be as in Section (6.18) and let $e \in A \subseteq A_K$ be an idempotent. Let $S \in \mathbf{mod}\text{-}A_K$ be a simple A_K-module and $T \in \mathbf{mod}\text{-}A_F$ be a simple A_F-module, such that $\{0\} \neq T\tilde{e} \in \mathbf{mod}\text{-}\tilde{e}A_F\tilde{e}$. Then we have

$$d_{S,T} = d^e_{Se,T\tilde{e}}.$$

In particular, if $Se = \{0\}$ then we have $d_{ST} = 0$.

Proof. Let $\hat{S} \in \mathbf{mod}_R\text{-}A$ such that $\hat{S} \otimes_R K \cong S$ as A_K-modules. By Proposition (6.19), for $\hat{S}e \in \mathbf{mod}_R\text{-}eAe$ we hence have $\hat{S}e \otimes_R K \cong Se$ as eA_Ke-modules. Thus the decomposition number $d^e_{Se,T\tilde{e}} \in \mathbb{N}_0$ is the multiplicity of the constituent $T\tilde{e}$ in an $\tilde{e}A_F\tilde{e}$-module composition series of $\widetilde{\hat{S}e} \in \mathbf{mod}\text{-}\tilde{e}A_F\tilde{e}$. By Proposition (6.19) we have $\widetilde{\hat{S}e} \cong \widetilde{\hat{S}}\tilde{e}$ as $\tilde{e}A_F\tilde{e}$-modules. As $C_{\tilde{e}} \colon \mathbf{mod}\text{-}A_F \to \mathbf{mod}\text{-}\tilde{e}A_F\tilde{e}$ is an exact functor, by Proposition (6.7) we conclude that the multiplicity of the constituent $T\tilde{e}$ in an $\tilde{e}A_F\tilde{e}$-module composition series of $\widetilde{\hat{S}}\tilde{e}$ equals the multiplicity of the constituent T in an A_F-module composition series of $\widetilde{\hat{S}} \in \mathbf{mod}\text{-}A_F$, where the latter by definition is the decomposition number $d_{S,T} \in \mathbb{N}_0$. ♯

(6.21) Remark. The statements of Proposition (2.11) are a special case of those of Proposition (6.20).

To see this let K, R and F, as well as λ be as in Section (2.10), where in particular the characteristic of F is coprime to $|H|$, and let $A := RG$. Then we have $\epsilon_\lambda \in RG$, and $\epsilon_\lambda KG\epsilon_\lambda \cong (E^\lambda_K)^\circ$ as K-algebras, as well as $\epsilon_\lambda RG\epsilon_\lambda \cong (E^\lambda_R)^\circ$ as R-algebras, and $\epsilon_{\tilde{\lambda}} FG\widetilde{\epsilon_{\tilde{\lambda}}} \cong (E^{\tilde{\lambda}}_F)^\circ$ as F-algebras, see Proposition (2.2). For $\chi \in \mathrm{Irr}_K(G)$ let $S_\chi \in \mathbf{mod}\text{-}KG$ denote the simple KG-module affording χ, see Section (2.8). Then $S_\chi \cdot \epsilon_\lambda \cong \mathrm{Hom}_{KG}(\epsilon_\lambda KG, S_\chi) \neq \{0\}$ as K-vectors spaces, if and only if $\chi \in \mathrm{Irr}^\lambda_K(G)$.

Let $\chi_\varphi \in \mathrm{Irr}_K^\lambda(G)$ denote the Fitting correspondent of $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, see Proposition (2.7). Hence we have $S_{\chi_\varphi} \cong \epsilon_\lambda KG \cdot e_\varphi \leq \epsilon_\lambda KG$ as KG-modules, where $e_\varphi \in E_K^\lambda$ is an idempotent as in Section (2.6). Thus we have $S_{\chi_\varphi}\epsilon_\lambda \cong \epsilon_\lambda KGe_\varphi \cdot \epsilon_\lambda = \epsilon_\lambda e_\varphi \cdot \epsilon_\lambda KG\epsilon_\lambda$ as $\epsilon_\lambda KG\epsilon_\lambda$-modules, where $\epsilon_\lambda e_\varphi \in \epsilon_\lambda KG\epsilon_\lambda$ is an idempotent. By Proposition (2.2) the latter $\epsilon_\lambda KG\epsilon_\lambda$-module can be identified with the $(E_K^\lambda)^\circ$-module $e_\varphi(E_K^\lambda)^\circ = E_K^\lambda e_\varphi$. Let $S_\varphi \in \textbf{mod}\text{-}E_K^\lambda$ denote the simple E_K^λ-module affording φ, and let $S_\varphi^* := \mathrm{Hom}_K(S_\varphi, K)$ be the $(E_K^\lambda)^\circ$-module dual to the E_K^λ-module S_φ. As E_K^λ is a symmetric K-algebra, we have $E_K^\lambda e_\varphi \cong (e_\varphi E_K^\lambda)^* \cong S_\varphi^*$ as $(E_K^\lambda)^\circ$-modules.

Similarly, for $\varphi \in \mathrm{Irr}_F(E_F^{\tilde\lambda})$ we have $P_\varphi \cong \epsilon_{\tilde\lambda} FG \cdot e_\varphi$ as FG-modules, and hence analogously the $\epsilon_{\tilde\lambda}FG\epsilon_{\tilde\lambda}$-module $P_\varphi \cdot \epsilon_{\tilde\lambda}$ can be identified with the $(E_F^{\tilde\lambda})^\circ$-module $E_F^{\tilde\lambda} e_\varphi \cong (e_\varphi E_F^{\tilde\lambda})^*$. Let $\chi_\varphi \in \mathrm{Irr}_F^{\tilde\lambda}(G)$ denote the Fitting correspondent of φ and let $S_{\chi_\varphi} := P_\varphi/\mathrm{rad}(P_\varphi) \in \textbf{mod}\text{-}FG$ be the simple FG-module affording χ_φ. As P_φ is an FG-direct summand of $\epsilon_{\tilde\lambda}FG$, we have $\{0\} \neq \mathrm{Hom}_{FG}(\epsilon_{\tilde\lambda}FG, S_{\chi_\varphi}) \cong S_{\chi_\varphi} \cdot \epsilon_{\tilde\lambda} \in \textbf{mod}\text{-}\epsilon_{\tilde\lambda}FG\epsilon_{\tilde\lambda}$. As $E_F^{\tilde\lambda}$ is a symmetric F-algebra, the latter $\epsilon_{\tilde\lambda}FG\epsilon_{\tilde\lambda}$-module can be identified with the $(E_F^{\tilde\lambda})^\circ$-module $S_\varphi^* \cong (e_\varphi E_F^{\tilde\lambda})^*/\mathrm{rad}((e_\varphi E_F^{\tilde\lambda})^*)$.

We have a decomposition map $D_{E^\circ}\colon G((E_K^\lambda)^\circ) \to G((E_F^{\tilde\lambda})^\circ)$, where the corresponding decomposition numbers are denoted by $d_{\cdot,\cdot}^{E^\circ} \in \mathbb{N}_0$. Let $S \in \textbf{mod}\text{-}E_K^\lambda$ be a simple E_K^λ-module and let $T \in \textbf{mod}\text{-}E_F^{\tilde\lambda}$ be a simple $E_F^{\tilde\lambda}$-module. Thus $S^* \in \textbf{mod}\text{-}(E_K^\lambda)^\circ$ and $T^* \in \textbf{mod}\text{-}(E_F^{\tilde\lambda})^\circ$ are simple modules and we have $d_{S^*,T^*}^{E^\circ} = d_{S,T}^E$.

(6.22) We conclude Section 6 with a few general remarks on computational applications of condensation functors; for more specific applications see Section 9.

Relative uncondensation functors, see Definition (6.17), have been used heavily as a constructive tool; for example to construct irreducible representations of the larger sporadic simple groups over finite fields using the MeatAxe, see [59, 78]. Condensation functors inducing equivalences between **mod**-A and **mod**-eAe, where A is a group algebra over a finite field have been studied in [46].

The other extreme, where $e \in A$ is a primitive idempotent, has been used in [47] to give an algorithm to compute submodule lattices. An implementation for algebras over finite fields is available in the MeatAxe, of which we make heavy use in the analysis of the examples dealt with in Part III. Other applications of condensation functors with respect to primitive idempotents are the computation of socle series [49] and the computation of endomorphism rings [74] of modules. Implementations for algebras over finite fields are available in the MeatAxe as well, these are also used in Part III.

7 Orbital graphs

In Section 7 we give an application using the information collected in the database, see Section (11.1). We begin by fixing the notation and giving the necessary definitions. We assume the reader familiar with the basic notions of graph theory, as a general reference see [4, 24].

(7.1) Definition.
a) A *(simple non-directed) graph* $\mathfrak{G}$ is a tuple $(\mathfrak{V}, \mathfrak{E}, \iota)$, where $\mathfrak{V} = \{v_1, \ldots, v_n\}$ is a finite set of *vertices* and $\mathfrak{E} = \{e_1, \ldots, e_m\}$ is a finite set of *edges*, as well as $\iota\colon \mathfrak{E} \to \{\{v, w\}; v, w \in \mathfrak{V}, v \neq w\}$ is an injective *incidence map*. If $\iota(e) = \{v, w\}$, for $e \in \mathfrak{E}$, then the edge $e \in \mathfrak{E}$ and the vertices $v, w \in \mathfrak{V}$ are called *incident*. If for $v, w \in \mathfrak{V}$ there is an $e \in \mathfrak{E}$ such that $\iota(e) = \{v, w\}$, then the vertices $v, w \in \mathfrak{V}$ are called *adjacent*, denoted by $v \sim_{\mathfrak{G}} w$. A pair $(v, e) \in \mathfrak{V} \times \mathfrak{E}$ such that $v \in \iota(e)$ is called a *flag* of $\mathfrak{G}$.

The number of vertices adjacent to a vertex $v \in \mathfrak{V}$ is called the *valency* of v. If all vertices of $\mathfrak{G}$ have the same valency, then $\mathfrak{G}$ is called *regular*. If the vertex set $\mathfrak{V}$ can be partitioned into $\mathfrak{V} = \mathfrak{V}_1 \,\dot\cup\, \mathfrak{V}_2$ such that $|\iota(e) \cap \mathfrak{V}_i| = 1$ for $i \in \{1, 2\}$ and for all $e \in \mathfrak{E}$, then $\mathfrak{G}$ is called *bipartite*.

b) A *path of length* d in $\mathfrak{G}$, for $d \in \mathbb{N}_0$, is a sequence $\{v_0, \ldots, v_d\} \subseteq \mathfrak{V}$ of vertices such that $v_{i-1} \sim_{\mathfrak{G}} v_i$, for $i \in \{1, \ldots, d\}$. The *distance* $d(v, w) = d_{\mathfrak{G}}(v, w) \in \mathbb{N}_0 \cup \{\infty\}$ of $v, w \in \mathfrak{V}$ in $\mathfrak{G}$ is the minimum length of a path such that $v_0 = v$ and $v_d = w$, if such a path exists, and $d(v, w) = d_{\mathfrak{G}}(v, w) = \infty$ otherwise. The *diameter* $d(\mathfrak{G}) \in \mathbb{N}_0 \cup \{\infty\}$ of $\mathfrak{G}$ is the maximum distance $d(v, w)$ of vertices $v, w \in \mathfrak{V}$. If $d(\mathfrak{G}) < \infty$, then $\mathfrak{G}$ is called *connected*. The largest connected subgraph of $\mathfrak{G}$ having $v \in \mathfrak{V}$ as one of its vertices is called the *connected component* of v.

For $d \in \mathbb{N}_0$ and $v \in \mathfrak{V}$ let the *distance sets* $\mathfrak{G}_d(v) := \{w \in \mathfrak{V}; d(v, w) = d\} \subseteq \mathfrak{V}$ and $\mathfrak{G}_{\leq d}(v) := \{w \in \mathfrak{V}; d(v, w) \leq d\} \subseteq \mathfrak{V}$. For $d \in \mathbb{N}_0$ the d-th *distance graph* $\mathfrak{G}_d$ of $\mathfrak{G}$ is defined by having vertex set $\mathfrak{V}$, and vertices $v, w \in \mathfrak{V}$ being adjacent if $w \in \mathfrak{G}_d(v)$.

c) A connected graph $\mathfrak{G}$ is called *distance-transitive*, if the group $\mathrm{Aut}(\mathfrak{G})$ of graph automorphisms of $\mathfrak{G}$ acts transitively on the the distance sets $\mathfrak{G}_d(v)$, for all $v \in \mathfrak{V}$ and $d \in \{0, \ldots, d(\mathfrak{G})\}$.

A regular connected graph $\mathfrak{G}$ of valency $k \in \mathbb{N}$ is called *distance-regular*, if
i) for all $d \in \{1, \ldots, d(\mathfrak{G})\}$ as well as $v \in \mathfrak{V}$ and $u \in \mathfrak{G}_d(v)$ the cardinality $|\{w \in \mathfrak{G}_{d-1}(v); w \sim_{\mathfrak{G}} u\}|$ is independent of the particular choice of $v \in \mathfrak{V}$ and $u \in \mathfrak{G}_d(v)$, and
ii) for all $d \in \{0, \ldots, d(\mathfrak{G}) - 1\}$ as well as $v \in \mathfrak{V}$ and $u \in \mathfrak{G}_d(v)$ the cardinality $|\{w \in \mathfrak{G}_{d+1}(v); w \sim_{\mathfrak{G}} u\}|$ is independent of the particular choice of $v \in \mathfrak{V}$ and $u \in \mathfrak{G}_d(v)$.
If both of these conditions are fulfilled, then for $v \in \mathfrak{V}$ we let $k_{\mathfrak{G}_d} := |\mathfrak{G}_d(v)| \in \mathbb{N}$ denote the valency of $\mathfrak{G}_d$, for $d \in \{0, \ldots, d(\mathfrak{G})\}$; and for $u \in \mathfrak{G}_d(v)$ we let

$$c_d := |\{w \in \mathfrak{G}_{d-1}(v); w \sim_{\mathfrak{G}} u\}| \in \mathbb{N}_0 \text{ for } d \in \{1, \ldots, d(\mathfrak{G})\},$$

as well as

$$b_d := |\{w \in \mathfrak{G}_{d+1}(v); w \sim_{\mathfrak{G}} u\}| \in \mathbb{N}_0 \text{ for } d \in \{0, \ldots, d(\mathfrak{G}) - 1\}.$$

Hence we have $b_0 = k$ and $c_1 = 1$. The sequence $[k, b_1, \ldots, b_{d(\mathfrak{G})}; 1, c_2, \ldots, c_{d(\mathfrak{G})}]$ of non-negative integers is called the *intersection array* of $\mathfrak{G}$.

A distance-regular graph $\mathfrak{G}$ is called *primitive*, if all the distance graphs $\mathfrak{G}_d$, for $d \in \{0, \ldots, d(\mathfrak{G})\}$, are connected, otherwise it is called *imprimitive*. A distance-regular graph $\mathfrak{G}$ is called *antipodal* if $d(\mathfrak{G}) \geq 2$ and if the relation $\{(v, w) \in \mathfrak{V} \times \mathfrak{V}; d(v, w) \in \{0, d(\mathfrak{G})\}\}$ is an equivalence relation on $\mathfrak{V}$.

(7.2) Remark.
a) A distance-transitive graph $\mathfrak{G}$ is distance-regular, and the group $\mathrm{Aut}(\mathfrak{G})$ acts flag-transitively, hence in particular edge-transitively and vertex-transitively.
b) Let $\mathfrak{G}$ be a distance-regular graph. If $\mathfrak{G}$ is bipartite then the distance graph $\mathfrak{G}_2$ is not connected. If $\mathfrak{G}$ is antipodal then the distance graph $\mathfrak{G}_{d(\mathfrak{G})}$ is not connected. If $\mathfrak{G}$ is imprimitive of valency $k \geq 3$ then by [8, Thm.4.2.1] it is bipartite or antipodal or both.
c) Let $\mathfrak{G}$ be a distance-regular graph of diameter $d(\mathfrak{G}) \geq 3$. Then by [8, Prop.5.1.1] the sequence $[k_{\mathfrak{G}_0}, \ldots, k_{\mathfrak{G}_{d(\mathfrak{G})}}]$ of positive integers is *unimodal*, hence there are $i, j \in \{1, \ldots, d\}$ such that $i \leq j$ and

$$1 = k_{\mathfrak{G}_0} < k_{\mathfrak{G}_1} < \ldots < k_{\mathfrak{G}_i} = \ldots = k_{\mathfrak{G}_j} > \ldots > k_{\mathfrak{G}_{d(\mathfrak{G})}}.$$

Furthermore, if for some $d, e \in \{0, \ldots, d(\mathfrak{G})\}$ such that $d < e$ and $d + e \leq d(\mathfrak{G})$ we have $k_{\mathfrak{G}_d} = k_{\mathfrak{G}_e}$, then we also have $k_{\mathfrak{G}_{d+1}} = k_{\mathfrak{G}_{e-1}}$.

(7.3) Definition. Let $\mathfrak{G}$ be a graph.
a) The symmetric matrix $A_{\mathfrak{G}} := [a_{ij}; i, j \in \{1, \ldots, n\}] \in \mathbb{Z}^{n \times n}$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in \mathrm{im}\,(\iota), \\ 0, & \text{if } \{v_i, v_j\} \notin \mathrm{im}\,(\iota), \end{cases}$$

is called the *adjacency matrix* of $\mathfrak{G}$. As the matrix $A_{\mathfrak{G}}$ is diagonalisable over $\mathbb{R}$, let $\rho_1 > \ldots > \rho_s$ for some $s \in \mathbb{N}$ denote the pairwise different eigenvalues of $A_{\mathfrak{G}}$. The set of eigenvalues $\{\rho_1, \ldots, \rho_s\} \subseteq \mathbb{R}$ of $A_{\mathfrak{G}}$, together with their multiplicities, is called the *spectrum* of $\mathfrak{G}$.
b) If $\mathfrak{G}$ is a regular graph of valency $k \in \mathbb{N}$, then the number

$$\rho_{\mathfrak{G}} := \max\{|\rho_i| \in \mathbb{R}; i \in \{1, \ldots, s\}, |\rho_i| < k\} \in \mathbb{R}$$

is called the *graph spectral radius* of $\mathfrak{G}$. A connected regular graph $\mathfrak{G}$ of valency $k \in \mathbb{N}$ is called a *Ramanujan graph* if $\rho_{\mathfrak{G}} \leq 2 \cdot \sqrt{k-1}$.

(7.4) Remark.
a) If $\mathfrak{G}$ is a regular graph of valency $k \in \mathbb{N}$, then by [4, Prop.I.3.1] we have $|\rho_i| \leq k$, for all $i \in \{1, \ldots, s\}$, where $\rho_1 = k$, whose multiplicity equals the number of connected components of $\mathfrak{G}$. Furthermore, if $\mathfrak{G}$ is connected, then by [24, Thm.8.8.2] $\mathfrak{G}$ is a bipartite graph if and only if $\rho_s = -k$. In this case, $-\rho \in \mathbb{R}$ is an eigenvalue of $A_{\mathfrak{G}}$ whenever $\rho \in \mathbb{R}$ is.
b) The notion of Ramanujan graphs is related to the notion of *expander graphs*. For a discussion of these notions, in particular how groups come into play in some of the constructions, and further references, see for example [44, Ch.1,Ch.4.5] and [76, Ch.II.19].

The following definition introduces the graphs we deal with in the sequel, orbital graphs. We show how some of the properties of orbital graphs can be deduced from the data collected in the database, see Section (11.1). We keep the notation of Section (1.1), where in all of Section 7 we assume $\lambda = 1$ and K to be as in Section 3.

(7.5) Definition. Let $1 \neq i \in \mathcal{I}$, let $\alpha_i \in \mathcal{A}$ be the corresponding Schur basis element of $E_{\mathbb{Z}}$, and let $[\alpha_i] \in \mathbb{Z}^{n\times n}$ be the representing matrix of its action on $\mathbb{Z}\Omega$, with respect to the basis Ω, see Proposition (1.10).
a) If $i = i^*$ corresponds to a self-paired orbital, then the graph $\mathfrak{O}_i$ with vertex set Ω, defined by the adjacency matrix $[\alpha_i] \in \mathbb{Z}^{n\times n}$, is called the i-th *orbital graph* of Ω.
b) If $i \neq i^*$ corresponds to a not self-paired orbital, then the graph $\mathfrak{O}_i$ with vertex set Ω, defined by the adjacency matrix $[\alpha_i] + [\alpha_{i^*}] = [\alpha_i] + [\alpha_i]^T \in \mathbb{Z}^{n\times n}$, see Corollary (1.14), is called the i-th *orbital graph* of Ω. It coincides with the i^*-th orbital graph $\mathfrak{O}_{i^*}$ of Ω.

Let $\mathfrak{O}_i^0$ denote the connected component of $\mathfrak{O}_i$ containing the vertex $\omega_1 \in \Omega$.

(7.6) Remark.
a) Let $1 \neq i \in \mathcal{I}$. As G acts transitively on the i-th orbital $\mathcal{O}_i \subseteq \Omega\times\Omega$, the group G acts as a vertex-transitive and edge-transitive group of graph automorphisms on $\mathfrak{O}_i$. If $i = i^*$ then G acts as a flag-transitive group of graph automorphisms on $\mathfrak{O}_i$, while if $i \neq i^*$ then G does not act flag-transitively. The connected components of $\mathfrak{O}_i$ are all isomorphic to $\mathfrak{O}_i^0$ as graphs and permuted transitively by G.
b) As the vertices adjacent to ω_1 in $\mathfrak{O}_i$ are $(\mathfrak{O}_i)_1(\omega_1) = \Omega_i$, the orbital graph $\mathfrak{O}_i$ is a regular graph of valency k_i. As H acts transitively on the suborbits $\Omega_k \subseteq \Omega$, for $k \in \mathcal{I}$, the distance sets $(\mathfrak{O}_i)_d(\omega_1) \subseteq \mathfrak{O}_i^0$ and $(\mathfrak{O}_i)_{\leq d}(\omega_1) \subseteq \mathfrak{O}_i^0$, for $d \in \mathbb{N}_0$, are unions of suborbits.
c) For a subset $\mathcal{J} \subseteq \mathcal{I} \setminus \{1\}$, such that for $i \in \mathcal{J}$ we also have $i^* \in \mathcal{J}$, the graph $\mathfrak{O}_{\mathcal{J}}$ with vertex set Ω, defined by the adjacency matrix $\sum_{i\in\mathcal{J}}[\alpha_i] \in \mathbb{Z}^{n\times n}$, is called the *generalised orbital graph of* Ω *with respect to* $\mathcal{J}$. In particular, for $\mathcal{J} = \mathcal{I} \setminus \{1\}$, the generalised orbital graph $\mathfrak{O}_{\mathcal{I}\setminus\{1\}}$ is the complete graph

with vertex set Ω. Note that the group G does not act edge-transitively on a generalised orbital graph which is not an orbital graph.

(7.7) Proposition. Let $k \in \mathcal{I}$ and $d \in \mathbb{N}_0$.
a) Let $1 \neq i \in \mathcal{I}$ such that $i = i^*$. Then the matrix entry $[(P_i)^d]_{1,k} \in \mathbb{N}_0$ equals the number of paths of length d in $\mathfrak{O}_i$ connecting ω_1 and $\omega_k \in \Omega$. Letting $\mathcal{K}_{i,\leq -1} := \emptyset$ as well as

$$\mathcal{K}_{i,\leq d} := \{k \in \mathcal{I}; [(P_i)^s]_{1,k} > 0 \text{ for some } s \in \{0, \ldots, d\}\} \subseteq \mathcal{I},$$

and $\mathcal{K}_{i,d} := \mathcal{K}_{i,\leq d} \setminus \mathcal{K}_{i,\leq(d-1)} \subseteq \mathcal{I}$, we have $(\mathfrak{O}_i)_d(\omega_1) = \coprod_{k \in \mathcal{K}_{i,d}} \Omega_k$, and hence

$$d(\mathfrak{O}_i^0) = \min\{d \in \mathbb{N}; \mathcal{K}_{i,d} = \emptyset\} - 1.$$

In particular $\mathfrak{O}_i$ is connected if and only if $\mathcal{K}_{i,\leq d(\mathfrak{O}_i^0)} = \mathcal{I}$.
b) Let $j \in \mathcal{I}$ such that $j \neq j^*$. Then the matrix entry $[(P_j + P_{j^*})^d]_{1,k} \in \mathbb{N}_0$ equals the number of paths of length d in $\mathfrak{O}_j$ connecting ω_1 and $\omega_k \in \Omega$. Letting $\mathcal{K}^*_{j,\leq -1} := \emptyset$ as well as

$$\mathcal{K}^*_{j,\leq d} := \{k \in \mathcal{I}; [(P_j + P_{j^*})^s]_{1,k} > 0 \text{ for some } s \in \{0, \ldots, d\}\} \subseteq \mathcal{I},$$

and $\mathcal{K}^*_{j,d} := \mathcal{K}^*_{j,\leq d} \setminus \mathcal{K}^*_{j,\leq(d-1)} \subseteq \mathcal{I}$, we have $(\mathfrak{O}_j)_d(\omega_1) = \coprod_{k \in \mathcal{K}^*_{j,d}} \Omega_k$, and hence

$$d(\mathfrak{O}_j^0) = \min\{d \in \mathbb{N}_0; \mathcal{K}^*_{j,d} = \emptyset\} - 1.$$

In particular $\mathfrak{O}_j$ is connected if and only if $\mathcal{K}^*_{j,\leq d(\mathfrak{O}_j^0)} = \mathcal{I}$.

Proof. By Definition (1.18) we have $\alpha_i^d = \alpha_1 \cdot \alpha_i^d = \sum_{k \in \mathcal{I}} [(P_i)^d]_{1,k} \cdot \alpha_k \in E_K$. By [4, La.I.2.5] the matrix entry $[(\alpha_i)^d]_{\omega_1,\omega_k} \in \mathbb{N}_0$, for $k \in \mathcal{I}$, is the number of paths of length d connecting ω_1 and ω_k. By Remark (7.6) the distance set $(\mathfrak{O}_i)_{\leq d}(\omega_1)$ is a union of suborbits. From this the assertions in a) follow. The assertions in b) are proved analogously. ♯

(7.8) Proposition. Let $1 \neq i \in \mathcal{I}$ such that $i = i^*$, and such that the orbital graph $\mathfrak{O}_i = \mathfrak{O}_i^0$ is connected.
a) Then $\mathfrak{O}_i$ is distance-regular if and only if
i) for all $d \in \{1, \ldots, d(\mathfrak{O}_i)\}$ and $k \in \mathcal{K}_{i,d}$, see Proposition (7.7), the number $\sum_{l \in \mathcal{K}_{i,d-1}} [P_i]_{lk} \in \mathbb{N}_0$ is independent of the particular choice of $k \in \mathcal{K}_{i,d}$, and
ii) for all $d \in \{0, \ldots, d(\mathfrak{O}_i) - 1\}$ and $k \in \mathcal{K}_{i,d}$ the number $\sum_{l \in \mathcal{K}_{i,d+1}} [P_i]_{lk} \in \mathbb{N}_0$ is independent of the particular choice of $k \in \mathcal{K}_{i,d}$.
If both of these conditions are fulfilled, then the entries of the intersection array are given as

$$c_d = \sum_{l \in \mathcal{K}_{i,d-1}} [P_i]_{lk} \in \mathbb{N}_0 \text{ for } d \in \{1, \ldots, d(\mathfrak{O}_i)\} \text{ and } k \in \mathcal{K}_{i,d},$$

as well as

$$b_d := \sum_{l \in \mathcal{K}_{i,d+1}} [P_i]_{lk} \in \mathbb{N}_0 \text{ for } d \in \{0, \ldots, d(\mathfrak{O}_i) - 1\} \text{ and } k \in \mathcal{K}_{i,d},$$

while for $d \in \{0, \ldots, d(\mathfrak{O}_i)\}$ the valency of the distance graph $(\mathfrak{O}_i)_d$ is given as

$$k_{(\mathfrak{O}_i)_d} = \sum_{l \in \mathcal{K}_{i,d}} k_l,$$

where the $k_l = |\Omega_l|$, for $l \in \mathcal{I}$, are the index parameters of Ω.
b) The group G acts distance-transitively on the graph $\mathfrak{O}_i$ if and only if we have $|\mathcal{K}_{i,d}| = 1$ for all $d \in \{0, \ldots, d(\mathfrak{O}_i)\}$.
c) If $\mathfrak{O}_i$ is distance-regular, then it is primitive if and only if for all $d \in \{1, \ldots, d(\mathfrak{O}_i)\}$ the eigenvalue $\sum_{l \in \mathcal{K}_{i,d}} k_l \in \mathbb{Z}$ of the matrix $\sum_{l \in \mathcal{K}_{i,d}} [\alpha_l] \in \mathbb{Z}^{n \times n}$ has multiplicity 1.
d) If $\mathfrak{O}_i$ is distance-regular, then it is bipartite if and only if $-\sum_{l \in \mathcal{K}_{i,1}} k_l \in \mathbb{Z}$ is an eigenvalue of the matrix $\sum_{l \in \mathcal{K}_{i,1}} [\alpha_l] \in \mathbb{Z}^{n \times n}$.
e) If $\mathfrak{O}_i$ is distance-regular, then it is antipodal if and only if $d(\mathfrak{O}_i) \geq 2$ and for all $l, s \in \mathcal{K}_{i,d(\mathfrak{O}_i)}$ and $k \in \mathcal{I} \setminus (\mathcal{K}_{i,d(\mathfrak{O}_i)} \cup \mathcal{K}_{i,0}) = \mathcal{I} \setminus (\mathcal{K}_{i,d(\mathfrak{O}_i)} \cup \{1\})$ we have $p_{lsk} = 0$.

As in Proposition (7.7), similar statements hold for $j \in \mathcal{I}$ such that $j \neq j^*$ and such that $\mathfrak{O}_j = \mathfrak{O}_j^0$ is connected.

Proof. By Definitions (7.5) and (1.18) we have

$$[P_i]_{lk} = p_{lik} = p_{li^*k} = |\{\omega \in \Omega_l; \omega \sim_{\mathfrak{O}_i} \omega_k\}|.$$

Thus the assertion in a) follows from Definition (7.1) and the definition of the sets $\mathcal{K}_{i,d}$ in Proposition (7.7). The assertion in b) is clear. By definition of the sets $\mathcal{K}_{i,d}$ the matrix $\sum_{l \in \mathcal{K}_{i,d}} [\alpha_l] \in \mathbb{Z}^{n \times n}$ is the adjacency matrix of the distance graph $(\mathfrak{O}_i)_d$. Hence the assertions in c) and d) follow from Remark (7.4). Finally, let $A := \sum_{s \in \mathcal{K}_{i,d(\mathfrak{O}_i)}} [\alpha_s] \in \mathbb{Z}^{n \times n}$. Then the relation

$$\{(\omega, \omega') \in \Omega \times \Omega; \omega = \omega' \text{ or } \omega' \in (\mathfrak{O}_i)_{d(\mathfrak{O}_i)}\}$$

is an equivalence relation if and only if $A^2 \in \mathbb{Z}^{n \times n}$ is a $\mathbb{Z}$-linear combination of $[\alpha_1] = [\mathrm{id}_{\mathbb{Z}\Omega}] \in \mathbb{Z}^{n \times n}$ and A. Hence the assertion in e) follows from Definition (1.18) and the non-negativity of the structure constants. The statements for $j \neq j^*$ are proved analogously. ♯

(7.9) Proposition. See also [8, Prop.4.1.11].
Let $1 \neq i \in \mathcal{I}$ such that G acts distance-transitively on the orbital graph $\mathfrak{O}_i$.
a) We have $j = j^*$ for all $j \in \mathcal{I}$.
b) The endomorphism ring E_K is as a K-algebra generated by the Schur basis element $\alpha_i \in E_K$. In particular, E_K is a commutative ring.

Proof. By distance-transitivity we have $i = i^* \in \mathcal{I}$. By Propositions (7.7) and (7.8) we have $|\mathcal{K}_{i,d}| = 1$ for all $d \in \{0, \ldots, d(\mathfrak{O}_i)\}$ and $\mathcal{K}_{i,\leq d(\mathfrak{O}_i)} = \mathcal{I}$, hence all suborbits are self-paired. Furthermore, we have $d(\mathfrak{O}_i) = r - 1$, and from the proof of Proposition (7.7) we conclude that the minimum polynomial of the structure constants matrix P_i has degree at least r. Hence we have $\dim_K(\langle P_i \rangle_{K\text{-algebra}}) = r = \dim_K(E_K)$. ♯

(7.10) Let $1 \neq i \in \mathcal{I}$ such that $i = i^*$. Then the spectrum of the graph $\mathfrak{O}_i$ is the set of eigenvalues in $\mathbb{R}$ of $[\alpha_i] \in \mathbb{Z}^{n \times n}$, together with their multiplicities. Analogously, if $j \in \mathcal{I}$ such that $j \neq j^*$, then the spectrum of the graph $\mathfrak{O}_j$ is the set of eigenvalues in $\mathbb{R}$ of $[\alpha_j] + [\alpha_{j^*}] \in \mathbb{Z}^{n \times n}$, together with their multiplicities.

As the regular K-representation of E_K is a faithful representation, the eigenvalues of $[\alpha_i]$, for $1 \neq i \in \mathcal{I}$ such that $i = i^*$, are precisely the eigenvalues in $\mathbb{R}$ of the matrix $P_i \in \mathbb{Z}^{r \times r}$ representing the action of α_i on the regular module E_K, see Definition (1.18), where by Remark (1.19) the matrix P_i is diagonalisable over $\mathbb{R}$. Analogously, the eigenvalues of $[\alpha_j] + [\alpha_{j^*}]$, for $j \in \mathcal{I}$ such that $j \neq j^*$, are precisely the eigenvalues in $\mathbb{R}$ of $(P_j + P_{j^*}) \in \mathbb{Z}^{r \times r}$, where again by Remark (1.19) the matrix $(P_j + P_{j^*})$ is diagonalisable over $\mathbb{R}$. Furthermore, the eigenvalues of $\sum_{l \in \mathcal{K}_{i,d}} [\alpha_l] \in \mathbb{Z}^{n \times n}$ and $\sum_{l \in \mathcal{K}^*_{j,d}} [\alpha_l] \in \mathbb{Z}^{n \times n}$ for $d \in \mathbb{N}_0$ and $i, j \in \mathcal{I}$ as above, see Proposition (7.8), are precisely the eigenvalues in $\mathbb{R}$ of $\sum_{l \in \mathcal{K}_{i,d}} P_l \in \mathbb{Z}^{r \times r}$ and $\sum_{l \in \mathcal{K}^*_{j,d}} P_l \in \mathbb{Z}^{r \times r}$, respectively, where by Proposition (1.13) the sets $\mathcal{K}_{i,d}$ and $\mathcal{K}^*_{j,d}$ are invariant under $*: \mathcal{I} \to \mathcal{I}$, hence by Remark (1.19) the latter matrices are diagonalisable over $\mathbb{R}$.

To determine the eigenvalues of $P_i \in \mathbb{Z}^{r \times r}$ and their multiplicities as eigenvalues of $[\alpha_i] \in \mathbb{Z}^{n \times n}$, for $1 \neq i \in \mathcal{I}$ such that $i = i^*$, we proceed as follows. By first decomposing the regular K-representation of E_K as a direct sum of siple E_K-modules S_φ, for $\varphi \in \mathrm{Irr}_K(E_K)$, and subsequently diagonalising the action of α_i on the simple E_K-summands, each eigenvalue of α_i is attached to one or more of the $\varphi \in \mathrm{Irr}_K(E_K)$, see Section (2.8). Hence we are reduced to finding the eigenvalues of the action of α_i on the simple E_K-modules S_φ, for $\varphi \in \mathrm{Irr}_K(E_K)$. If E_K is commutative, then the eigenvalues of the action of α_i on S_φ are precisely the entries of the character table of E_K in the column corresponding to $i \in \mathcal{I}$. The multiplicity of an eigenvalue of $[\alpha_i] \in \mathbb{Z}^{n \times n}$ is the sum of the degrees of the Fitting correspondents of the $\varphi \in \mathrm{Irr}_K(E_K)$ attached to it. By Remark (3.9) these degrees can also be determined from the character table of E_K.

The sums $[\alpha_j] + [\alpha_{j^*}]$ and $\sum_{l \in \mathcal{K}_{i,d}} [\alpha_l]$ as well as $\sum_{l \in \mathcal{K}^*_{j,d}} [\alpha_l]$, for $d \in \mathbb{N}_0$ and $i, j \in \mathcal{I}$ as above, are dealt with analogously, using the sums of the columns corresponding to $\{j, j^*\}$ and $\mathcal{K}_{i,d}$ as well as $\mathcal{K}^*_{j,d}$, respectively.

We conclude Section 7 by presenting two classification results using the data collected in the database, being concerned with distance-regular orbital graphs, and Ramanujan orbital graphs, respectively.

(7.11) Using the data contained in the database, see Section (11.1), it is straightforward to implement the technique described in Propositions (7.7) and (7.8) and Section (7.10) into GAP. Hence for the sporadic simple groups, their automorphism groups and their Schur covering groups we obtain a classification of their distance-regular orbital graphs afforded by a multiplicity-free permutation action, up to the single exception $G = 2.B$ and $H = Fi_{23}$ not yet dealt with; as soon as the data for the bicyclic extensions of the sporadic simple groups is available, these cases can be dealt with as well, see Section (11.1).

By Proposition (7.9) this covers all distance-transitive graphs having one of the above-mentioned groups as a group of automorphisms. The primitive distance-transitive graphs amongst them have been classified in [34], hence we do not list them here. The imprimitive cases are given in Table 1. Below we rule out the existence of a distance-transitive orbital graph for the exceptional case $G = 2.B$ and $H = Fi_{23}$ not dealt with in Section (11.1), hence the latter list indeed is complete.

Let us assume to the contrary that one of the orbital graphs, $\mathcal{O}_i$ say, afforded by the permutation action of $G = 2.B$ on the right cosets of $H = Fi_{23}$ is distance-transitive. As this permutation action has rank $r = 34$, see Section (17.11), by Proposition (7.8) we conclude that $\mathcal{O}_i$ has diameter $d(\mathcal{O}_i) = 33$ and that the sequence of index parameters can be reordered to yield the sequence $[k_{(\mathcal{O}_i)_0}, \ldots, k_{(\mathcal{O}_i)_{33}}]$ of the valencies of the corresponding distance graphs $(\mathcal{O}_i)_d$, for $d \in \{0, \ldots, 33\}$. Using Remark (5.15), the index parameters can be derived from the splitting of suborbits as given in Table 27, see also Section (17.11). By Proposition (7.2) we conclude that $k_{(\mathcal{O}_i)_0} = k_{(\mathcal{O}_i)_{33}} = 1$ and furthermore that $k_{(\mathcal{O}_i)_d} = k_{(\mathcal{O}_i)_{33-d}}$ for $d \in \{0, \ldots, 16\}$, a contradiction to the sequence of index parameters derived from Table 27. Hence none of the orbital graphs afforded by this permutation action are distance-transitive.

For the distance-regular orbital graphs, afforded by a multiplicity-free permutation action where the group under consideration does not act distance-transitively, we restrict ourselves to the edge-transitive cases, which are shown in Table 2. For the non-edge-transitive cases we would have to consider all the generalised orbital graphs, see Remark (7.6). This would be doable, but the author does not expect interesting results.

In Tables 1 and 2, we indicate the rank $r \in \mathbb{N}$ of the permutation action under consideration, the orbital $i \in \mathcal{I}$ leading to the corresponding distance-regular orbital graph, its valency $k \in \mathbb{N}$, the cardinality $n \in \mathbb{N}$ of its vertex set, its diameter $d \in \mathbb{N}$, its intersection array, and whether it is primitive p, bipartite b, or antipodal a, see Definition (7.1).

Using the data given in Tables 1 and 2, it is possible to identify the corresponding graphs. The imprimitive distance-transitive orbital graphs of diameter 5 of $HS.2$ and $M_{22}.2$ are described in [8, Ch.6.11]. The non-distance-transitive orbital graphs of diameter 8 of $3.M_{22}$ and of diameter 4 of $3.Fi'_{24}$ are described in [8, Ch.6.12]. The non-distance-transitive orbital graph of diameter 4 of $3.Suz$ is a 3-

fold antipodal cover, see [8, Ch.4.2.A], of the primitive distance-transitive Suzuki graph of diameter 2. The orbital graph of diameter 4 of J_2 is the J_2-graph, see [8, Thm.13.6.1], whose full graph automorphism group is isomorphic to $J_2.2$ and acts distance-transitively. The orbital graphs of diameter 3 of M_{24} and M_{12} are, by [8, Thm.6.1.1], the Johnson graphs $J(12, 3)$ and $J(24, 3)$, see [8, Ch.9.1], whose full graph automorphism groups act distance-transitively. The imprimitive distance-regular graphs of diameter 3 are described in [8, Ch.14,pp.431–432]. Finally, the distance-regular graphs of diameter 2 are precisely the *strongly regular graphs*, see [8, Ch.A.1], as a general reference see [9, 33].

(7.12) Using the data contained in the database, see Section (11.1), the technique described in Section (7.10) and GAP, it is straightforward to obtain a classification of the Ramanujan orbital graphs for the sporadic simple groups, their automorphism groups and their Schur covering groups, coming from a multiplicity-free permutation action, up to the single exception $G = 2.B$ and $H = Fi_{23}$ not yet dealt with; as soon as the data for the bicyclic extensions of the sporadic simple groups is available, these cases can be dealt with as well, see Section (11.1).

By the discussion of Ramanujan graphs in [44], a Ramanujan graph tends to be the more interesting the smaller its valency is, compared to the cardinality of its vertex set. Accordingly, a subset of the Ramanujan connected orbital graphs of the above-mentioned groups and permutation actions is shown in Table 3; complete results for the generalised orbital graphs such that $n \leq 10^7$ have been compiled in [32]. In Table 3 we indicate the rank $r \in \mathbb{N}$ of the permutation action under consideration, the orbital $i \in \mathcal{I}$ leading to the corresponding Ramanujan orbital graph, its valency $k \in \mathbb{N}$, the cardinality $n \in \mathbb{N}$ of its vertex set, and its diameter $d \in \mathbb{N}$.

II Computational techniques

8 Intersection numbers and character tables

In Section 8 we discuss computational techniques useful to deal with structure constants matrices, character tables of endomorphism rings, and the Fitting correspondence. We keep the notation of Sections 1 and 3. In particular let Φ_λ be the character table of the endomorphism ring E_K^λ, see Definition (3.7) and Section (1.5), where K is as in Section 3.

Throughout Section 8 we assume E_K^λ to be commutative.

(8.1) If Φ_λ is known, then the structure constants matrices P_j^λ, for $j \in \mathcal{I}_\lambda$, see Definitions (1.6) and (1.18), can be determined using Proposition (3.18). As this is particularly nice and straightforward to implement in GAP, we show the relevant GAP code in Table 4.

Table 1: Imprimitive distance-transitive orbital graphs.

G	H	i	r	k	n	d	intersection array	
$HS.2$	$M_{22} < M_{22}.2$	3	6	22	200	5	[22, 21, 16, 6, 1; 1, 6, 16, 21, 22]	ba
$M_{22}.2$	$2^4{:}A_6 < 2^4{:}S_6$	3	6	16	154	5	[16, 15, 12, 4, 1; 1, 4, 12, 15, 16]	ba
Co_3	$McL < McL.2$	3	4	275	552	3	[275, 112, 1; 1, 112, 275]	a
Co_3	$McL < McL.2$	4	4	275	552	3	[275, 162, 1; 1, 162, 275]	a
HS	$U_3(5) < U_3(5).2$	3	4	175	352	3	[175, 72, 1; 1, 72, 175]	a
HS	$U_3(5) < U_3(5).2$	4	4	175	352	3	[175, 102, 1; 1, 102, 175]	a
HS	$U_3(5) < U_3(5).2$	3	4	175	352	3	[175, 72, 1; 1, 72, 175]	a
HS	$U_3(5) < U_3(5).2$	4	4	175	352	3	[175, 102, 1; 1, 102, 175]	a
$HS.2$	$U_3(5).2$	3	4	126	352	3	[126, 125, 36; 1, 90, 126]	b
$HS.2$	$U_3(5).2$	2	4	50	352	3	[50, 49, 36; 1, 14, 50]	b
$M_{22}.2$	$L_3(4) < L_3(4).2_2$	4	4	21	44	3	[21, 20, 1; 1, 20, 21]	ba
$2.M_{12}$	M_{11}	3	3	22	24	2	[22, 1; 1, 22]	a
$2.M_{12}$	M_{11}	3	3	22	24	2	[22, 1; 1, 22]	a
M_{11}	$A_6 < A_6.2_3$	3	3	20	22	2	[20, 1; 1, 20]	a
$M_{12}.2$	M_{11}	3	3	12	24	2	[12, 11; 1, 12]	ba

Table 2: Non-distance-transitive distance-regular orbital graphs.

G	H	i	r	k	n	d	intersection array	
$3.M_{22}$	$2^3{:}\,L_3(2)$	5	13	7	990	8	$[7,6,4,4,4,1,1,1;1,1,1,2,4,4,6,7]$	a
$3.Fi'_{24}$	Fi_{23}	5	7	31671	920808	4	$[31671,28160,2160,1;1,1080,28160,31671]$	a
$3.Suz$	$G_2(4)$	6	7	416	5346	4	$[416,315,64,1;1,32,315,416]$	a
J_2	$2_-^{1+4}{:}\,A_5$	2	6	10	315	4	$[10,8,8,2;1,1,4,5]$	p
$Fi_{22}.2$	$O_7(3)$	6	6	12636	28160	3	$[12636,12635,1296;1,11340,12636]$	b
$M_{22}.2$	A_7	6	6	126	352	3	$[126,125,36;1,90,126]$	b
M_{24}	$L_3(4).3.2_2$	2	5	63	2024	3	$[63,40,19;1,4,9]$	p
M_{12}	$3^2.2.S_4$	3	5	27	220	3	$[27,16,7;1,4,9]$	p
M_{12}	$3^2.2.S_4$	3	5	27	220	3	$[27,16,7;1,4,9]$	p
Fi_{22}	$O_8^+(2){:}\,S_3$	3	4	22400	61776	2	$[22400,14175;1,8064]$	p
$Fi_{22}.2$	$O_8^+(2){:}\,S_3 \times 2$	3	4	22400	61776	2	$[22400,14175;1,8064]$	p
Fi_{22}	$O_8^+(2){:}\,S_3$	2	4	1575	61776	2	$[1575,1376;1,36]$	p
$Fi_{22}.2$	$O_8^+(2){:}\,S_3 \times 2$	2	4	1575	61776	2	$[1575,1376;1,36]$	p
M_{23}	M_{11}	4	4	792	1288	2	$[792,315;1,504]$	p
J_2	$3.A_6.2_2$	4	4	135	280	2	$[135,64;1,60]$	p
$J_2.2$	$3.A_6.2^2$	4	4	135	280	2	$[135,64;1,60]$	p
J_2	$3.A_6.2_2$	2	4	36	280	2	$[36,27;1,4]$	p
$J_2.2$	$3.A_6.2^2$	2	4	36	280	2	$[36,27;1,4]$	p
M_{22}	$2^4{:}\,S_5$	3	4	40	231	2	$[40,19;1,4]$	p
$M_{22}.2$	$2^5{:}\,S_5$	3	4	40	231	2	$[40,19;1,4]$	p
M_{22}	$2^4{:}\,S_5$	2	4	30	231	2	$[30,20;1,3]$	p
$M_{22}.2$	$2^5{:}\,S_5$	2	4	30	231	2	$[30,20;1,3]$	p
M_{12}	$L_2(11)$	5	5	66	144	2	$[66,35;1,30]$	p
M_{12}	$L_2(11) < M_{11}$	5	5	66	144	2	$[66,35;1,30]$	p
$M_{12}.2$	$L_2(11).2$	4	4	66	144	2	$[66,35;1,30]$	p
$M_{12}.2$	$L_2(11).2$	4	4	66	144	2	$[66,35;1,30]$	p
M_{11}	$11{:}\,5 < L_2(11)$	5	6	55	144	2	$[55,32;1,20]$	p
M_{12}	$L_2(11) < M_{11}$	4	5	55	144	2	$[55,32;1,20]$	p
$M_{12}.2$	$L_2(11).2$	3	4	55	144	2	$[55,32;1,20]$	p
$M_{12}.2$	$L_2(11).2$	2	4	22	144	2	$[22,11;1,2]$	p
M_{11}	$A_5.2$	3	4	20	66	2	$[20,9;1,4]$	p

Table 3: Ramanujan orbital graphs of valency $k \leq \sqrt{n}$.

G	H	i	r	n	k	d
$M_{12}.2$	$3^2.2.S_4$	2	9	440	4	6
M_{22}	$2^3{:}\,L_3(2)$	2	5	330	7	4
$M_{22}.2$	$2^3{:}\,L_3(2) \times 2$	2	5	330	7	4
$M_{22}.2$	$2^3{:}\,L_3(2) < 2^3{:}\,L_3(2) \times 2$	3	10	660	7	5
J_2	$2_-^{1+4}{:}\,A_5$	2	6	315	10	4
$J_2.2$	$2_-^{1+4}{:}\,S_5$	2	5	315	10	4
M_{12}	$3^2.2.S_4$	2	5	220	12	3
M_{12}	$3^2.2.S_4$	2	5	220	12	3
$J_2.2$	$(A_5 \times D_{10}).2$	2	8	1008	12	5
$M_{22}.2$	A_7	2	6	352	15	4
J_2	$A_4 \times A_5$	2	7	840	15	4
$J_2.2$	$(A_4 \times A_5).2$	2	7	840	15	4
$J_2.2$	$A_4 \times A_5 < (A_4 \times A_5).2$	3	14	1680	15	5
$HS.2$	$5_+^{1+2}{:}\,[2^5]$	2	15	22176	50	3
M_{24}	$2^6{:}\,(L_3(2) \times S_3)$	3	5	3795	56	3
M_{24}	$2^6{:}\,(L_3(2) \times 3) < 2^6{:}\,(L_3(2) \times S_3)$	3	8	7590	56	3
M_{24}	$2^6{:}\,(L_3(2) \times 3) < 2^6{:}\,(L_3(2) \times S_3)$	4	8	7590	56	3

Table 4: GAP code: Finding the P_j^λ from Φ_λ.

```
# tbl:  Φ_λ for E_K^λ commutative,
#       a matrix with entries in the cyclotomic field K
# mats:  the P_j^λ for j ∈ I_λ, a list of matrices over K
IntersectionMatsFromCharTable:=function(tbl)
    local mats, trtbl, itrtbl, j, diag;
    mats:=[];
    trtbl:=TransposedMat(tbl);
    itrtbl:=trtbl^(-1);
    for j in [1..Length(tbl)] do
        diag:=DiagonalMat(List([1..Length(tbl)],i->tbl[i][j]));
        mats[j]:=trtbl*diag*itrtbl;
    od;
    return mats;
end;
```

Table 5: GAP code: Finding the m_φ from Φ.

```
# tbl:  Φ for E_K commutative,
#       a matrix with entries in the cyclotomic field K,
#       φ_1 is the first character in Φ
# degs:  the m_φ for φ ∈ Irr_K(E_K), a list of positive integers
CharDegrees:=function(tbl)
    local degs, n, j, s, i;
    degs:=[1];
    n:=Sum(tbl[1]);
    for j in [2..Length(tbl)] do
        s:=0;
        for i in [1..Length(tbl)] do
            s:=s+tbl[j][i]*GaloisCyc(tbl[j][i],-1)/tbl[1][i];
        od;
        degs[j]:=n/s;
    od;
    return degs;
end;
```

For the case $\lambda = 1$, if Φ is known, then the Fitting correspondent φ_1 of the trivial character KG is found by Remark (3.21). This yields the index parameters k_i, for $i \in \mathcal{I}$, see Definition (1.2), from Φ. Furthermore, if Φ_λ is known for arbitrary λ, by the first orthogonality relations, see Remark (3.9), the character degrees $\chi_\varphi(1) = m_\varphi$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, can be determined from Φ_λ and the index parameters k_i, for $i \in \mathcal{I}_\lambda$. For the case $\lambda = 1$ we show the relevant GAP code in Table 5.

For the case $\lambda = 1$, if Φ is known, then the Krein parameters q_{ijk}, see Definition (4.5), can be determined using Proposition (4.6). We show the relevant GAP code in Table 6.

(8.2) We discuss the strategy to find the character table of E_K^λ from the structure constants matrices. Let $\mathcal{E}_\lambda$ be the set of all centrally primitive idempotents of E_K^λ. By Proposition (3.14), the rows of $[\mathcal{E}_\lambda]_{\mathcal{A}_\lambda} \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$ are a K-basis of $K^{1\times|\mathcal{I}_\lambda|}$, consisting of simultaneous eigenvectors of all the structure constants matrices $P_j^\lambda \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$, for $j \in \mathcal{I}_\lambda$. Up to reordering and scalar multiples, this is the only K-basis of $K^{1\times|\mathcal{I}_\lambda|}$ consisting of simultaneous eigenvectors of all the P_j^λ.

Furthermore, the corresponding eigenvalues are the character values $\varphi(\alpha_j^\lambda)$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $j \in \mathcal{I}_\lambda$. Hence to determine the character table Φ_λ, we could just determine a K-basis consisting of simultaneous eigenvectors of all the P_j^λ, and subsequently compute the corresponding eigenvalues. But for the latter we would have to determine all the P_j^λ, for $j \in \mathcal{I}_\lambda$. Indeed, we can do better.

Table 6: GAP code: Finding the Krein parameters q_{ijk} from Φ.

```
# tbl:  Φ for E_K commutative,
#       a matrix with entries in the cyclotomic field K,
#       φ_1 is the first character in Φ
# q:  the q_ijk, a list of lists of lists over K
KreinParameters:=function(tbl)
    local q, n, m, i, j, k, t, s;
    q:=[];
    n:=Sum(tbl[1]);
    m:=CharDegrees(tbl); # see Table 5
    for i in [1..Length(tbl)] do
        q[i]:=[];
        for j in [1..Length(tbl)] do
            q[i][j]:=[];
            for k in [1..Length(tbl)] do
                t:=0;
                for s in [1..Length(tbl)] do
                    t:=t+GaloisCyc(tbl[i][s],-1)
                        *GaloisCyc(tbl[j][s],-1)
                        *tbl[k][s]/tbl[1][s]^2;
                od;
                q[i][j][k]:=t*m[i]*m[j]/n^2;
            od;
        od;
    od;
    return q;
end;
```

By Proposition (3.18), the rows of $[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda} \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$ are a K-basis of $K^{1\times|\mathcal{I}_\lambda|}$, consisting of simultaneous eigenvectors of all the dual structure constants matrices $\hat{P}_j^\lambda = (P_j^\lambda)^T \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$, for $j \in \mathcal{I}_\lambda$, see Proposition (3.17). Still, the corresponding eigenvalues are the character values $\varphi(\alpha_j^\lambda)$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $j \in \mathcal{I}_\lambda$. Furthermore, the row of $[\mathcal{E}_\lambda]_{\hat{\mathcal{A}}_\lambda}$ corresponding to $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ is equal to $\frac{m_\varphi}{|G|} \cdot [\varphi(\alpha_j^\lambda); j \in \mathcal{I}_\lambda] \in K^{1\times|\mathcal{I}_\lambda|}$, hence up to a scalar multiple is equal to the corresponding row of Φ_λ.

Because of $\varphi(\alpha_1^\lambda) = 1$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, to determine the character table Φ_λ, it hence is sufficient to find a K-basis consisting of simultaneous eigenvectors of all the $\hat{P}_j^\lambda$, for $j \in \mathcal{I}_\lambda$, and to rescale these vectors to have an entry 1 in position $i = 1 \in \mathcal{I}_\lambda$. In turn, to find a K-basis consisting of simultaneous eigenvectors of all the $\hat{P}_j^\lambda$, it is sufficient to find a subset $\mathcal{J} \subseteq \mathcal{I}_\lambda$, such that $\mathcal{C} := \langle \alpha_j^\lambda; j \in \mathcal{J}\rangle_{K-\text{algebra}}$ equals E_K^λ, and to compute $\{\hat{P}_j^\lambda; j \in \mathcal{J}\}$ only. By Corollary (3.15), we have $\mathcal{C} = E_K^\lambda$ if and only if the simultaneous eigenspaces of $\{\hat{P}_j^\lambda; j \in \mathcal{J}\}$ in $K^{1\times|\mathcal{I}_\lambda|}$ are 1-dimensional.

A similar algorithm is well-known for the group character table case, see [17, 71]. In that case, the character degrees usually are not known in advance. Hence, in addition to finding simultaneous eigenvalues, the scaling factors to yield the correct character degrees have to be determined as well.

To find the eigenspaces of $\hat{P}_j^\lambda$, for $j \in \mathcal{I}_\lambda$, we proceed as follows. Let $\mu_{\hat{P}_j^\lambda} \in K[X]$ be the minimum polynomial of $\hat{P}_j^\lambda \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$. As E_K^λ is a commutative split semisimple K-algebra, $\mu_{\hat{P}_j^\lambda} \in K[X]$ is a separable polynomial. Hence to find the eigenspaces of $\hat{P}_j^\lambda$, we have to find the irreducible factors of $\mu_{\hat{P}_j^\lambda}$ in $K[X]$, which are linear. As $\hat{P}_j^\lambda \in \mathbb{Q}(\lambda(H))^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$, we also have $\mu_{\hat{P}_j^\lambda} \in \mathbb{Q}(\lambda(H))[X]$. Hence we first compute the irreducible factors of $\mu_{\hat{P}_j^\lambda}$ in $\mathbb{Q}(\lambda(H))[X]$, and subsequently factorize the latter into linear factors in $K[X]$.

Algorithms for polynomial factorisation over algebraic number fields are known, see [12, Ch.3.6.2]. By Proposition (3.10) we even have $\mu_{\hat{P}_j^\lambda} \in \mathbb{Z}[\lambda(H)][X]$; note that by [50, Cor.2.2] the ring $\mathbb{Z}[\lambda(H)]$ coincides with the ring of algebraic integers in $\mathbb{Q}(\lambda(H))$. For the case $\lambda = 1$ the polynomial $\mu_{\hat{P}_j^\lambda}$ has to be factorized in $\mathbb{Z}[X]$. Algorithms for polynomial factorisation over $\mathbb{Z}$ are known as well, see [12, Ch.3.5], and are available in GAP. Furthermore, as the zeroes of $\mu_{\hat{P}_j^\lambda} \in K[X]$ are exactly the character values $\varphi(\alpha_j^\lambda)$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, the factorisation of $\mu_{\hat{P}_j^\lambda}$ into linear factors can be done in the polynomial ring $K'[X]$, where $K' := \mathbb{Q}(\lambda(H))[\chi(C); \chi \in \mathrm{Irr}_K^\lambda(G), C \in \mathcal{C}l(G)]$, which is a splitting field for E_K^λ, see Remark (3.21).

(8.3) We briefly digress, and consider the case where E_K^λ is non-commutative. As the irreducible characters $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ are no longer necessarily linear, we are faced with the problem to determine representing matrices for the action of the Schur basis elements $\alpha_i^\lambda \in \mathcal{A}_\lambda$ on the simple E_K^λ-modules S_φ, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ and $i \in \mathcal{I}_\lambda$. Still, it suffices to find the structure constants matrices $P_i^\lambda \in K^{|\mathcal{I}_\lambda|\times|\mathcal{I}_\lambda|}$, for $i \in \mathcal{I}_\lambda$. But the technique described in Section (8.2), to find the character table Φ_λ of E_K^λ from possibly only part of the structure constants matrices, does no longer work. Furthermore, Proposition (3.18) no longer holds. Hence it seems to be unavoidable to compute all of the structure constants matrices explicitly. For larger examples this might be a considerable task. If the structure constants matrices are available, there are at least two strategies to proceed.

Firstly, in particular if the degrees $m_\varphi = \varphi(1)$ of the $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ are small, we could use the strategy described in Section (8.2). For $i, j \in \mathcal{I}_\lambda$, such that $i = i^*$ and $j \neq j^*$, by Remark (1.19) the structure constants matrices P_i^λ and $P_j^\lambda \pm P_{j^*}^\lambda$ are diagonalisable over a suitable algebraic extension field of K. Hence we again could compute the irreducible factors of the minimum polynomials $\mu_{P_i^\lambda} \in K[X]$, find the corresponding characteristic spaces in $K^{1\times|\mathcal{I}_\lambda|}$, intersect them, and compute the action of the structure constants matrices on these K-subspaces. Secondly, in particular for the case $\lambda = 1$, where $P_i \in \mathbb{Z}^{n\times n}$, for $i \in \mathcal{I}$, we could use general MeatAxe techniques over the rationals and the rational integers, see [55, 66], to find the constituents of the regular E_K^λ-module. For the time being, no substantial examples have been dealt with computationally.

(8.4) In the remaining parts of Section 8 we discuss the strategy to determine the Fitting correspondence explicitly. Let again E_K^λ be commutative.

Without loss of generality we may assume that K is a splitting field for KG. If the full character table $\mathcal{X} = [\chi(C); \chi \in \mathrm{Irr}_K(G), C \in \mathcal{C}l(G)] \in K^{|\mathcal{C}l(G)|\times|\mathcal{C}l(G)|}$ of G as well as Φ_λ are known, then necessary conditions to find the Fitting correspondent $\chi_\varphi \in \mathrm{Irr}_K(G)$ of $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$ are given as follows. Note that, although in many cases $\mathrm{Irr}_K^\lambda(G)$ is known in advance and only the Fitting correspondence has to be determined, $\mathrm{Irr}_K^\lambda(G)$ need not be known for the following approach.

By Section (8.1), the character degree $\chi_\varphi(1) = m_\varphi$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$, can be determined from Φ_λ. Furthermore, by Remark (3.24), the matrix $\Gamma_\lambda \in \mathbb{Q}(\lambda(H))^{|\mathcal{I}_\lambda|\times|\mathcal{C}l(G)|}$, see Definition (3.19), can be determined from Φ_λ and $\mathcal{X}_\lambda$. Now the $\gamma_i^\lambda(C) \in \mathbb{Q}(\lambda(H))$, for $i \in \mathcal{I}_\lambda$ and $C \in \mathcal{C}l(G)$, are algebraic integers, and in particular for $\lambda = 1$ we even have $\gamma_i(C) \in \mathbb{N}_0$, for $i \in \mathcal{I}$.

We first determine the sets $\mathrm{Irr}_K^\varphi(G) := \{\chi \in \mathrm{Irr}_K(G); \chi(1) = m_\varphi\}$, for $\varphi \in \mathrm{Irr}_K(E_K^\lambda)$. Thus $\prod_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} \mathrm{Irr}_K^\varphi(G)$ can be considered as a set of candidate cases for the Fitting correspondence searched for, where we restrict ourselves to the cases where $[\chi_1, \ldots, \chi_{|\mathcal{I}_\lambda|}] \in \prod_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} \mathrm{Irr}_K^\varphi(G)$ has pairwise different entries. From $[\chi_1, \ldots, \chi_{|\mathcal{I}_\lambda|}]$ we obtain the submatrix $\mathcal{X}_{\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}}$ of $\mathcal{X}$ consisting of the rows corresponding to $\chi_1, \ldots, \chi_{|\mathcal{I}_\lambda|}$. Then we compute the matrix

$\Gamma_{\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}} \in K^{|\mathcal{I}_\lambda|\times|\mathcal{C}l(G)|}$ defined by

$$\Gamma_{\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}} := \frac{1}{n}\cdot \mathrm{diag}[k_i^{-1}; i\in\mathcal{I}_\lambda]\cdot \Phi_\lambda^T \cdot \overline{\mathcal{X}_{\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}}}\cdot \mathrm{diag}[|C|; C\in\mathcal{C}l(G)].$$

By Remark (3.24), if $\Gamma_{\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}}$ has an entry which is not an element of $\mathbb{Q}(\lambda(H))$ or which is not an algebraic integer or which for the case $\lambda = 1$ is a negative integer, then we discard the candidate case $[\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}]$, otherwise $[\chi_1,\ldots,\chi_{|\mathcal{I}_\lambda|}]$ is an admissible case. Let $\mathcal{F}_\lambda \subseteq \prod_{\varphi\in\mathrm{Irr}_K(E_K^\lambda)} \mathrm{Irr}_K^\varphi(G)$ denote the set of admissible candidate cases.

(8.5) Definition.
a) Let $\mathcal{S}_{\mathcal{C}l(G)}$ be the symmetric group on the set $\mathcal{C}l(G)$ of conjugacy classes of G, let $\pi\in\mathcal{S}_{\mathcal{C}l(G)}$, and for $\chi\in\mathrm{Irr}_K(G)$ let $\chi^\pi\colon\mathcal{C}l(G)\to K$ be the class function defined by $\chi^\pi\colon C\mapsto\chi(C\pi^{-1})$. For $s\in\mathbb{Z}$ the *s-th power map* $\mathcal{C}l(G)\to\mathcal{C}l(G)$ is defined as the map induced by the map $G\to G\colon g\mapsto g^s$. Then π is called a *table automorphism* of $\mathrm{Irr}_K(G)$, if π commutes with the s-th power maps on $\mathcal{C}l(G)$, for all $s\in\mathbb{Z}$, and $\chi^\pi\in\mathrm{Irr}_K(G)$, for all $\chi\in\mathrm{Irr}_K(G)$. Let $\mathrm{Aut}(\mathrm{Irr}_K(G))\leq\mathcal{S}_{\mathcal{C}l(G)}$ denote the group of table automorphisms of $\mathrm{Irr}_K^\lambda(G)$. Furthermore, $\pi\in\mathrm{Aut}(\mathrm{Irr}_K(G))$ is a *table automorphism* of $\mathrm{Irr}_K^\lambda(G)$, if additionally $\chi^\pi\in\mathrm{Irr}_K^\lambda(G)$, for all $\chi\in\mathrm{Irr}_K^\lambda(G)$.

b) Let $\mathcal{S}_{\mathcal{I}_\lambda}$ be the symmetric group on the set $\mathcal{I}_\lambda$, let $\pi\in\mathcal{S}_{\mathcal{I}_\lambda}$, and for $\varphi\in\mathrm{Irr}_K(E_K^\lambda)$ let $\varphi^\pi\colon\mathcal{I}_\lambda\to K$ be the class function defined by $\varphi^\pi\colon i\mapsto\varphi(i\pi^{-1})$. Then π is called a *table automorphism* of $\mathrm{Irr}_K(E_K^\lambda)$, if $\varphi^\pi\in\mathrm{Irr}_K(E_K^\lambda)$, for all $\varphi\in\mathrm{Irr}_K(E_K^\lambda)$. Let $\mathrm{Aut}(\mathrm{Irr}_K(E_K^\lambda))\leq\mathcal{S}_{\mathcal{I}_\lambda}$ denote the the group of table automorphisms of $\mathrm{Irr}_K(E_K^\lambda)$.

(8.6) Remark.
a) Given the character table $\mathcal{X}\in K^{|\mathcal{C}l(G)|\times|\mathcal{C}l(G)|}$ of $\mathrm{Irr}_K(G)$, there are programs available in GAP to compute $\mathrm{Aut}(\mathrm{Irr}_K(G))$. Note that, by the orthogonality relations for $\mathcal{X}$, a table automorphism $\pi\in\mathrm{Aut}(\mathrm{Irr}_K(G))$ leaves the sets $\mathcal{C}l(G)_c := \{C\in\mathcal{C}l(G); |C| = c\}$, for $c\in\mathbb{N}$, invariant.

Furthermore, given the character table $\Phi\in K^{|\mathrm{Irr}_K(E_K)|\times|\mathcal{I}|}$ for the case $\lambda = 1$, by Remark (3.21), each $\pi\in\mathrm{Aut}(\mathrm{Irr}_K(E_K))$ fixes the Fitting correspondent φ_1 of the trivial KG-character. Hence π leaves the sets $\mathcal{I}_k := \{i\in\mathcal{I}; k_i = k\}$, for $k\in\mathbb{N}$, invariant. Thus we have $\mathrm{Aut}(\mathrm{Irr}_K(E_K))\leq\prod_{k\in\{k_i; i\in\mathcal{I}\}}\mathcal{S}_{\mathcal{I}_k}$. For the examples occurring in the present work, see Section (11.1), this turns out to be a sufficiently small group such that we are able to check for all of its elements whether they are in $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$ or not. In particular, if all the index parameters k_i, for $i\in\mathcal{I}$, are pairwise different, then we have $\mathrm{Aut}(\mathrm{Irr}_K(E_K)) = \{1\}$.

b) For table automorphisms $\pi_G\in\mathrm{Aut}(\mathrm{Irr}_K^1(G))$ and $\pi_E\in\mathrm{Aut}(\mathrm{Irr}_K(E_K))$ let $[\pi_G]\in\mathbb{Z}^{|\mathcal{C}l(G)|\times|\mathcal{C}l(G)|}$ and $[\pi_E]\in\mathbb{Z}^{|\mathcal{I}|\times|\mathcal{I}|}$ denote the permutation matrices inducing the corresponding column permutations of $\mathcal{X}_1$ and Φ, respectively. If

$\Gamma_{\chi_1,\ldots,\chi_{|\mathcal{I}|}} \in K^{|\mathcal{I}|\times|\mathcal{C}l(G)|}$ fulfils the admissibility conditions in Section (8.4), then

$$[\pi_E]^{-1} \cdot \Gamma_{\chi_1,\ldots,\chi_{|\mathcal{I}|}} \cdot [\pi_G] = \frac{1}{n} \cdot \mathrm{diag}[k_i^{-1}] \cdot (\Phi \cdot [\pi_E])^T \cdot \overline{\mathcal{X} \cdot [\pi_G]} \cdot \mathrm{diag}[|C|]$$

also is an admissible matrix. Hence $\mathrm{Aut}(\mathrm{Irr}_K(E_K)) \times \mathrm{Aut}(\mathrm{Irr}_K^1(G))$ acts on the set of admissible candidate cases $\mathcal{F}_1$ for the Fitting correspondence, and the strategy described in Section (8.4) yields unions of orbits under this action.

The strategy described in Section (8.4) and Remark (8.6) is applied in Section (11.3).

9 Condensation

In Section 9 we discuss aspects of practical computational applications of condensation functors. We keep the notation of Section 6.

(9.1) Let Θ be as in Section (2.1), let λ be a representation of ΘH, such that the underlying ΘH-module Θ_λ is Θ-free of degree 1, and let $\epsilon_\lambda \in \Theta H \subseteq \Theta G$ denote the corresponding idempotent. We have $\epsilon_\lambda \Theta G \cong \lambda^G$ as ΘG-modules. Hence, using Definition (6.4) and Remark (6.5), for $V \in \mathbf{mod}\text{-}\Theta G$ we have $V\epsilon_\lambda = C_{\epsilon_\lambda}(V) \cong \mathrm{Hom}_{\Theta G}(\lambda^G, V) \cong \mathrm{Hom}_{\Theta H}(\lambda, V_H)$ as Θ-modules. Hence, if Θ is a field, then the underlying set of the condensed module $V\epsilon_\lambda \in \mathbf{mod}\text{-}\epsilon_\lambda\Theta G\epsilon_\lambda$ is the isotypic component of V_H belonging to λ.

From the computational point of view, for given $V \in \mathbf{mod}_\Theta\text{-}\Theta G$, we have to find a Θ-basis of $V\epsilon_\lambda$, and subsequently, for given $g \in G$, we have to find the action of $\epsilon_\lambda g \epsilon_\lambda \in \epsilon_\lambda \Theta G \epsilon_\lambda$ on $V\epsilon_\lambda$ with respect to this basis. In practice, this has to be done without having available explicit representing matrices for the action of the elements of G on V, since typically $\mathrm{rk}_\Theta(V)$ is so large that we would not be able to deal computationally with these matrices.

If $\lambda = 1$, let $\epsilon := \epsilon_1$. In this case we have $V\epsilon = \mathrm{Fix}_H(V)$, the set of the H-fixed points in V. This particular condensation functor is called a *fixed point condensation functor*. The latter have been applied to different types of FG-modules over finite fields F. Historically, the first application [77] has been to permutation modules. We give the corresponding *condensation formula* in Proposition (9.5). An implementation is available as the ZKD program in the MeatAxe. Originally, this program returns representing matrices for the action of $\epsilon g \epsilon$ on $V\epsilon$, for $g \in G$. We have generalised it slightly to return, optionally, orbit counting matrices $C(g)$ having integral entries, see Definition (9.4).

Applying fixed point condensation functors to tensor product modules has been sketched in [52], and has been worked out in [79, 48]; an implementation, with a few improvements [58], is also available in the MeatAxe. Arbitrary induced modules have been dealt with in [59], an implementation being available in GAP. Great improvements for the permutation module case have been made by the invention of the direct condense technique [67], which has subsequently been developed into a parallelised version in [45]; a modified version has been used in [57] and we further elaborate on this technique in Section 10.

(9.2) Remark.
a) Let $\Theta \in \{K, F\}$ be as in Section (2.10), where in particular the characteristic of F is coprime to $|H|$, and let $\chi_V \in \mathbb{Z}\mathrm{IBr}_\Theta(G)$ denote the Brauer character of $V \in \textbf{mod-}\Theta G$. If $\Theta = F$, then χ_V is a K-valued class function on the p'-classes of G, which can be extended to a class function on G by letting $\chi_V(g) = \chi_V(g_{p'})$, where $g = g_p \cdot g_{p'} \in G$ denote the p-part and the p'-part of $g \in G$, respectively. If $\Theta = K$, then χ_V is a K-valued class function on G anyway. Hence we have

$$\dim_\Theta(V\epsilon_\lambda) = \langle (\chi_V)_H, \lambda\rangle_H = \langle \chi_V, \lambda^G\rangle_G,$$

where $\langle\cdot,\cdot\rangle_G$ and $\langle\cdot,\cdot\rangle_H$ denote the hermitian products on the K-valued class functions on G and H, respectively. Hence the Θ-dimension of the condensed module $V\epsilon_\lambda$ of V can be determined from purely character theoretic information without actually applying the condensation functor.
b) Let $k \in \mathcal{I}$ and $g \in Hg_kH$, and let $\mathrm{tr}_{V\epsilon_\lambda}$ and tr_V denote the Θ-valued trace functions on V and $V\epsilon_\lambda$, respectively. As in the proof of Proposition (3.20) we have

$$\mathrm{tr}_{V\epsilon_\lambda}(\epsilon_\lambda g \epsilon_\lambda) = \mathrm{tr}_V(\epsilon_\lambda g \epsilon_\lambda) = \frac{1}{|H|} \cdot \sum_{C \in \mathcal{C}l(G)} \gamma_k(C) \cdot \mathrm{tr}_V(C),$$

where $\gamma_k(C) \in \Theta$ is as in Definition (3.19). We have $\mathrm{tr}_V(C) = \chi_V(C)$, if $\Theta = K$, and $\mathrm{tr}_V(C) = \widetilde{\chi_V(C)}$, if $\Theta = F$, respectively.

This has been applied to solve problems concerned with the determination of decomposition numbers of algebraically conjugate ordinary characters, see [57, 65, 70].

(9.3) We proceed to prove the *condensation formula*, see Proposition (9.5), to which fixed point condensation of permutation modules boils down.

Let $\lambda = 1$ and $\epsilon = \epsilon_1$. Let $U \leq G$ be another subgroup and $\Xi := U|G$. Let $\mathcal{J} := \{1, \ldots, \tilde{r}\}$, where $\tilde{r} \in \mathbb{N}$ is the number of U-H-double cosets in G, and let $\{\tilde{\tilde{g}}_j \in G; j \in \mathcal{J}\}$ be a set of representatives of the U-H-double cosets in G, where $\tilde{\tilde{g}}_1 := 1_G$.

As in Section (5.2) we have

$$\mathrm{Hom}_{\Theta G}(1_H^G, 1_U^G) \cong \mathrm{Hom}_{\Theta H}(1, (1_U^G)_H) \cong \bigoplus_{j \in \mathcal{J}} \mathrm{Hom}_{\Theta H}(1, (1_{U^{\tilde{\tilde{g}}_j} \cap H})^H).$$

(9.4) Definition.
a) For $j \in \mathcal{J}$ let $\Xi_j := \{U\tilde{\tilde{g}}_j h \in \Xi; h \in H\} \subseteq \Xi$ and $\Xi_j^+ := \sum_{\xi \in \Xi_j} \xi \in \Theta\Xi$. Hence $\Xi^+ := \{\Xi_j^+; j \in \mathcal{J}\}$ is a Θ-basis of $\Theta\Xi \cdot \epsilon$.

b) For $g \in G$ and $i, j \in \mathcal{J}$ let the *orbit counting numbers* $c_{ij}(g) \in \mathbb{N}_0$ *with respect to* $\Xi = \coprod_{j\in\mathcal{J}} \Xi_j$ be defined by

$$c_{ij}(g) := |\{\xi \in \Xi_i; \xi g \in \Xi_j\}| = |\Xi_i g \cap \Xi_j| = |\Xi_i \cap (\Xi_j g^{-1})|.$$

Let the *orbit counting matrix* $C(g) \in \mathbb{N}_0^{\tilde{r}\times\tilde{r}}$ *with respect to* $\Xi = \coprod_{j\in\mathcal{J}} \Xi_j$, belonging to g, be defined by $C(g)_{ij} := c_{ij}(g)$, for $i, j \in \mathcal{J}$.

(9.5) Proposition. Let $g \in G$. Then the representing matrix for the action of $\epsilon g\epsilon \in \epsilon\Theta G\epsilon$ on $\Theta\Xi \cdot \epsilon$ with respect to the Θ-basis Ξ^+ is given as

$$[\epsilon g\epsilon]_{\Xi^+} = C(g) \cdot \mathrm{diag}[|\Xi_j|^{-1}; j \in \mathcal{J}] \in \Theta^{\tilde{r}\times\tilde{r}}.$$

Proof. We have $\Xi_i^+ \cdot \epsilon g\epsilon = \frac{1}{|H|} \cdot \sum_{j\in\mathcal{J}} |\{\xi \in \Xi_i; \xi g \in \Xi_j\}| \cdot \frac{|H|}{|\Xi_j|} \cdot \Xi_j^+$. ♯

(9.6) Proposition. Let $U = H$, hence $\Xi = \Omega$ and $\mathcal{J} = \mathcal{I}$. Let $g \in Hg_lH \subseteq G$ for $l \in \mathcal{I}$. Then for $i, j \in \mathcal{I}$ and the structure constants $p_{lij} \in \Theta$, see Definition (1.18), we have,

$$p_{lij} = \frac{k_l}{k_j} \cdot c_{ij}(g) = \frac{k_l}{k_j} \cdot |\Omega_i \cap (\Omega_j g^{-1})| = \frac{k_l}{k_j} \cdot |\Omega_i g \cap \Omega_j| \in \Theta,$$

independent of the particular choice of $g \in Hg_lH$.

Proof. As in the proof of Proposition (2.2), we have

$$\Omega_i^+ \sigma = \sum_{i'\in\{1,\ldots,k_i\}} \epsilon g_i h_{ii'} = \frac{|H|}{|H_i|} \cdot \epsilon g_i \epsilon = k_i \cdot \epsilon g_i \epsilon \in \epsilon\Theta G.$$

We may without loss of generality assume that $g = g_l$. Thus we have

$$(k_i \cdot \epsilon g_i \epsilon) \cdot \epsilon g_l \epsilon = \sum_{j\in\mathcal{I}} \frac{c_{ij}(g_l)}{k_j} \cdot (k_j \cdot \epsilon g_j \epsilon).$$

Furthermore, by Proposition (2.2) we have $(k_i \cdot \epsilon g_i \epsilon) \cdot (k_l \cdot \epsilon g_l \epsilon) = \sum_{j\in\mathcal{I}} p_{lij} \cdot (k_j \cdot \epsilon g_j \epsilon)$. This yields the assertion. ♯

(9.7) Remark. Let still $U = H$, hence $\Xi = \Omega$ and $\mathcal{J} = \mathcal{I}$.
a) If $g \in G$ is given, the row of $C(g) = [c_{ij}(g); i, j \in \mathcal{I}]$ corresponding to $i = 1 \in \mathcal{I}$ has exactly one non-vanishing entry. If this is in the column corresponding to $k \in \mathcal{I}$, then by Definition (9.4) we have $g \in Hg_kH$. Furthermore, for the row sums and column sums of $C(g)$ we have $\sum_{j\in\mathcal{I}} c_{ij}(g) = k_i$ and $\sum_{i\in\mathcal{I}} c_{ij}(g) = k_j$, for fixed $i \in \mathcal{I}$ and fixed $j \in \mathcal{I}$, respectively.
b) If E_K is commutative, we have $p_{lij} = p_{ilj}$, for $i, j, l, \in \mathcal{I}$. Hence by Proposition (9.6) we have

$$P_l = k_l \cdot C(g_l) \cdot \mathrm{diag}[k_j^{-1}; j \in \mathcal{I}] = k_l \cdot [\epsilon g_l \epsilon]_{\Omega^+}.$$

(9.8) For the computation of orbit counting matrices, see Definition (9.4), for the case $\Xi = \Omega$, which is most interesting in the present work, we occasionally use the following strategy, whose usefulness becomes clear in Section (10.3).

Let $U \leq H$, let $\mathcal{J} := \{1, \ldots, \tilde{r}\}$, where $\tilde{r} \in \mathbb{N}$ is the number of H-U-double cosets in G, and let $\{\tilde{g}_j \in G; j \in \mathcal{J}\}$ be a set of representatives of the H-U-double cosets in G, where $\tilde{g}_1 := 1_G$. For $j \in \mathcal{J}$ let $\tilde{\Omega}_j := \{H\tilde{g}_j u \in \Omega; u \in U\} \subseteq \Omega$ Then, for each $j \in \mathcal{J}$ there is $i \in \mathcal{I}$ such that $\tilde{\Omega}_j \subseteq \Omega_i$. This defines a surjective map $\alpha_{U,H} \colon \mathcal{J} \rightarrow \mathcal{I}$.

Let $\tilde{c}_{ij}(g) := |\{\omega \in \tilde{\Omega}_i; \omega g \in \tilde{\Omega}_j\}| \in \mathbb{N}_0$, for $i, j \in \mathcal{J}$ and $g \in G$, be the orbit counting numbers with respect to $\Omega = \coprod_{j \in \mathcal{J}} \tilde{\Omega}_j$. Furthermore, let $c_{ij}(g) \in \mathbb{N}_0$, for $i, j \in \mathcal{I}$ and $g \in G$, be the orbit counting numbers with respect to $\Omega = \coprod_{i \in \mathcal{I}} \Omega_i$. Hence we have

$$c_{ij}(g) = \sum_{\tilde{i} \in \alpha_{U,H}^{-1}(i)} \sum_{\tilde{j} \in \alpha_{U,H}^{-1}(j)} \tilde{c}_{\tilde{i}\tilde{j}}(g).$$

To determine the sets $\alpha_{U,H}^{-1}(i) \subseteq \mathcal{J}$, for $i \in \mathcal{I}$, we additionally compute orbit counting matrices $\tilde{C}(h) = [\tilde{c}_{ij}(h); i, j \in \mathcal{J}] \in \mathbb{N}_0^{\tilde{r} \times \tilde{r}}$, for $h \in \mathcal{H}$, where $\mathcal{H} \subseteq H$ is a set of generators of H. From these we compute the finest set partition $\{\mathcal{J}_k \subseteq \mathcal{J}; k \in \{1, \ldots, s\}\}$ of $\mathcal{J}$, hence $\mathcal{J} = \coprod_{k=1}^{s} \mathcal{J}_k$, such that we have $j \in \mathcal{J}_k$, whenever $i \in \mathcal{J}_k$ and $j \in \mathcal{J}$ such that $\tilde{c}_{ij}(h) \neq 0$ for some $h \in \mathcal{H}$. As $\langle \mathcal{H} \rangle = H$ and by the definition of the orbit counting numbers we conclude that $s = r$ and $\{\alpha_{U,H}^{-1}(i); i \in \mathcal{I}\} = \{\mathcal{J}_k; k \in \{1, \ldots, r\}\}$.

(9.9) We return to the case of λ arbitrary. In practice we only compute representing matrices for the action of a few elements $\{\epsilon_\lambda g \epsilon_\lambda; g \in \mathcal{G}\} \subseteq \epsilon_\lambda \Theta G \epsilon_\lambda$, for some subset $\mathcal{G} \subseteq G$, on the module $V\epsilon_\lambda \in \mathbf{mod}_\Theta\text{-}\epsilon_\lambda \Theta G \epsilon_\lambda$, where $V \in \mathbf{mod}_\Theta\text{-}\Theta G$. Hence we only know the action of the Θ-subalgebra

$$\mathcal{C}_\mathcal{G} := \langle \epsilon_\lambda g \epsilon_\lambda; g \in \mathcal{G} \rangle_{\Theta-\text{algebra}} \subseteq \epsilon_\lambda \Theta G \epsilon_\lambda$$

on $V\epsilon_\lambda$, which poses the problem to infer the structure of the $\epsilon_\lambda \Theta G \epsilon_\lambda$-module $V\epsilon_\lambda$ from an explicit analysis of the $\mathcal{C}_\mathcal{G}$-module structure of $V\epsilon_\lambda$. Different strategies to tackle this problem have been developed, see for example [26, 38, 59]. The following idea and the criterion in Proposition (9.11) might be helpful as well, although for the time being they have not yet been applied to substantial examples.

Let $V \in \mathbf{mod}_{\epsilon_\lambda}\text{-}FG$ be a trivial source FG-module; for example this holds for $V \cong \epsilon_\lambda FG$ as FG-modules. By [39, Thm.II.12.4] we have $\dim_F \mathrm{End}_{FG}(V) = \langle \chi_{\hat{V}}, \chi_{\hat{V}} \rangle_G$, where $\hat{V} \in \mathbf{mod}_R\text{-}RG$ is the uniquely defined trivial source RG-module such that $\widetilde{\hat{V}} \cong V$ as FG-modules, and where $\langle \cdot, \cdot \rangle_G$ is the hermitian product on the K-valued class functions on G. Hence in this situation $\dim_F \mathrm{End}_{FG}(V)$ can be determined from purely character theoretic information, and as the assumptions of Remark (6.13) are fulfilled, $\dim_F \mathrm{End}_{\epsilon_\lambda FG\epsilon_\lambda}(V\epsilon_\lambda)$ is also known.

If $\mathcal{C}_\mathcal{G} \subseteq \epsilon_\lambda FG\epsilon_\lambda$ is an F-subalgebra, and the restriction of $V\epsilon_\lambda$ to $\mathcal{C}_\mathcal{G}$ is given, we can explicitly determine $\mathrm{End}_{\mathcal{C}_\mathcal{G}}(V\epsilon_\lambda)$, using the algorithms described in [74] to compute endomorphism rings available in the MeatAxe. If $\dim_F \mathrm{End}_{\mathcal{C}_\mathcal{G}}(V\epsilon_\lambda) > \dim_F \mathrm{End}_{\epsilon_\lambda FG\epsilon_\lambda}(V\epsilon_\lambda)$, we compute representing matrices for the action of additional elements $\epsilon_\lambda g\epsilon_\lambda \in \epsilon_\lambda FG\epsilon_\lambda$ on $V\epsilon_\lambda$, and thus enlarge the set $\mathcal{G} \subseteq G$ and the F-subalgebra $\mathcal{C}_\mathcal{G} \subseteq \epsilon_\lambda FG\epsilon_\lambda$, until we have $\dim_F \mathrm{End}_{\mathcal{C}_\mathcal{G}}(V\epsilon_\lambda) = \dim_F \mathrm{End}_{\epsilon_\lambda FG\epsilon_\lambda}(V\epsilon_\lambda)$. Hence we have explicitly determined $\mathrm{End}_{\epsilon_\lambda FG\epsilon_\lambda}(V\epsilon_\lambda) = \mathrm{End}_{\mathcal{C}_\mathcal{G}}(V\epsilon_\lambda)$.

Knowing $\mathrm{End}_{\epsilon_\lambda FG\epsilon_\lambda}(V\epsilon_\lambda)$, we may for example determine a direct sum decomposition of the $\epsilon_\lambda FG\epsilon_\lambda$-module $V\epsilon_\lambda$ into indecomposable summands and the isomorphism types of the summands, using the relevant algorithms described in [74], available in the MeatAxe. Furthermore we may infer the existence of certain $\epsilon_\lambda FG\epsilon_\lambda$-modules, namely those which are images of $\epsilon_\lambda FG\epsilon_\lambda$-endomorphisms of $V\epsilon_\lambda$.

Letting $D_{V\epsilon_\lambda}\colon \epsilon_\lambda FG\epsilon_\lambda \to \mathrm{End}_F(V\epsilon_\lambda)$ denote the corresponding representation, in general we might still have a proper inclusion $D_{V\epsilon_\lambda}(\mathcal{C}_\mathcal{G}) \subset D_{V\epsilon_\lambda}(\epsilon_\lambda FG\epsilon_\lambda)$, as we might have a proper inclusion $D_{V\epsilon_\lambda}(\epsilon_\lambda FG\epsilon_\lambda) \subset \mathrm{End}_{\mathrm{End}_{\epsilon_\lambda FG\epsilon_\lambda}(V\epsilon_\lambda)}(V\epsilon_\lambda) = \mathrm{End}_{\mathrm{End}_{\mathcal{C}_\mathcal{G}}(V\epsilon_\lambda)}(V\epsilon_\lambda)$. To the knowledge of the author, the known general criteria to ensure equality here, hence the *double centralizer property*, are quite restrictive, see [14, Thm.VIII.59.6].

(9.10) Let $H' \leq H$ as well as λ and λ' be as in Section (5.3). In particular, we keep the condition that λ^G and $(\lambda'^H - \lambda)^G$ have no KG-constituents in common, and that the characteristic of F is coprime to $|H|$. Let $\Theta \in \{K, R\}$. Then we have $\lambda^G\epsilon_{\lambda'} \cong \mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G)$ as Θ-modules, and $\tilde{\lambda}^G\epsilon_{\widetilde{\lambda'}} \cong \mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G)$ as F-vector spaces. Furthermore let $D_\Theta^{\lambda'\lambda}\colon (E_\Theta^{\lambda'})^\circ \to \mathrm{End}_\Theta \mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G)$ denote the corresponding representation of $(E_\Theta^{\lambda'})^\circ \cong \epsilon_{\lambda'}\Theta G\epsilon_{\lambda'}$, and let $D_F^{\widetilde{\lambda'}\tilde{\lambda}}\colon (E_F^{\widetilde{\lambda'}})^\circ \to \mathrm{End}_F\mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G)$ denote the analogous one of $(E_F^{\widetilde{\lambda'}})^\circ \cong \epsilon_{\widetilde{\lambda'}}FG\epsilon_{\widetilde{\lambda'}}$. We give an admittedly rather restrictive criterion to ensure the equality $D_\Theta^{\lambda'\lambda}(\mathcal{C}) = D_\Theta^{\lambda'\lambda}(E_\Theta^{\lambda'})$, for a Θ-subalgebra $\mathcal{C} \subseteq E_\Theta^{\lambda'}$, and an analogous statement for $E_F^{\widetilde{\lambda'}}$.

(9.11) Proposition. We keep the notation of Section (9.10).
a) Let $\alpha_1^{\lambda'\lambda} \in \mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G)$ be as in Remark (5.7) and let $\mathcal{C} \subseteq E_\Theta^{\lambda'}$ be a Θ-subalgebra. Then we have

$$E_\Theta^{\lambda'} \cdot \alpha_1^{\lambda'\lambda} = \mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G),$$

and $D_\Theta^{\lambda'\lambda}(\mathcal{C}) = D_\Theta^{\lambda'\lambda}(E_\Theta^{\lambda'})$ holds if and only if $\mathcal{C} \cdot \alpha_1^{\lambda'\lambda} = E_\Theta^{\lambda'} \cdot \alpha_1^{\lambda'\lambda}$.
b) Let $\alpha_1^{\widetilde{\lambda'}\tilde{\lambda}} \in \mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G)$ be as in Remark (5.7) and let $\mathcal{C} \subseteq E_F^{\widetilde{\lambda'}}$ be an F-subalgebra. Then we have

$$E_F^{\widetilde{\lambda'}} \cdot \alpha_1^{\widetilde{\lambda'}\tilde{\lambda}} = \mathrm{Hom}_{FG}(\widetilde{\lambda'}^G, \tilde{\lambda}^G),$$

and $D_F^{\tilde{\lambda}'\tilde{\lambda}}(\mathcal{C}) = D_F^{\tilde{\lambda}'\tilde{\lambda}}(E_F^{\tilde{\lambda}'})$ holds if and only if $\mathcal{C} \cdot \alpha_1^{\tilde{\lambda}'\tilde{\lambda}} = E_F^{\tilde{\lambda}'} \cdot \alpha_1^{\tilde{\lambda}'\tilde{\lambda}}$.

Proof. Let $i' \in \mathcal{I}_{\lambda'}$ and $\alpha_{i'}^{\lambda'} \in \mathcal{A}_{\lambda'}$. Then by Section (1.7) and Definition (5.6) we have

$$\alpha_{i'}^{\lambda'} \cdot \alpha_1^{\lambda'\lambda} \colon \omega_1' \mapsto \sum_{j' \in \{1,\ldots,k'_{i'}\}} \lambda'(h'^{-1}_{i'j'}) \cdot \lambda(h''_{i'j'}) \cdot \omega_{ij},$$

where $i = \alpha_{H',H}(i')$, and $j \in \{1, \ldots, k_i\}$ depends on j'. For $i \in \mathcal{I}_\lambda$, by Corollary (5.11), we have $\alpha_{H',H}^{-1}(i) \subseteq \mathcal{I}_{\lambda'}$. Hence from $g'_{i'}h'_{i'j'} = h''_{i'j'}g_i h_{ij}$ and $g'_{i'} = h''_{i'}g_i h_{ij_1}$ for some $j_1 \in \{1, \ldots, k_i\}$, see Definition (5.6), we obtain

$$g_i^{-1}h''_{i'}g_i \cdot h_{ij_1} \cdot h'_{i'j'} = g_i^{-1}h''_{i'j'}g_i \cdot h_{ij}$$

and thus $\lambda'(h'^{-1}_{i'j'}) \cdot \lambda(h''_{i'j'}) = \zeta'_{i'} \cdot \lambda(h_{ij}^{-1})$. Hence, by Proposition (5.4), for $i \in \mathcal{I}_\lambda$ and $i' \in \alpha_{H',H}^{-1}(i)$ we have

$$\alpha_{i'}^{\lambda'} \cdot \alpha_1^{\lambda'\lambda} = \frac{k'_{i'}}{k_i} \cdot \zeta'_{i'} \cdot \alpha_i^{\lambda'\lambda} \in \mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G).$$

This shows the first assertion in a). By Remark (5.9) the quotient $\frac{k'_{i'}}{k_i} \in \mathbb{N}$ is coprime to p. Hence an analogous argument shows the first assertion in b).

For the second assertion in a), we only have to show sufficiency. Let $\alpha \in E_\Theta^{\lambda'}$. Because of $\mathcal{C} \cdot \alpha_1^{\lambda'\lambda} = \mathrm{Hom}_{\Theta G}(\lambda'^G, \lambda^G)$ there is $\beta \in \mathcal{C}$ such that $\beta \cdot \alpha_1^{\lambda'\lambda} = \alpha \cdot \alpha_1^{\lambda'\lambda}$. By Proposition (5.4) for $i \in \mathcal{I}_\lambda$ we have $\alpha_i^{\lambda'\lambda} = \alpha_1^{\lambda'\lambda} \cdot \alpha_i^\lambda$. Hence we have $(\alpha - \beta) \cdot \alpha_i^{\lambda'\lambda} = (\alpha - \beta) \cdot \alpha_1^{\lambda'\lambda} \cdot \alpha_i^\lambda = 0$. Thus $D_\Theta^{\lambda'\lambda}(\alpha) = D_\Theta^{\lambda'\lambda}(\beta)$. An analogous argument shows the second assertion in b). ♯

(9.12) For the special case $H = H'$ and $\lambda = \lambda' = 1$, the representations $D_\Theta^{\lambda\lambda}$ and $D_F^{\tilde{\lambda}\tilde{\lambda}}$ are the left regular representations of E_Θ and E_F, respectively. These hence are faithful representations. We have $\alpha_1^{\lambda\lambda} = \alpha_1^\lambda = \mathrm{id}_{\Theta_\lambda\Omega}$ and $\alpha_1^{\tilde{\lambda}\tilde{\lambda}} = \alpha_1^{\tilde{\lambda}} = \mathrm{id}_{F_{\tilde{\lambda}}\Omega}$, and the criteria in Proposition (9.11) boil down to the trivial statements that $\mathcal{C} = E_\Theta$ if and only if $\mathcal{C} \cdot \mathrm{id}_{\Theta\Omega} = E_\Theta \cdot \mathrm{id}_{\Theta\Omega}$, and $\mathcal{C} = E_F$ if and only if $\mathcal{C} \cdot \mathrm{id}_{F\Omega} = E_F \cdot \mathrm{id}_{F\Omega}$, respectively.

This special case has found practical applications, for example see [46] and also Section (19.2). Another generalisation of this special case different from the one given in Proposition (9.11) has been derived in [79, 46].

We conclude Section 9 with an observation concerning symmetric algebras, which proves useful in Section (19.2).

(9.13) Proposition. Let Θ be a perfect field, and let A be a symmetric finite-dimensional Θ-algebra. For $\varphi \in \mathrm{Irr}_\Theta(A)$ let $S_\varphi \in$ **mod**-A be the simple A-module affording φ, let $d_\varphi := \dim_\Theta(S_\varphi) \in \mathbb{N}_0$ and $f_\varphi := \dim_\Theta \mathrm{End}_A(S_\varphi) \in \mathbb{N}_0$, and let $P_\varphi \in$ **mod**-A be the projective cover of S_φ.

a) Then the multiplicity of the constituent S_φ in an A-module composition series of the regular A-module A equals $\frac{1}{f_\varphi} \cdot \dim_\Theta(P_\varphi)$.
b) The simple A-module S_φ is a projective A-module if and only if the above multiplicity of S_φ equals $\frac{d_\varphi}{f_\varphi}$, otherwise the multiplicity is at least $2 \cdot \frac{d_\varphi}{f_\varphi}$.

Proof. For $\psi \in \mathrm{Irr}_\Theta(A)$ let the *Cartan number* $c_{\varphi\psi} \in \mathbb{N}_0$ be the multiplicity of the constituent S_ψ in an A-module composition series of P_φ. Hence we have

$$\dim_\Theta(P_\varphi) = \sum_{\psi \in \mathrm{Irr}_\Theta(A)} c_{\varphi\psi} \cdot d_\psi.$$

Let $e_\psi \in A$ be a primitive idempotent such that $e_\psi A/\mathrm{rad}(e_\psi A) \cong S_\psi$ as A-modules. By Remark (6.5) we have, as Θ-vector spaces,

$$S_\psi e_\psi \cong \mathrm{Hom}_A(e_\psi A, S_\psi) \cong \mathrm{Hom}_A(e_\psi A/\mathrm{rad}(e_\psi A), S_\psi) \cong \mathrm{End}_A(S_\psi),$$

and hence $f_\psi = \dim_\Theta(S_\psi e_\psi)$. Furthermore, by Propositions (6.6) and (6.7) we have $\dim_\Theta(e_\varphi A \cdot e_\psi) = \dim_\Theta(P_\varphi \cdot e_\psi) = c_{\varphi\psi} \cdot f_\psi$. By [18, La.I.16.6] we have $\dim_\Theta(e_\varphi A e_\psi) = \dim_\Theta(e_\psi A e_\varphi)$. Hence we conclude $c_{\varphi\psi} \cdot f_\psi = c_{\psi\varphi} \cdot f_\varphi$.

As $A \cong \bigoplus_{\varphi \in \mathrm{Irr}_\Theta(A)} \left(\bigoplus_{i=1}^{\frac{d_\varphi}{f_\varphi}} P_\varphi \right)$ as A-modules, the multiplicity of S_φ in the regular A-module A is equal to

$$\sum_{\psi \in \mathrm{Irr}_\Theta(A)} \frac{d_\psi}{f_\psi} \cdot c_{\psi\varphi} = \sum_{\psi \in \mathrm{Irr}_\Theta(A)} \frac{d_\psi}{f_\varphi} \cdot c_{\varphi\psi} = \frac{1}{f_\varphi} \cdot \sum_{\psi \in \mathrm{Irr}_\Theta(A)} d_\psi \cdot c_{\varphi\psi} = \frac{1}{f_\varphi} \cdot \dim_\Theta(P_\varphi).$$

This proves the assertion in a), the assertion in b) is clear. ♯

(9.14) Remark. Proposition (9.13) can be applied to the regular module of the Θ-algebra $A := \epsilon\Theta G\epsilon$, where $\epsilon = \epsilon_1 \in \Theta H \subseteq \Theta G$, hence the situation of Section (9.12). If Θ is a finite field, then the MeatAxe finds the simple A-modules S_φ, for $\varphi \in \mathrm{Irr}_\Theta(A)$, the Θ-dimensions d_φ, the Θ-dimensions f_φ, and the multiplicities of the S_φ as constituents in an A-module composition series of the regular A-module. Hence from these standard MeatAxe results the projective simple A-modules can be determined, as well as the Θ-dimensions of the projective indecomposable A-modules, without actually decomposing the regular A-module into indecomposable summands. For such an application, see Section (19.2)

10 Enumeration of long orbits

In Section 10 we describe strategies to enumerate long and ultra-long orbits. Different variants of these are the main workhorses to collect the data necessary to compute structure constants matrices. We elaborate on the basic idea invented in [45], where the exposition given here is inspired by [57].

(10.1) Let $U \leq G$, let $\tilde{r} \in \mathbb{N}$ be the number of H-U-double cosets in G and $\mathcal{J} := \{1, \ldots, \tilde{r}\}$, and let $\{\tilde{g}_j \in G; j \in \mathcal{J}\}$ be a set of representatives of the H-U-double cosets in G, where $\tilde{g}_1 := 1_G$. For $j \in \mathcal{J}$ let $\tilde{\Omega}_j := \{H\tilde{g}_j u \in \Omega; u \in U\} \subseteq \Omega$ and $\tilde{\omega}_j := H\tilde{g}_j \in \tilde{\Omega}_j \subseteq \Omega$, where Ω still is the set $H|G$ of right cosets of H in G. Note that in Section (9.8) we have assumed additionally that $U \leq H$ holds, which we do not do here. In the sequel of Section 10 we do not distinguish between the G-set $\Omega = H|G$ and other G-sets isomorphic to Ω, such as sets of vectors.

Let $\mathcal{G} \subset G$ be a set of generators of G, and let $\mathcal{U} \subseteq U$ be a set of generators of U. We use a modification of the standard breadth-first orbit algorithm using $\mathcal{G}$ to enumerate the G-orbit Ω. Namely, whenever we compute an element $\omega \in \Omega$ which has not been encountered earlier in the orbit enumeration, then we first compute its whole U-orbit $\omega \cdot U \subseteq \Omega$, using $\mathcal{U}$, which hence is one of the $\tilde{\Omega}_j$, for $j \in \mathcal{J}$, and then proceed with the general orbit algorithm. Thus the G-orbit Ω is enumerated piecewise, U-orbit by U-orbit. For each U-orbit $\omega \cdot U \subseteq \Omega$ we encounter, we store a word in the set of generators $\mathcal{G}$ of G mapping the start point $\omega_1 \in \Omega$ to $\omega \in \Omega$. To actually enumerate long and ultra-long orbits we cannot afford to store all elements of Ω. Instead we only store a certain subset of Ω, which is done as follows.

We choose a subgroup $U_1 \leq U \leq G$, small enough such that the elements of U_1 can be enumerated explicitly, and objects representing the action of all of these elements on Ω can be stored; these objects could be permutations, or matrices if Ω is a set of vectors. Furthermore we choose a U_1-set Ξ_1, such that there is a homomorphism $q\colon \Omega_{U_1} \to \Xi_1$ of U_1-sets, where Ω_{U_1} denotes the U_1-set Ω defined by restricting the G-action to U_1. We do not assume that q is surjective, nor that U_1 acts transitively on Ξ_1, but we assume that $|\Xi_1|$ is small enough such that the elements of Ξ_1 can be enumerated explicitly, and all of them can be stored.

For each U_1-orbit $\xi \cdot U_1 \subseteq \Xi_1$ we choose an element $\xi_0 \in \xi \cdot U_1 \subseteq \Xi_1$, which is called the *strongly minimal* element of $\xi \cdot U_1$. For $\xi_0 \neq \xi' \in \xi \cdot U_1 \subseteq \Xi_1$ we store an element of U_1 mapping ξ' to ξ; in practice this means a pointer to that element of U_1. For the strongly minimal element ξ_0 we store the elements of $\mathrm{Stab}_{U_1}(\xi_0) \leq U_1$; in practice this again means pointers to the elements of $\mathrm{Stab}_{U_1}(\xi_0)$.

For a U_1-orbit $\omega \cdot U_1 \subseteq \Omega$ let $\xi_0 \in q(\omega \cdot U_1) = q(\omega) \cdot U_1 \in \Xi_1$ be the strongly minimal element of the U_1-orbit $q(\omega) \cdot U_1$. Then the set $q^{-1}(\xi_0) \subseteq \omega \cdot U_1 \subseteq \Omega$ is called the set of *weakly minimal* elements of the U_1-orbit $\omega \cdot U_1$. The weakly minimal elements of $\omega \cdot U_1$ are given as $\omega \cdot u \cdot \mathrm{Stab}_{U_1}(\xi_0) \subseteq \Omega$, where $u \in U_1$ is the element stored with $q(\omega) \in \Xi_1$ if $q(\omega) \in \Xi_1$ is not strongly minimal, or $u = 1$ if $q(\omega) \in \Xi_1$ is strongly minimal, while $\mathrm{Stab}_{U_1}(\xi_0)$ is stored with the strongly minimal element $\xi_0 \in \Xi_1$ belonging to $q(\omega) \cdot U_1$.

To enumerate Ω piecewise, U-orbit by U-orbit, we have to enumerate all the U-orbits $\tilde{\Omega}_j \subseteq \Omega$, for $j \in \mathcal{J}$, in turn. The latter again are enumerated piecewise,

U_1-orbit by U_1-orbit. We store exactly the weakly minimal elements of the U_1-orbits in Ω, together with the information to which of the U-orbits $\tilde{\Omega}_j$, for $j \in \mathcal{J}$, they belong.

Hence, during the enumeration of Ω, for $\omega \in \Omega$ we have to decide whether we already have encountered the U_1-orbit $\omega \cdot U_1 \subseteq \Omega$ earlier. To do this we compute $\omega \cdot u \in \Omega$, where $u \in U_1$ is the element stored with $q(\omega) \in \Xi_1$ if $q(\omega) \in \Xi_1$ is not strongly minimal, or $u = 1$ if $q(\omega) \in \Xi_1$ is strongly minimal. If $\omega \cdot u \in \Omega$ is already stored, then we have encountered $\omega \cdot U_1$ earlier. If $\omega \cdot u \in \Omega$ is not yet stored, then we store all the weakly minimal elements of $\omega \cdot U_1 \subseteq \Omega$.

(10.2) A few comments on this general strategy are in order.

a) As we also store the information to which of the U-orbits $\tilde{\Omega}_j$, for $j \in \mathcal{J}$, the weakly minimal elements belong, this is sufficient to compute orbit counting numbers with respect to $\Omega = \coprod_{j \in \mathcal{J}} \tilde{\Omega}_j$, see Definition (9.4).

b) To avoid to store too many elements of Ω, the proportion of weakly minimal elements of Ω should be small. Hence there is a tendency of choosing U_1 such that the U_1-orbits in Ξ_1 are long, at least on average; this makes the proportion of strongly minimal elements of Ξ_1 small. Furthermore, the sets $\omega \cdot u \cdot \mathrm{Stab}_{U_1}(\xi_0) \subseteq \Omega$ of weakly minimal elements of Ω tend to be smaller, if the stabilizers $\mathrm{Stab}_{U_1}(\xi_0) \leq U_1$ are small. Hence at best we have some large subgroup $U_1 \leq U$ and some large set Ξ_1, having a tendency to contain mostly regular U_1-orbits. Contrary to this, as we require both the elements of U_1 and of Ξ_1 to be explicitly enumerable, this poses upper bounds on how large U_1 and Ξ_1 might possibly be chosen.

c) If Ω is a set of vectors in an FG-module V, where F is a finite field, then a standard choice of the U_1-set Ξ_1 is as follows. Let $\hat{q}\colon V_{U_1} \to V_1$ be a homomorphism of FU_1-modules, let $\Xi_1 = V_1$ be the set of vectors in V_1 and let $q := \hat{q}_\Omega\colon \Omega_{U_1} \to \Xi_1$. Note that one possible pitfall here is that the zero vector $0 \in \Xi_1 = V_1$ is a strongly minimal element of Ξ_1 and we have $\mathrm{Stab}_{U_1}(0) = U_1$, hence all elements of $q^{-1}(0) \cap \Omega$ are weakly minimal elements of Ω and have to be stored.

d) To store and recover elements quickly we use a hashing technique. If the elements of Ω are vectors over some finite field F, one technique to find a suitable hash function is to view the entries of a vector as the coefficients of the $|F|$-adic expansion of an integer, and to take the latter as hash value. If this yields a hash function whose range is too large, compared to the expected number of weakly minimal elements of Ω, then we are content with only using part of the entries of the vectors to compute the hash function. Hash functions of that type have indeed proven to be suitable for the kind of computations done in the present work, although no attempt of a formal analysis has been made.

e) Depending on the example being under consideration, different amounts of memory to store an element of Ω are needed. Hence we have to make choices, fulfilling the requirements described above, such that we obtain a sufficiently small number of weakly minimal elements of Ω, which actually have to be stored. We do not go into a detailed analysis of memory requirements here; some numerical considerations of these issues are given in [57] for the example examined there, where in particular the problem, that the zero vector has a relatively large stabilizer, cannot be neglected.

(10.3) We briefly discuss implementational details.

a) The strategy described in Sections (10.1) and (10.2) has been implemented, for Ω being a set of vectors in a vector space V over a finite field, in the programs described in [45]. These are also parallelised in the sense that the different suborbits $\tilde{\Omega}_j \subseteq \Omega$ are treated in parallel. To use the full strength of this parallelisation, we have to ensure that $|\mathcal{J}|$ is large compared to the number of processors we want to use.

b) We also make use of modified versions of these programs [56], for Ω being a set of subspaces of a vector space V over a finite field F, where we both allow for 1-dimensional and higher dimensional F-subspaces. In the former case, the necessary modifications of the programs are straightforward.

In the latter case, let $\Omega \subseteq V$ consist of F-subspaces of V of F-dimension $d \in \mathbb{N}$. Then a standard choice of the U_1-set Ξ_1 is as follows. Let again $\hat{q}: V_{U_1} \to V_1$ be a homomorphism of FU_1-modules. One possible choice of Ξ_1 is the set of all F-subspaces of V_1. But it turns out that typically these sets are too large to be enumerated explicitly. Instead we assume $\dim_F \mathrm{im}\,(\hat{q}) \geq d+1$ and let Ξ_1 be the set of all F-subspaces of V_1 of F-dimension d. Still we have to ensure that the elements of Ξ_1 can be enumerated explicitly. Thus we only have a map of U_1-sets $\tilde{q} := \hat{q}_{\tilde{\Omega}}: \tilde{\Omega}_{U_1} \to \Xi_1$, where $\tilde{\Omega} := \{\omega \in \Omega; \hat{q}(\omega) \in \Xi_1\} \subseteq \Omega$, which might be a proper subset of Ω. Hence this only allows to treat the $\omega \in \tilde{\Omega}$ as described above, while all $\omega \in \Omega \setminus \tilde{\Omega}$ are simply defined to be weakly minimal and hence have to be stored.

To remedy this, we proceed as follows. Let $\hat{q}_i: V_{U_1} \to V_1^i$, for $i \in \{1, \ldots, k\}$ and $k \in \mathbb{N}$, be homomorphisms of FU_1-modules, such that $\dim_F \mathrm{im}\,(\hat{q}_i) \geq d+1$, for $i \in \{1, \ldots, k\}$. As becomes clear below, the different FU_1-homomorphisms $\hat{q}_i$ should be as independent from each other as possible. Hence we additionally require that

$$\mathrm{codim}_F(\bigcap_{i \in \{1,\ldots,k\}} \ker \hat{q}_i) = \sum_{i \in \{1,\ldots,k\}} \mathrm{codim}_F \ker(\hat{q}_i).$$

Let again Ξ_1^i be the set of all F-subspaces of V_1^i of F-dimension d, and $\tilde{\Omega}^i := \{\omega \in \Omega; \hat{q}_i(\omega) \in \Xi_1^i\} \subseteq \Omega$, as well as $\tilde{q}_i := (\hat{q}_i)_{\tilde{\Omega}^i}: (\tilde{\Omega}^i)_{U_1} \to \Xi_1^i$, for $i \in \{1, \ldots, k\}$. An element $\omega \in \Omega$ is processed as follows. If $\omega \in \tilde{\Omega}^1$ then we may and do use $\tilde{q}_1$ as described above. If $\omega \notin \tilde{\Omega}^1$, but $\omega \in \tilde{\Omega}^2$, then we may and do use $\tilde{q}_2$, and so on. Hence only the elements $\omega \in \Omega \setminus (\bigcup_{i \in \{1,\ldots,k\}} \tilde{\Omega}^i)$, cannot be treated this way, and are simply defined to be weakly minimal and hence have to be stored. Thus we have to choose k big enough such that we can afford to do so. A more detailed description of this idea, together with some numerical considerations for the example examined there, is given in [57].

(10.4) For ultra-long orbits Ω the assumptions in Section (10.1) on the explicit enumerability of the subgroup $U_1 < U$ and the U_1-set Ξ_1 turn out to be too strict. We still follow the strategy to enumerate Ω piecewise, U-orbit by U-orbit, but instead of one subgroup $U_1 < U$ we use a whole chain of subgroups iteratively.

Let $\{1\} := U_0 < U_1 < U_2 < \ldots < U_k < U_{k+1} =: U$ be a chain of subgroups, for some $k \in \mathbb{N}$, given by sets of generators $\mathcal{U}_i$, respectively. Hence Section (10.1) deals with the case $k = 1$. For all $i \in \{1, \ldots, k\}$ let Ξ_i be a U_i-set, such that there are homomorphisms $q_{i,i-1}\colon (\Xi_i)_{U_{i-1}} \to \Xi_{i-1}$ of U_{i-1}-sets, for $i \in \{1, \ldots, k+1\}$, where we let $\Xi_{k+1} := \Omega$ and Ξ_0 is the trivial U_0-set with $|\Xi_0| = 1$.

For $i \in \{1, \ldots, k\}$ let $\mathcal{T}_i \subseteq U_i$ be a set of representatives of the left cosets $U_i|U_{i-1}$ of U_{i-1} in U_i. Thus $u \in U_k$ can be written as $u = t_k(u) \cdot t_{k-1}(u) \cdot \cdots \cdot t_1(u)$, where $t_i(u) \in \mathcal{T}_i$, for $i \in \{1, \ldots, k\}$. We assume that the sets $\mathcal{T}_i \subseteq U_i$ can be enumerated explicitly, but we do not assume that this can be done for the left cosets $U|U_k$ of $U = U_{k+1}$ in U_k.

For $i \in \{0, \ldots, k+1\}$ we by induction define certain distinguished elements of the U_i-orbits in Ξ_i. For $i \in \{0, \ldots, k\}$ we define *strongly minimal* elements such that each U_i-orbit in Ξ_i contains exactly one strongly minimal element, while for $i \in \{1, \ldots, k+1\}$ we define *weakly minimal* elements of the U_i-orbits in Ξ_i, where each U_i-orbit in Ξ_i contains at least one, but possibly more than one, weakly minimal element. For $i = 0$ the U_0-set Ξ_0 is a U_0-orbit and Ξ_0 is the set of strongly minimal elements of Ξ_0.

Let $i \in \{1, \ldots, k\}$. By induction we may assume that we have already defined the strongly minimal elements of Ξ_{i-1}. Let $\xi \in \Xi_i$. Using the set of coset representatives $\mathcal{T}_i$ we obtain $\xi \cdot U_i = \coprod_{t \in \mathcal{T}_i'} \xi \cdot t \cdot U_{i-1} \subseteq \Xi_i$, where t runs through a suitable subset $\mathcal{T}_i' \subseteq \mathcal{T}_i$. For $t \in \mathcal{T}_i'$ let $\tilde{\xi}_{t,0} \in \Xi_{i-1}$ be the strongly minimal element of the U_{i-1}-orbit $q_{i,i-1}(\xi \cdot t \cdot U_{i-1}) = q_{i,i-1}(\xi \cdot t) \cdot U_{i-1} \subseteq \Xi_{i-1}$. The set of weakly minimal elements of $\xi \cdot U_i$ is defined as $q_{i,i-1}^{-1}(\{\tilde{\xi}_{t,0}; t \in \mathcal{T}_i'\}) \subseteq \xi \cdot U_i \subseteq \Xi_i$. In particular, for $i = 1$ this means that all elements of Ξ_1 are weakly minimal. We choose one of the weakly minimal elements of $\xi \cdot U_i \subseteq \Xi_i$ as the strongly minimal element $\xi_0 \in \xi \cdot U_i$, and for each weakly minimal element $\xi_0 \neq \xi' \in \xi \cdot U_i \subseteq \Xi_i$ we store an element of U_i, as a word in the set of generators $\mathcal{U}_i$, mapping ξ' to ξ_0, while for the strongly minimal element ξ_0 we store a set of generators of $\mathrm{Stab}_{U_i}(\xi_0)$, again as a set of words in the set of generators $\mathcal{U}_i$.

Let finally $i = k+1$, hence we have $U_{k+1} = U$ and $\Xi_{k+1} = \Omega$. The set of weakly minimal elements of a U_k-orbit $\xi \cdot U_k \subseteq \Xi_{k+1}$ is defined as the set $q^{-1}(\tilde{\xi}_0) \subseteq \xi \cdot U_k$, where $\tilde{\xi}_0 \in q_{k+1,k}(\xi \cdot U_k) = q_{k+1,k}(\xi) \cdot U_k \subseteq \Xi_k$ is the strongly minimal element of the U_k-orbit $q_{k+1,k}(\xi) \cdot U_k \subseteq \Xi_k$. The set of weakly minimal elements of a U_{k+1}-orbit $\xi \cdot U_{k+1} \subseteq \Xi_{k+1}$ is defined as the union of the sets of weakly minimal elements of the U_k-orbits contained in $\xi \cdot U_{k+1}$.

By induction, for $i \in \{1, \ldots, k+1\}$, each U_i-orbit in Ξ_i encountered is enumerated piecewise, U_{i-1}-orbit by U_{i-1}-orbit. Exactly the weakly minimal elements are stored, where for $i \in \{1, \ldots, k\}$ we store the additional information as described above, while for $i = k+1$ we additionally store the information to which of the U-orbits $\tilde{\Omega}_j$, for $j \in \mathcal{J}$, the weakly minimal elements belong. Finally, we store elements of G mapping the start point $\omega_1 \in \Omega$ to $\tilde{\omega}_j \in \tilde{\Omega}_j \subseteq \Omega$, for $j \in \mathcal{J}$, as well as elements of U mapping $\tilde{\omega}_j \in \tilde{\Omega}_j$ to representatives of the U_k-orbits in $\tilde{\Omega}_j$. These elements are stored as words in the sets of generators $\mathcal{G}$ and $\mathcal{U}$, respectively.

During the enumeration of Ω, for $\omega \in \Omega$ we have to decide whether we have already encountered the U_k-orbit $\omega \cdot U_k \subseteq \Omega$ earlier. To do this, as in Section (10.1), we compute $\omega \cdot u \in \Omega$, where $u \in U_k$ is the element stored with $q_{k+1,k}(\omega) \in \Xi_k$ if $q_{k+1,k}(\omega) \in \Xi_k$ is not strongly minimal, or $u = 1$ if $q_{k+1,k}(\omega) \in \Xi_k$ is strongly minimal. If $\omega \cdot u \in \Omega$ is already stored, then we have encountered $\omega \cdot U_k$ earlier. If $\omega \cdot u \in \Omega$ is not yet stored, then we store all the weakly minimal elements of $\omega \cdot U_k \subseteq \Omega$, which are again given as $\omega \cdot u \cdot \mathrm{Stab}_{U_k}(\xi_0) \subseteq \Omega$, where $\mathrm{Stab}_{U_k}(\xi_0)$ is stored with the strongly minimal element $\xi_0 \in \Xi_k$ belonging to $q_{k+1,k}(\omega) \cdot U_k \subseteq \Xi_k$.

(10.5) Again, a few comments on this general strategy are in order.

a) Let $i \in \{1, \ldots, k\}$. Deviating from the strategy described in Section (10.1), we do not store the weakly minimal elements in Ξ_i in advance, and we even do not store all of them. We only store those weakly minimal elements which actually belong to $q_{i+1,i} \circ \cdots \circ q_{k,k-1} \circ q_{k+1,k}(\Omega_{U_i})$. Such an element is stored if a preimage of it is encountered during the enumeration of Ω.

b) To find the sets $\mathcal{T}_i$ of representatives of the left cosets $U_i|U_{i-1}$ of U_{i-1} in U_i, for $i \in \{1, \ldots, k\}$, as a set of words in the set of generators $\mathcal{U}_i$, we proceed as follows. Let Ξ'_i be the regular transitive U_i-set, hence we have $\Xi'_i \cong U_i$ as U_i-sets. For $i = 1$ we use a standard breadth-first orbit algorithm using $\mathcal{U}_1$ to enumerate the elements of $\Xi'_1 \cong U_1 =: \mathcal{T}_1$.

Let by induction $i \geq 2$. We enumerate Ξ'_i piecewise, U_{i-1}-orbit by U_{i-1}-orbit, but using a *left orbit algorithm*. Let $\xi_1 \in \Xi'_i$ be fixed. Using the isomorphism $\Xi'_i \cong U_i$ of U_i-sets, the element $\xi_1 \in \Xi'_i$ corresponds to $1 \in U_i$. Whenever we compute an element $\xi \in \Xi'_i$, whose U_{i-1}-orbit $\xi \cdot U_{i-1} \subseteq \Xi'_i$ has not been encountered before, we store an element $u_\xi \in U_i$, as a word in the set of generators $\mathcal{U}_i$, mapping ξ_1 to ξ, where we let $u_{\xi_1} := 1$, and then enumerate the U_{i-1}-orbit $\xi \cdot U_{i-1} \subseteq \Xi'_i$. Thus we obtain a sequence $\{u_{\xi_1}, u_{\xi_2}, \ldots\} \subseteq U_i$, and the left orbit algorithm is now performed by running through this list, multiplying from the left with the elements of $\mathcal{U}_i$, hence forming successively the products $u \cdot u_{\xi_j}$, for $u \in \mathcal{U}_i$, and computing the U_{i-1}-orbits $\xi_1 \cdot u \cdot u_{\xi_j} \cdot U_{i-1} \subseteq \Xi'_i$. As $\Xi'_i \cong U_i$ as U_i-sets, on termination the set $\mathcal{T}_i := \{u_{\xi_1}, u_{\xi_2}, \ldots\} \subseteq U_i$ is a set of representatives of the left cosets $U_i|U_{i-1}$ of U_{i-1} in U_i.

To actually do these orbit enumerations we in turn may use the strategies described in Section (10.1) or in Section (10.4), for the truncated subgroup chain $U_0 < U_1 < \ldots < U_{i-1} < U_i$. Having built up the U_{i-1}-orbit structure on Ξ_i', for $i \in \{1, \ldots, k\}$, this can be used to compute in U_k, hence multiply or invert elements of U_k, and writing the results again as a product of elements of the $\mathcal{T}_i$.

c) Let $\xi_0 \in \Xi_i$ be a strongly minimal element. A set of generators of $\mathrm{Stab}_{U_i}(\xi_0)$ is found as follows. We may assume $\xi_0 \cdot U_i = \coprod_{t \in \mathcal{T}_i'} \xi_0 \cdot t \cdot U_{i-1} \subseteq \Xi_i$, where $\mathcal{T}_i' \subseteq \mathcal{T}_i$ is a suitable subset as in Section (10.4). If $t \cdot u \in \mathrm{Stab}_{U_i}(\xi_0)$, where $t \in \mathcal{T}_i$ and $u \in U_{i-1}$, then $\xi_0 \cdot t \in \xi_0 \cdot U_{i-1} \subseteq \Xi_i$. Hence we only have to consider the coset representatives $t \in \mathcal{T}_i'' := \{t \in \mathcal{T}_i; \xi_0 \cdot t \in \xi_0 \cdot U_{i-1}\}$. Conversely, for $t \in \mathcal{T}_i''$ there is a $u_t \in U_{i-1}$ such that $\xi_0 \cdot t \cdot u_t = \xi_0$, and we have $\mathrm{Stab}_{U_i}(\xi_0) \cap (t \cdot U_{i-1}) = t \cdot u_t \cdot \mathrm{Stab}_{U_{i-1}}(\xi_0)$. Hence we have to find the sets $\mathcal{T}_i'' \subseteq \mathcal{T}_i$, the elements $u_t \in U_{i-1}$, and a set of generators of $\mathrm{Stab}_{U_{i-1}}(\xi_0)$, where we have $\mathrm{Stab}_{U_{i-1}}(\xi_0) \leq \mathrm{Stab}_{U_{i-1}}(q_{i,i-1}(\xi_0))$, and the latter group is known by induction.

d) As we do not assume that a set of representatives of the left cosets $U_{k+1}|U_k$ of U_k in $U_{k+1} = U$ can be enumerated explicitly, the machinery using regular transitive sets described above cannot immediately be extended to U. Occasionally, we use another U-set Ξ_{k+1}', which we choose to be faithful, together with randomised Schreier-Sims techniques, to obtain results on certain subgroups of U, such as stabilizers $\mathrm{Stab}_U(\omega)$, for $\omega \in \Omega$. This tends to be helpful to find break conditions, where some U-orbit $\omega \cdot U \subseteq \Omega$ is too long to be enumerated completely, but where it suffices to know some substantial part of it, see Section (17.8).

e) If Ω is a set of vectors in an FG-module V, where F is a finite field, then again a standard choice of the U_i-sets Ξ_i, for $i \in \{1, \ldots, k\}$, is as follows. Let $V_{k+1} := V$, and let $\hat{q}_{i+1,i} \colon (V_{i+1})_{U_i} \to V_i$ be homomorphisms of FU_i-modules, let $\Xi_i = V_i$ be the set of the vectors in V_i, and let $q_{i+1,i} := (\hat{q}_{i+1,i})_{\Xi_{i+1}} \colon (\Xi_{i+1})_{U_i} \to \Xi_i$. Furthermore, a standard choice of the regular transitive set Ξ_i', for $i \in \{1, \ldots, k\}$, is a regular U_i-orbit in the FU_i-module $(V_j)_{U_i}$, for some $j \in \{i, \ldots, k+1\}$.

(10.6) The strategy described in Sections (10.4) and (10.5) has been implemented in GAP, for Ω being a set of vectors in an FG-module V, where F is a finite field. We make heavy use of the fast arithmetic for vectors over finite fields, available in GAP, which employs the techniques also used in the MeatAxe. Altogether, the relevant GAP code implementing the hashing techniques, computations in U_k using the regular transitive U_i-sets, for $i \in \{1, \ldots, k\}$, the different necessary orbit enumeration algorithms and the randomised Schreier-Sims algorithms, keeping track of transversals and subgroup generators as words in the given sets of generators, amounts to some 2000 lines of GAP code. A more detailed description of this, including some numerical considerations of memory requirements and running times, will be given elsewhere [54].

III Explicit results

For all of Part III, let $\lambda = 1$ be the trivial character of the subgroup $H \leq G$ under consideration, and let K be as in Section 3, depending on the group under consideration. We keep the notation of Sections 1 and 3. Occasionally we need another subgroup $H' \leq H$, where we keep the notation of Section 5 and let $\lambda' = 1$ as well.

11 The database

(11.1) We have compiled a database containing the character tables of the endomorphism rings of the multiplicity-free permutation representations of the sporadic simple groups, their automorphism groups and their Schur covering groups, see [7]. Up to now, there still is a single exceptional case, where the character table is not known, namely for $G := 2.B$ and $H := Fi_{23}$. As some partial information is already known, see Section (17.11), there is hope that this case will be successfully treated completely in the near future, see Section (17.12). Furthermore, an examination of the multiplicity-free permutation representations of the bicyclic extensions of the sporadic simple groups currently is under way [7].

In the present work we provide proofs for the cases for the sporadic simple groups, their automorphism groups and their Schur covering groups where $n = |\Omega| \geq 10^7$, see Table 7. In Sections 12–17 we deal with the different groups G and subgroups H as indicated in Table 7. But before doing so, in the remaining parts of Section (11.1) we comment on the smaller cases, on earlier results used and on the explicit determination of the Fitting correspondence.

a) The multiplicity-free permutation representations of the sporadic simple groups, their automorphism groups, their Schur covering groups and their bicyclic extensions have been classified in [6, 43, 5].

b) The work of systematically computing structure constants matrices related to the sporadic simple groups and their automorphism groups has been begun in [68]. In the thesis [32], which the author has had the opportunity to co-supervise, these and other earlier results, scattered in the literature, have been collected. Furthermore, the remaining cases of multiplicity-free permutation actions of the sporadic simple groups and their automorphism groups for $n = |\Omega| \leq 10^7$ have been dealt with. We comment briefly on the methods used in [32], which we refer to for more details.

For the sporadic simple groups up to group order 10^9, hence the largest one being McL, and a few of their automorphism groups, the tables of marks are known and available in GAP. Together with the corresponding table of marks, GAP provides the smallest faithful permutation representation of the corresponding group, given in terms of a set of standard generators in the sense of [81], and for each conjugacy class of subgroups a set of generators of a representative of this class is given as words in the set of standard generators. Using this information, and the programs dealing with permutation groups available in GAP, it is straightforward to compute the necessary permutation representations and sufficiently many of the related structure constants matrices. Thus by the technique described in Section (8.2) the character tables of the corresponding endomorphism rings can be determined.

For quite a few of these cases, this strategy is not sufficient. Instead we have to apply other standard techniques from the MeatAxe to construct the necessary permutation representations, such as finding orbits of vectors, as is implemented in the ZVP program of the MeatAxe. Here we use the database [83] as a source of explicitly given representations for the sporadic simple groups and related groups, where these as well are given in terms of sets of standard generators in the sense of [81], and as a source of words describing sets of generators of maximal subgroups.

c) For the Schur covering groups of the sporadic simple groups, the corresponding permutation representations have been constructed in [43], with the exception of the cases $G := 3.Fi_{24}'$ and $H := O_{10}^-(2)$ as well as $G := 2.B$ and $H := Fi_{23}$, and sufficiently many of the related structure constants matrices have been computed [41]. For the case $G := 3.Fi_{24}'$ and $H := O_{10}^-(2)$ see Section (12.2), for the case $G := 2.B$ and $H := Fi_{23}$ see Section (17.11).

(11.2) We briefly comment on the cases in Table 7 related to Fi_{22}, to Fi_{23}, to Co_1 and to M. Here, either explicit permutations are known, and hence all the structure constants matrices can be computed using Remark (1.19), or part of the structure constants matrices have been computed elsewhere. In all of these cases the character tables of the endomorphism rings can be computed using the technique described in Section (8.2), since the known structure constants matrices are sufficient to get 1-dimensional eigenspaces.

a) Let $G := 3.Fi_{22}$ and $H := {}^2F_4(2)'$. By [43], explicit permutations are known, as well as the character table of the endomorphism ring [41].

b) Let $G := Fi_{23}$ and $H := S_8(2)$. The index parameters and the structure constants matrices for the two smallest non-trivial suborbits Ω_2 and Ω_3 with $k_2 = 2295$ and $k_3 = 13056$ have been computed in [42].

c) Let $G := Fi_{23}$ and $H := 2^{11}.M_{23}$. The index parameters and the structure constants matrix for the smallest non-trivial suborbit Ω_2 with $k_2 = 506$ have been computed in [42].

d) Let $G := Co_1$ and $H := 2_+^{1+8}.O_8^+(2)$. The index parameters and the structure constants matrix for the smallest non-trivial suborbit Ω_2 with $k_2 = 270$ have been computed in [34].

e) Let $G := 2.Co_1$ and $H := Co_3$. By [43], explicit permutations are known, as well as the character table of the endomorphism ring [41].

f) Let $G := M$ and $H := 2.B$. The index parameters and the structure constants matrix for the smallest non-trivial suborbit Ω_2 with $k_2 = 27\,143\,910\,000$ have

Table 7: Large multiplicity-free permutation representations.

G	H	n	r	Section
$3.Fi_{22}$	${}^2F_4(2)'$	10 777 536	25	(11.2)
HN	A_{11}	13 680 000	19	(13.2)
HN	$U_3(8).3_1$	16 500 000	19	(13.4)
$HN.2$	S_{11}	13 680 000	17	(13.1)
$HN.2$	$U_3(8).6$	16 500 000	15	(13.3)
Ly	$3.McL$	19 212 250	8	(14.1)
Th	${}^3D_4(2).3$	143 127 000	11	(15.1)
Th	$2^5.L_5(2)$	283 599 225	11	(15.2)
Fi_{23}	$S_8(2)$	86 316 516	13	(11.2)
Fi_{23}	$2^{11}.M_{23}$	195 747 435	16	(11.2)
Co_1	$2_+^{1+8}.O_8^+(2)$	46 621 575	11	(11.2)
$2.Co_1$	Co_3	16 773 120	12	(11.2)
J_4	$2^{11}{:}\,M_{24}$	173 067 389	7	(16.1)
J_4	$2^{11}{:}\,M_{23}$	4 153 617 336	11	(16.2)
Fi_{24}'	$O_{10}^-(2)$	50 177 360 142	17	(12.1)
Fi_{24}'	$3^7.O_7(3)$	125 168 046 080	18	(12.4)
$Fi_{24}'.2$	$O_{10}^-(2).2$	50 177 360 142	17	(12.1)
$Fi_{24}'.2$	$O_{10}^-(2)$	100 354 720 284	34	(12.1)
$Fi_{24}'.2$	$3^7.O_7(3).2$	125 168 046 080	17	(12.3)
$3.Fi_{24}'$	$O_{10}^-(2)$	150 532 080 426	43	(12.2)
B	$2.{}^2E_6(2).2$	13 571 955 000	5	(17.1)
B	$2.{}^2E_6(2)$	27 143 910 000	8	(17.1)
B	$2^{1+22}.Co_2$	11 707 448 673 375	10	(17.2) ff.
B	Fi_{23}	1 015 970 529 280 000	23	(17.6) ff.
$2.B$	Fi_{23}	2 031 941 058 560 000	34	(17.11) f.
M	$2.B$	97 239 461 142 009 186 000	9	(11.2)

been computed in [64].

(11.3) The character tables of endomorphism rings E_K contained in the database, and in particular the indicated Fitting correspondence from the characters $\varphi \in \mathrm{Irr}(E_K)$ to the irreducible characters $\chi_\varphi \in \mathrm{Irr}_K^1(G)$ of the corresponding group G, have been compiled taking the following point of view into account.

If a set $\mathcal{G} \subseteq G$ of standard generators of G in the sense of [81] is given, then the conjugacy classes $\mathcal{C}l(G)$ can be defined by giving representatives as words in the set $\mathcal{G}$ of generators. Such sets of standard generators and definitions of the conjugacy classes are available for the sporadic simple groups, their automorphism groups and their Schur covering groups in [83]. This hence also defines the irreducible characters $\mathrm{Irr}_K(G)$ uniquely. Note that the character table $\mathcal{X}$ of $\mathrm{Irr}_K(G)$ alone leaves ambiguities which are described by the group $\mathrm{Aut}(\mathrm{Irr}_K(G))$ of table automorphisms of $\mathrm{Irr}_K(G)$, see Definition (8.5).

To determine the Fitting correspondence, we hence first find all the admissible candidate cases $\mathcal{F} = \mathcal{F}_1$ using the technique described in Section (8.4). The set $\mathcal{F}$ is a union of orbits under the action of $\mathrm{Aut}(\mathrm{Irr}_K(E_K)) \times \mathrm{Aut}(\mathrm{Irr}_K^1(G))$, see Remark (8.6), hence also is a union of orbits under the action of the possibly strictly smaller group $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$. To obey the point of view introduced above, we have to determine which of the $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$-orbits in $\mathcal{F}$ gives the Fitting correspondence, but then we are allowed to choose freely within this orbit. In particular, we are done if $\mathcal{F}$ consists of exactly one such orbit.

Actually, for a few of the cases dealt with the determination of the Fitting correspondence in the above sense would pose rather hard problems. Hence we loosen our assumptions as follows. Let $\{H_1, \ldots, H_k\}$, for some $k \in \mathbb{N}$, be a set of representatives of the conjugacy classes of proper subgroups affording a multiplicity-free permutation character $1_{H_i}^G$. Furthermore, let $\mathrm{Aut}(\mathrm{Irr}_K^{1_{H_i}}(G))$ and $\mathrm{Aut}(\mathrm{Irr}_K(E_K^{1_{H_i}}))$, for $i \in \{1, \ldots, k\}$, be the corresponding table automorphism groups of $\mathrm{Irr}_K^{1_{H_i}}(G)$ and $\mathrm{Irr}_K(E_K^{1_{H_i}})$, respectively, see Definition (8.5). We consider the sets $\mathcal{F}^i$ of admissible candidate cases for the Fitting correspondence for the subgroups H_i, for $i \in \{1, \ldots, k\}$, at the same time. Namely,

$$\left(\prod_{i=1}^{k} \mathrm{Aut}(\mathrm{Irr}_K(E_K^{1_{H_i}}))\right) \times \left(\bigcap_{i=1}^{k} \mathrm{Aut}(\mathrm{Irr}_K^{1_{H_i}}(G))\right)$$

acts on $\prod_{i=1}^k \mathcal{F}^i$, where the first direct factor acts componentwise, while the second one acts diagonally. Still, we have to determine which of the orbits in $\prod_{i=1}^k \mathcal{F}^i$ under the action of the above group gives the k-tuple of Fitting correspondences, but then we are allowed to choose freely within this orbit. In particular, we are done if $\prod_{i=1}^k \mathcal{F}^i$ consists of exactly one such orbit.

(11.4) We comment on the computations involved in the explicit determination of the Fitting correspondence. The most complicated case is dealt with in

Section (11.5).

a) Actually, for the cases dealt with the technique described in Section (8.4), applied to a fixed subgroup H, rather often yields a set $\mathcal{F}$ of admissible candidate cases consisting of a single $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$-orbit, or $\mathcal{F}$ even consists of a unique solution. In particular, the latter case occurs if the degrees $\chi(1)$ for $\chi \in \mathrm{Irr}^1_K(G)$ are pairwise different. Furthermore, Corollary (5.13) and Remarks (5.15), (5.16) and (5.18) can be applied to delete inadmissible orbits.

b) For the remaining cases of those groups G whose tables of marks are known, see Section (11.1), we use one of the faithful permutation representations of G and the programs dealing with permutation groups available in GAP to find representatives of the conjugacy classes $Cl(G)$, and to find the matrices $\Gamma := [|C \cap Hg_i|; i \in \mathcal{I}, C \in \mathcal{C}l(G)] \in \mathbb{Z}^{|\mathcal{I}| \times |\mathcal{C}l(G)|}$, see Definition (3.19), explicitly. The only general technique known to the author to find the numbers $|C \cap Hg_i| \in \mathbb{N}_0$ is to fix $C \in \mathcal{C}l(G)$ and $i \in \mathcal{I}$, to run through the elements of $h \in H$ explicitly and to find out to which conjugacy class $C \in \mathcal{C}l(G)$ the element hg_i belongs, using conjugacy tests in G. This admittedly not too clever strategy turns out to be doable for the present cases.

This also works for $G := HS.2$, as the relevant subgroups turn out to be $H_1 := 5^{1+2}_{+}{:}\,[2^5]$ and $H_2 := M_{11}$. Hence we have $|H_1| = 4000$ and $r_1 = 15$ as well as $|H_2| = 7920$ and $r_2 = 17$. The transitive permutation representation of G on 100 points is available in [83], in terms of a set of standard generators of G in the sense of [81]. Using the programs dealing with permutation groups available in GAP, we find the subgroups H_1 and H_2, representatives of the conjugacy classes $Cl(G)$, and the matrices $\Gamma \in \mathbb{Z}^{|\mathcal{I}| \times |\mathcal{C}l(G)|}$.

The same strategy works for $3.M_{22}$ and $6.M_{22}$, where we use the table of marks of M_{22} available in GAP and the permutation representations available in [83].

c) Let $G := Ru$ and $H := (2^2 \times Sz(8)){:}\,3$. By the technique described in Section (8.4), we find 2 admissible candidate cases for the Fitting correspondence. They differ in the preimages of $34944a/b$. For each of the other subgroups $\tilde{H}$ of G affording multiplicity-free permutation characters, namely ${}^2F_4(2)'.2$ and ${}^2F_4(2)'$ the technique described in Section (8.4) yields a set $\mathcal{F}^{\tilde{H}}$ of admissible candidate cases for the Fitting correspondence, which is exactly one orbit under the action of the corresponding table automorphism group $\mathrm{Aut}(\mathrm{Irr}_K(E_K^{1_{\tilde{H}}}))$. Hence we are allowed to use the action of the full group $\mathrm{Aut}(\mathrm{Irr}^1_K(G))$ of table automorphisms on the set of admissible candidate cases for the Fitting correspondence for E_K. Using GAP we find that the image of the action of $\mathrm{Aut}(\mathrm{Irr}^1_K(G))$ on the characters in $\mathrm{Irr}^1_K(G)$ is generated by the element $(34944a, 34944b)$. Hence we are allowed to choose freely from the set of admissible candidate cases for the subgroup H.

d) Let $G := ON$ and $H := L_3(7).2$, as well as $H' := L_3(7)$. For both cases, by the technique described in Section (8.4), we find 4 admissible candidate cases each for the Fitting correspondence. They differ in the preimages of $26752a^{\pm}, 52668a^{\pm} \in \mathrm{Irr}_K(E_K^{1_H}) \subseteq \mathrm{Irr}_K(E_K^{1_{H'}})$. Hence, as the assumptions of

Section (5.15) are fulfilled, the Fitting correspondence for $E_K^{1_{H'}}$ is determined by the one for $E_K^{1_H}$. As H and H' are the only subgroups of G affording multiplicity-free permutation characters, we are allowed to use the action of the full group $\mathrm{Aut}(\mathrm{Irr}_K^{1_H}(G))$ of table automorphisms on the set of admissible candidate cases for the Fitting correspondence for $E_K^{1_H}$. Using GAP we find that the image of the action of $\mathrm{Aut}(\mathrm{Irr}_K^{1_H}(G))$ on the characters in $\mathrm{Irr}_K^{1_H}(G)$ is generated by the elements $(26752a^+, 26752a^-)$ and $(52668a^+, 52668a^-)$. Hence we are allowed to choose freely from the set of admissible candidate cases for the subgroup H.

(11.5) Let $G := HN$ and $H := U_3(8).3_1$, see Section (13.4) and in particular Table 17. By the technique described in Section (8.4) we find 16 admissible candidate cases for the Fitting correspondence. These are given by

$$\begin{array}{rcl} \{\varphi_{5'}, \varphi_{5''}\} & \rightarrow & \{35112a, 35112b\} \\ \{\varphi_{8'}, \varphi_{8''}\} & \rightarrow & \{374528a, 374528b\} \\ \{\varphi_{12'}, \varphi_{12''}\} & \rightarrow & \{656250a, 656250b\} \\ \{\varphi_{13'}, \varphi_{13''}\} & \rightarrow & \{1361920b, 1361920c\} \end{array}$$

while for the other characters in $\mathrm{Irr}(E_K)$, their Fitting correspondent is uniquely determined and as shown in Table 17.

The group $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$ of table automorphisms of $\mathrm{Irr}_K(E_K)$, being defined by its action on the columns of the character table Φ, is generated by the set $\{(5', 5''), (8', 8''), (12', 12''), (13', 13'')\}$. Hence we have $|\mathrm{Aut}(\mathrm{Irr}_K(E_K))| = 16$. The image of the action of $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$ on the characters in $\mathrm{Irr}(E_K)$ is generated by $\{(\varphi_{5'}, \varphi_{5''})(\varphi_{8'}, \varphi_{8''}), (\varphi_{9'}, \varphi_{9''})(\varphi_{12'}, \varphi_{12''})\}$, thus the image has order 4.

For each of the other subgroups $\tilde{H}$ of G affording multiplicity-free permutation characters, namely A_{12} and A_{11} as well as $2.HS.2$, the technique described in Section (8.4) yields a set $\mathcal{F}^{\tilde{H}}$ of admissible candidate cases for the Fitting correspondence, which is exactly one orbit under the action of the corresponding table automorphism group $\mathrm{Aut}(\mathrm{Irr}_K(E_K^{1_{\tilde{H}}}))$. Hence we are allowed to use the action of the full group $\mathrm{Aut}(\mathrm{Irr}_K^1(G))$ of table automorphisms on the set of admissible candidate cases for the Fitting correspondence for E_K. Using GAP we find that the image of the action of $\mathrm{Aut}(\mathrm{Irr}_K^1(G))$ on the characters in $\mathrm{Irr}_K^1(G)$ is generated by $\{\pi_1, \pi_2, \pi_3\}$, where $\pi_1 := (35112a, 35112b)(374528a, 374528b)$ as well as $\pi_2 := (656250a, 656250b)$ and $\pi_3 := (1361920b, 1361920c)$.

Hence we may choose the Fitting correspondence, using $\mathrm{Aut}(\mathrm{Irr}_K(E_K))$, to be $\varphi_{5'} \mapsto 35112a$ and $\varphi_{5''} \mapsto 35112b$, and using $\mathrm{Aut}(\mathrm{Irr}_K^1(G))$ we may furthermore choose $\varphi_{12'} \mapsto 656250a$ and $\varphi_{12''} \mapsto 656250b$ as well as $\varphi_{13'} \mapsto 1361920b$ and $\varphi_{13''} \mapsto 1361920c$. Hence we have to decide whether $\varphi_{8'} \mapsto 374528a$ or $\varphi_{8'} \mapsto 374528b$.

Using Proposition (4.6), see also Section (8.1) and in particular Table 6, we find the Krein parameter $q_{2,5',8'} = \frac{1}{4400000} \neq 0$. Furthermore, using GAP we find that the tensor product $760a \cdot 35112a$ of irreducible characters of G decomposes in $\mathrm{Irr}_K(G)$ as

$$760a \cdot 35112a = 3344a + 2 \cdot 35112b + 267520a + 270864a + \mathbf{374528a} + 1185030a + 1361920a + 1575936a + 4561920a + \dots,$$

where we only give the constituents belonging to $\mathrm{Irr}_K^1(G)$. From this we conclude by Proposition (4.8) that we have $\varphi_{8'} \mapsto 374528a$ and $\varphi_{8''} \mapsto 374528b$.

12 The Fischer group Fi_{24}'

(12.1) Let $G := Fi_{24}'.2$ and $H := O_{10}^-(2).2$, as well as $G' := Fi_{24}'$ and $H' := O_{10}^-(2)$. We have $r = 17$ and $r' = 34$. The conditions of Remark (5.16) are fulfilled.

The index parameters and the structure constants matrices for the two smallest non-trivial suborbits Ω_2 and Ω_3 of $\Omega := H|G$ with $k_2 = 25245$ and $k_3 = 104448$ have been computed in [42]. Using the technique described in Section (8.2), where these structure constants matrices are sufficient to get eigenspaces of dimension 1, we obtain the character table of $E_K = E_K^{1_H^G}$ as given in Table 8. By Remark (5.16), $E_K^{1_{H'}^{G'}}$ and E_K have the same character table. The Fitting correspondents for $E_K^{1_{H'}^{G'}}$ are given as the restrictions $(\chi_\varphi)_{G'}$. Again by Remark (5.16), the character table of $E_K^{1_{H'}} = E_K^{1_{H'}^G}$ is determined by the character table of E_K. As the general pattern of this is shown in Example (5.17), the character table of $E_K^{1_{H'}}$ is not shown here.

(12.2) Let $G := 3.Fi_{24}'$ and $H' := O_{10}^-(2)$, as well as $H := Z(G) \times H' = 3 \times O_{10}^-(2)$. We have $r' = 43$. Let $\lambda' = 1$ and let $\lambda_3 \in \mathrm{Irr}_K^{1_{H'}}(H)$ be as in Remark (5.18).

The splitting of the suborbits $i \in \mathcal{I}$, the index parameters and the structure constants matrices for the non-trivial suborbits $\Omega'_{1''}$, $\Omega'_{1'''}$ as well as $\Omega'_{2'}$ and $\Omega'_{3'}$, $\Omega'_{3''}$, $\Omega'_{3'''}$ on $\Omega' := H'|G$, where $1, 3 \in \mathcal{I}_{\lambda_3}$ but $2 \notin \mathcal{I}_{\lambda_3}$, with $k'_{1'} = 1$, $k'_{2'} = 75735$ and $k'_{3'} = 104448$, have been computed in [41], using a technique similar to the one employed in [42]. Using the technique described in Section (8.2), and these structure constants matrices, we obtain a splitting of $K^{1 \times r'}$ into 39 eigenspaces of dimension 1, and two eigenspaces of dimension 2. One of the latter is contained in the K-span of $\mathrm{Irr}_K(E_K^{\lambda_3}) \subseteq \mathrm{Irr}_K(E_K^{1_{H'}})$, while the other one is contained in the K-span of $\mathrm{Irr}_K(E_K^{\lambda_3^{-1}}) \subseteq \mathrm{Irr}_K(E_K^{1_{H'}})$. Employing a technique similar to the one described in more detail for an analogous situation in Section (12.3), we also find the splitting of the eigenspaces of dimension 2.

By Remark (5.18), to describe the character table of $E_K^{1_{H'}}$, it is sufficient to give the character table of $E_K^{1_H}$, see Table 8, and the character table of $E_K^{\lambda_3}$, see Table 9. In the latter character table we have made the following choice for the

Table 8: The character table for $G := Fi_{24}'.2$ and $H := O_{10}^{-}(2).2$.

φ	χ_φ	1	2	3	4	5	6	7	8	9
1	$1a^+$	1	25245	104448	1570800	12773376	45957120	67858560	107233280	193881600
2	$8671a^-$	1	−5049	−13056	157080	798336	2010624	−3392928	−1340416	4847040
3	$57477a^+$	1	1755	16752	145740	145152	−145920	1955016	4102784	1639440
4	$249458a^+$	1	3195	5664	27300	−266112	798720	−302400	546560	2311200
5	$555611a^-$	1	−1485	8544	56100	−57024	337920	178200	1168640	−1782000
6	$1666833a^+$	1	2079	−2256	26400	−14256	489984	28512	−256960	−712800
7	$35873145a^+$	1	819	1776	10020	55296	26304	129816	−75520	179280
8	$48893768a^-$	1	−189	−2688	19380	16848	−37056	−132840	−65152	6480
9	$79452373a^+$	1	−45	3072	14340	−1728	−30720	26136	111104	−2160
10	$415098112a^-$	1	−639	−708	2730	−16632	34944	15120	−6832	57780
11	$1264015025a^+$	1	171	912	1596	−6912	−6528	−21816	−2944	23760
12	$1540153692a^+$	1	279	−816	3000	−5616	384	4752	2240	−21600
13	$2346900864a^-$	1	−315	588	1110	4752	10320	−4320	−10720	−19980
14	$3208653525a^+$	1	387	48	−780	3456	12480	−7560	7424	3024
15	$10169903744a^-$	1	−99	−276	30	3456	−1776	−1728	7424	5940
16	$13904165275a^+$	1	63	48	192	−432	−3072	6048	−8128	−864
17	$17161712568a^+$	1	−45	48	−780	−1728	−480	−1080	2240	−2160

10	11	12	13	14	15	16	17
263208960	579059712	1085736960	5147197440	5428684800	7238246400	12634030080	17371791360
−3290112	−18095616	27143424	80424960	−67858560	−90478080	−39481344	108573696
−1983744	2370816	16284240	−10730496	30119040	−7197120	−44706816	7983360
2419200	2161152	−393120	−5376000	5443200	−22377600	21288960	−6289920
1468800	−3269376	1568160	5913600	1425600	1900800	7050240	−15966720
−1237248	2318976	1012176	3480576	−498960	−665280	2073600	−6044544
262656	161856	20304	561408	−544320	665280	−587520	−867456
69120	22464	456192	−652800	−149040	501120	456192	−508032
−100224	66816	−120960	−316416	−855360	17280	635904	552960
−30240	−67536	−9828	−84000	−68040	279720	−66528	−39312
−6912	1152	−34128	167424	124416	63936	−152064	−152064
24192	14976	−54864	36096	19440	−60480	−172800	210816
8640	−2736	−25380	−58080	77760	−50760	−82080	151200
−6912	−17856	−5616	−74496	−15552	63936	−6912	44928
−6912	1584	−20196	36960	19440	−11880	16416	−48384
−6912	−12672	9936	−12288	19440	−29376	55296	−17280
8640	12384	15120	−1920	−38880	17280	−17280	8640

$i' \in \alpha^{-1}_{H',H}(i)$, for $i \in \mathcal{I}_{\lambda_3}$. As we have $i^* = i$, for $i \in \mathcal{I}$, see Section (12.1), we conclude that the pairing $*\colon \mathcal{I}' \to \mathcal{I}'$ leaves the sets $\alpha^{-1}_{H',H}(i)$ invariant, for $i \in \mathcal{I}$. Hence for each $i \in \mathcal{I}_{\lambda_3}$ we may without loss of generality choose $i' \in \alpha^{-1}_{H',H}(i)$ such that $i'^* = i'$ and $i''^* = i'''$.

(12.3) Let $G := Fi'_{24}.2$ and $H := 3^7.O_7(3).2$. We have $r = 17$.

The index parameters and the structure constants matrices for the two smallest non-trivial suborbits Ω_2 and Ω_3 of $\Omega := H|G$ with $k_2 = 1120$ and $k_3 = 49140$ have been computed in [42]. Using the technique described in Section (8.2) and the structure constants matrix P_3, we obtain a splitting of $K^{1\times r}$ into 15 eigenspaces of dimension 1, and an eigenspace of dimension 2. The structure constants matrix P_2 does not give a further splitting. As the characters in $\mathrm{Irr}^1_K(G)$ have pairwise different degrees, the Fitting correspondents of the characters in the 1-dimensional eigenspaces can be determined by Section (8.1). From this we conclude that we have found the irreducible characters $\{\varphi_1, \ldots, \varphi_5, \varphi_7, \ldots, \varphi_{15}, \varphi_{17}\} \subseteq \mathrm{Irr}_K(E_K)$, see Table 10.

A K-basis for the 2-dimensional eigenspace is given by $\{\psi_1, \psi_2\}$, see also Table 10. As we have $\varphi_6(\alpha_1) = \varphi_{16}(\alpha_1) = 1$, the missing characters are given by $\varphi_6 = \psi_1 + a\psi_2$ and $\varphi_{16} = \psi_1 + b\psi_2$, for $a, b \in K$. As all values of $\chi_{\varphi_6} = 79\,452\,373a^+$ and $\chi_{\varphi_{16}} = 17\,161\,712\,568a^+$ are rational integers, by Proposition (3.20) we have $a, b \in \mathbb{Q}$. By the first orthogonality relations, see Proposition (3.8), we obtain

$$\sum_{i \in \mathcal{I}} \frac{((\psi_1 + a\psi_2)(\alpha_i))^2}{k_i} = \frac{n}{\chi_{\varphi_6}(1)}.$$

This leads to a quadratic equation for a, with coefficients in $\mathbb{Q}$, which turns out to have the solutions $a = 2916$ and $a' = -\frac{96228}{31}$. As a' leads to a character whose values are not all integers, by Proposition (3.10) we have $\varphi_6 = \psi_1 + 2916 \cdot \psi_2$. An analogous argument for φ_{16} yields $b = -108$ and $b' = -\frac{2484}{31}$, and thus $\varphi_{16} = \psi_1 - 108 \cdot \psi_2$. The characters φ_6 and φ_{16} are also given in Table 10.

(12.4) Let still $G := Fi'_{24}.2$ and $H := 3^7.O_7(3).2$, as well as $G' := Fi'_{24}$ and $H' := 3^7.O_7(3)$. We have $r' = 18$ and $r = 17$.

Note that the condition on the KG-constituents of 1_H^G and $(1^-)_H^G$ in Remark (5.16) are not fulfilled. We may identify $H'|G'$ with $\Omega := H|G$. As the ranks of the G'-action and of the G-action on Ω are $r' = 18$ and $r = 17$, respectively, the G'-suborbits and the G-suborbits on Ω coincide, except exactly one G-suborbit which is the union of two G'-suborbits. It was shown in [42] that the G-suborbit Ω_{15} splits into G'-suborbits as $\Omega_{15} = \Omega_{15'} \dot\cup \Omega_{15''}$. As $H' \trianglelefteq H$, the group H interchanges the H'-orbits $\Omega_{15'}$ and $\Omega_{15''}$, and for the index numbers we hence have $k_{15'} = k_{15''} = \frac{k_{15}}{2} = 9\,183\,300\,480$. Using the above identification, analogous to Remark (5.16), we have an embedding $E_K = E_K^{1_H^G} \to E_K^{1_{H'}^{G'}}$ of

Table 9: The character table for $G := 3.Fi_{24}'$ and $H := 3 \times O_{10}^{-}(2)$, where $\lambda = \lambda_3$.

φ	χ_φ	$1'$	$3'$	$4'$	$5'$	$6'$	$7'$	$8'$
1	$783a$	1	35904	392700	−798336	2872320	8482320	16755200
2	$64584a$	1	−8160	89760	−413424	933504	−1332936	−694144
3	$306153a$	1	11208	84000	90720	49920	703080	1937600
4	$6724809a$	1	−3984	32340	.	14784	−299376	−142912
5	$19034730a$	1	4440	24240	−23328	−42240	91368	261440
6	$43779879a$	1	−1104	7860	67392	106944	32400	−53056
7	$195019461a$	1	1704	4800	14688	49920	−57240	36800
8	$203843871a$	1	336	4260	−23328	49344	58320	−70336
9	$1050717096a$	1	−1104	5340	−2160	−14016	−9936	−4672
10	$1818548820a$	1	912	2568	864	−9984	6696	−16768
11	$10726070355a$	1	192	−492	−3456	1536	−7344	512
12	$15016498497a$	1	−240	−60	−864	3264	3024	5696
13	$21096751104a$	1	30	−330	2376	−2676	1404	−2404

$10'$	$12'$	$13'$	$14'$	$15'$	$17'$
−8225280	101787840	−160849920	339292800	−226195200	542868480
4465152	10178784	−36765696	−20599920	35544960	19388160
665280	5103000	13009920	5896800	18295200	−8346240
−495936	1496880	2188032	−1995840	−1663200	1197504
1728	68040	−1026048	−1412640	−1740960	−2021760
292032	81648	215808	−51840	−833760	−461376
−95040	−171720	−168960	194400	237600	17280
−70848	42768	−37632	142560	73440	−513216
13824	−45360	−58368	174960	43200	−158976
13824	−27216	70656	−33696	43200	324864
13824	5184	24576	31104	−34560	−55296
−1728	−14256	17664	−38880	4320	95040
−6588	12474	−26616	4860	9180	−52380

Table 10: The character table for $G := Fi'_{24}.2$ and $H := 3^7.O_7(3).2$.

φ	χ_φ	1	2	3	4	5	6	7	8	9	10
1	$1a^+$	1	1120	49140	275562	816480	21228480	57316896	62178597	286584480	429876720
2	$57477a^+$	1	−40	3900	31266	−29160	56160	1617408	2814669	−463320	4801680
3	$249458a^+$	1	392	7644	20034	76104	812448	471744	85293	5798520	2240784
4	$35873145a^+$	1	200	2220	2322	13320	64800	−53568	73629	104760	343440
5	$40536925a^+$	1	224	2772	3402	18144	108864	54432	−37179	272160	81648
7	$112168056a^+$	1	32	−636	3834	3744	−29952	121824	114453	21600	−364176
8	$281380736a^+$	1	152	1224	1134	6264	6048	−20736	−16767	3240	−241056
9	$1069551175a^+$	1	−64	468	−1782	1728	−22464	33696	9477	67392	50544
10	$1264015025a^+$	1	−40	732	2754	−3816	5472	20736	−8019	−7128	11664
11	$3208653525a^+$	1	104	588	−270	936	−3168	5184	3645	−53352	22032
12	$3283490925a^+$	1	56	−84	882	504	−14112	−24192	3645	35784	3024
13	$5775278080a^+$	1	14	−294	1134	882	−5544	9072	−16767	−7938	27216
14	21553171200	1	−28	.	−378	756	.	.	−3159	−9828	.
15	$17068369920a^+$	1	44	36	−486	−1188	216	2592	−2187	10692	−1944
17	$54234085491a^+$	1	.	−140	42	.	2240	−224	2917	.	−1680
6	$79452373a^+$	1	−40	300	3186	−360	−1440	−41472	190269	−117720	265680
16	$17161712568a^+$	1	−40	300	162	−360	−1440	−5184	−243	3240	−6480
ψ_1		1	−40	300	270	−360	−1440	−6480	6561	−1080	3240
ψ_2		.	.	.	1	.	.	−12	63	−40	90

φ	11	12	13	14	15	16	17
1	2901667860	5158520640	6964002864	15475561920	18366600960	23213342880	52230021480
2	32411340	−30326400	77787216	40940640	−81881280	−37528920	−10235160
3	15125292	28304640	−9552816	−19105632	−30652992	11144952	−4776408
4	−811620	−1244160	1271376	−2838240	699840	4908600	−2536920
5	−591948	−326592	−244944	3592512	1679616	−2776032	−1837080
7	−144828	388800	128304	373248	373248	1220832	−2210328
8	190512	−155520	303264	−1065312	139968	−849528	1697112
9	−63180	−202176	−151632	−202176	.	−202176	682344
10	164268	−217728	−221616	−116640	233280	106920	29160
11	4860	62208	−128304	116640	−233280	126360	75816
12	74844	−72576	−81648	163296	46656	68040	−204120
13	−44226	68040	102060	−81648	5832	−78246	20412
14	58968	.	.	.	101088	29484	−176904
15	−2916	19440	58320	−33048	93312	−32076	−110808
17	−10220	−20160	−27216	2240	−46080	.	98280
6	43740	777600	104976	−1049760	2099520	−2536920	262440
16	−46980	51840	50544	38880	−77760	3240	−9720
ψ_1	−43740	77760	52488	.	.	−87480	.
ψ_2	30	240	18	−360	720	−840	90

K-algebras given by, for $j \in \mathcal{I}$,

$$\alpha_j \mapsto \begin{cases} \alpha_j^{1_{H'}^{G'}}, & \text{if } j \neq 15, \\ \alpha_{15'}^{1_{H'}^{G'}} + \alpha_{15''}^{1_{H'}^{G'}}, & \text{if } j = 15. \end{cases}$$

Furthermore, we have $(\chi_{\varphi_i})_{G'} \in \mathrm{Irr}_K^{1_{H'}^{G'}}(G')$ for $i \neq 14$, while $\chi_{\varphi_{14}}$ splits under restriction to G' as $(\chi_{\varphi_{14}})_{G'} = 10\,776\,585\,600a + 10\,776\,585\,600b$. Using the Fitting correspondence, see Proposition (2.7), for KG and E_K as well as for KG' and $E_K^{1_{H'}^{G'}}$, we obtain $\mathrm{Irr}_K(E_K^{1_{H'}^{G'}}) = \{\varphi_i'; i \in \mathcal{I}, i \neq 14\} \,\dot\cup\, \{\varphi_{14'}', \varphi_{14''}'\}$, where for $i \in \mathcal{I} \setminus \{14\}$ and $j \in \mathcal{I} \setminus \{15\}$ we have $\varphi_i'(\alpha_j^{1_{H'}^{G'}}) = \varphi_i(\alpha_j)$. Furthermore we have, for $j \neq 15$,

$$\varphi_{14'}'(\alpha_j^{1_{H'}^{G'}}) = \varphi_{14''}'(\alpha_j^{1_{H'}^{G'}}) = \varphi_{14}(\alpha_j),$$

and using the first orthogonality relations, see Proposition (3.8), with respect to φ_1' we obtain, for $i \neq 14$,

$$\varphi_i'(\alpha_{15'}^{1_{H'}^{G'}}) = \varphi_i'(\alpha_{15''}^{1_{H'}^{G'}}) = \frac{\varphi_i(\alpha_{15})}{2}.$$

Finally, we have

$$\varphi_{14'}'(\alpha_{15'}^{1_{H'}^{G'}} + \alpha_{15''}^{1_{H'}^{G'}}) = \varphi_{14''}'(\alpha_{15'}^{1_{H'}^{G'}} + \alpha_{15''}^{1_{H'}^{G'}}) = \varphi_{14}(\alpha_{15}) = 101088.$$

Again using the first orthogonality relations with respect to φ_1', where now $\chi_{\varphi_{14'}'}(1) = \chi_{\varphi_{14''}'}(1) = \frac{\chi_{\varphi_{14}}(1)}{2}$, we obtain a system of two linear equations for $\varphi_{14'}'(\alpha_{15'}^{1_{H'}^{G'}})$ and $\varphi_{14''}'(\alpha_{15'}^{1_{H'}^{G'}})$, which leads to

$$\{\varphi_{14'}'(\alpha_{15'}^{1_{H'}^{G'}}), \varphi_{14''}'(\alpha_{15'}^{1_{H'}^{G'}})\} = \{-112752, 213840\}.$$

This determines the values of the characters in $\mathrm{Irr}_K(E_K^{1_{H'}^{G'}})$ completely. The character table of $E_K^{1_{H'}^{G'}}$ is shown in Table 11, where the latter character values are indicated in bold type.

13 The Harada-Norton group HN

(13.1) Let $G := HN.2$ and $H := S_{11}$, as well as $G' := HN$ and $H' := A_{11}$. We have $r = 17$.

Let $\Omega := H|G$. As $n = |\Omega| = 13\,680\,000$ is small enough, using GAP, we construct explicit permutations for the action of G on Ω. Let additionally $\tilde{H} := S_{12}$, where we have $H < \tilde{H} < G$ and $[\tilde{H} \colon H] = 12$. Let $\tilde{\Omega} := \tilde{H}|G$, where

Table 11: The character table for $G' := Fi_{24}'$ and $H' := 3^7.O_7(3)$.

φ	χ_φ	1	2	3	4	5	6	7	8	9	10
1	$1a$	1	1120	49140	275562	816480	21228480	57316896	62178597	286584480	429876720
2	$57477a$	1	−40	3900	31266	−29160	56160	1617408	2814669	−463320	4801680
3	$249458a$	1	392	7644	20034	76104	812448	471744	85293	5798520	2240784
4	$35873145a$	1	200	2220	2322	13320	64800	−53568	73629	104760	343440
5	$40536925a$	1	224	2772	3402	18144	108864	54432	−37179	272160	81648
6	$79452373a$	1	−40	300	3186	−360	−1440	−41472	190269	−117720	265680
7	$112168056a$	1	32	−636	3834	3744	−29952	121824	114453	21600	−364176
8	$281380736a$	1	152	1224	1134	6264	6048	−20736	−16767	3240	−241056
9	$1069551175a$	1	−64	468	−1782	1728	−22464	33696	9477	67392	50544
10	$1264015025a$	1	−40	732	2754	−3816	5472	20736	−8019	−7128	11664
11	$3208653525a$	1	104	588	−270	936	−3168	5184	3645	−53352	22032
12	$3283490925a$	1	56	−84	882	504	−14112	−24192	3645	35784	3024
13	$5775278080a$	1	14	−294	1134	882	−5544	9072	−16767	−7938	27216
$14'$	$10776585600a$	1	−28	.	−378	756	.	.	−3159	−9828	.
$14''$	$10776585600b$	1	−28	.	−378	756	.	.	−3159	−9828	.
15	$17068369920a$	1	44	36	−486	−1188	216	2592	−2187	10692	−1944
16	$17161712568a$	1	−40	300	162	−360	−1440	−5184	−243	3240	−6480
17	$54234085491a$	1	.	−140	42	.	2240	−224	2917	.	−1680

11	12	13	14	$15'$	$15''$	16	17
2901667860	5158520640	6964002864	15475561920	9183300480	9183300480	23213342880	52230021480
32411340	−30326400	77787216	40940640	−40940640	−40940640	−37528920	−10235160
15125292	28304640	−9552816	−19105632	−15326496	−15326496	11144952	−4776408
−811620	−1244160	1271376	−2838240	349920	349920	4908600	−2536920
−591948	−326592	−244944	3592512	839808	839808	−2776032	−1837080
43740	777600	104976	−1049760	1049760	1049760	−2536920	262440
−144828	388800	128304	373248	186624	186624	1220832	−2210328
190512	−155520	303264	−1065312	69984	69984	−849528	1697112
−63180	−202176	−151632	−202176	.	.	−202176	682344
164268	−217728	−221616	−116640	116640	116640	106920	29160
4860	62208	−128304	116640	−116640	−116640	126360	75816
74844	−72576	−81648	163296	23328	23328	68040	−204120
−44226	68040	102060	−81648	2916	2916	−78246	20412
58968	.	.	.	**−112752**	**213840**	29484	−176904
58968	.	.	.	**213840**	**−112752**	29484	−176904
−2916	19440	58320	−33048	46656	46656	−32076	−110808
−46980	51840	50544	38880	−38880	−38880	3240	−9720
−10220	−20160	−27216	2240	−23040	−23040	.	98280

Table 12: The character table for $G := HN.2$ and $\tilde{H} := S_{12}$.

φ	χ_φ	1	2	3	4	5	6	7	8	9	10
1	$1a^+$	1	462	5040	10395	16632	30800	69300	311850	332640	362880
2	266	1	−198	.	2475	792	4400	−9900	−14850	.	17280
3	$760a^+$	1	132	−1080	1485	−1188	1100	4950	.	−11880	6480
4	$3344a^+$	1	12	240	495	1332	−2200	300	−900	−2160	2880
5	$8910a^+$	1	82	480	515	−88	400	900	−1650	1280	−1920
6	$16929a^+$	1	62	−240	155	632	400	−300	1050	160	−1920
7	70224	1	−48	.	225	−108	−100	−150	900	.	−720
8	$267520a^+$	1	12	60	−45	−18	50	−150	450	−540	180
9	$365750a^+$	1	−18	.	−45	72	80	180	−270	.	.
10	$406296a^+$	1	12	−40	5	−68	−100	−50	−200	360	80

$\tilde{n} = 1\,140\,000$. The character table of $E_K^{1_{\tilde{H}}^G}$ as is contained in the database, see Section (11.1), is given in Table 12.

Let $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. We start with explicitly known permutations for the action of the elements of $\mathcal{G}$ on $\tilde{\Omega}$ available in [83]. Using the randomised Schreier-Sims algorithm implemented in GAP, keeping track of transversals and subgroup generators as words in the given set of generators, see Section (10.6), we obtain a Schreier subgroup chain of G and a set $\tilde{\mathcal{H}}$ of generators of $\tilde{H}$, given explicitly as words in $\mathcal{G}$. Restricting to the smallest non-trivial suborbit $\tilde{\Omega}_2$, where $\tilde{k}_2 = 462$, we obtain a faithful permutation action of $\tilde{H}$. By a random search we find a subset of $\tilde{H}$, again explicitly as words in the set of generators of $\tilde{H}$, generating a subgroup of order $39\,916\,800 = 11!$, which we hence may choose as $H \cong S_{11}$. Using the programs implemented in GAP dealing with permutation groups, explicit permutations for the action of $\tilde{\mathcal{H}}$ on the set of right cosets $\Xi := H|\tilde{H}$ of H in $\tilde{H}$ can be determined.

Let $\{\tilde{g}_i; i \in \{1, \ldots, \tilde{n}\}\}$ be a set of representatives of the right cosets $\tilde{H}|G$ of $\tilde{H}$ in G, where $\tilde{g}_1 := 1$. Let $\{\tilde{h}_j; j \in \{1, \ldots, [\tilde{H}\colon H]\}\}$ be a set of representatives of the right cosets $H|\tilde{H}$ of H in $\tilde{H}$, where $\tilde{h}_1 := 1$. Hence we obtain a set of representatives $\{\tilde{h}_j\tilde{g}_i; j \in \{1, \ldots, [\tilde{H}\colon H]\}, i \in \{1, \ldots, \tilde{n}\}\}$ of the right cosets $H|G$ of H in G. Let $\pi_{\tilde{\Omega}}\colon G \to \mathcal{S}_{\tilde{n}}$ as well as $\pi_\Xi\colon \tilde{H} \to \mathcal{S}_{[\tilde{H}:H]}$ and $\pi_\Omega\colon G \to \mathcal{S}_n$ denote the group homomorphisms defined by the action of G on $\tilde{\Omega}$, by the action of $\tilde{H}$ on Ξ and by the action of G on Ω, respectively. As noted above both $\pi_{\tilde{\Omega}}$ and π_Ξ are given in terms of the sets $\mathcal{G}$ and $\tilde{\mathcal{H}}$ of generators of G and $\tilde{H}$, respectively. Explicit permutations for the action of $\mathcal{G}$ and of $\tilde{\mathcal{H}}$ on Ω are obtained as follows.

For $g \in G$ as well as $i \in \{1, \ldots, \tilde{n}\}$ and $j \in \{1, \ldots, [\tilde{H}\colon H]\}$, let $i' := i \cdot \pi_{\tilde{\Omega}}(g)$ and $j' := j \cdot \pi_\Xi(\tilde{g}_i \cdot g \cdot \tilde{g}_{i'}^{-1})$. Hence we have $\tilde{h}_j\tilde{g}_i \cdot g = h \cdot \tilde{h}_{j'}\tilde{g}_{i'}$, for some $h \in H$. Thus $\pi_\Omega(g)$ can be determined from $\pi_{\tilde{\Omega}}(g)$ and π_Ξ, where we have to write $\tilde{g}_i \cdot g \cdot \tilde{g}_{i'}^{-1} \in \tilde{H}$ as a word in the given set $\tilde{\mathcal{H}}$ of generators of $\tilde{H}$. This can be done in GAP using the Schreier subgroup chain of G obtained above, containing the transversals and subgroup generators as words in the given set $\tilde{\mathcal{H}}$ of generators.

Hence we are prepared to apply the ZKD program in the MeatAxe, see Section (9.1), to some arbitrarily chosen elements of G. Using Proposition (9.6) and Remark (9.7) we obtain some of the structure constants matrices P_k, for some $k \in \mathcal{I}$. Using the technique described in Section (8.2) and sufficiently many of the P_k, we obtain the character table of E_K as shown in Table 13. Rows and columns have been reordered and column indexing has been adjusted to exhibit the phenomena described in Section 5, see Example (5.14), where the character table of $E_K^{1_{\tilde{H}}^G}$ is given in Table 12.

Table 13: The character table for $G := HN.2$ and $H := S_{11}$.

φ	χ_φ	$1'$	$1''$	$2'$	$2''$	$3'$	$4'$	$4''$	$5'$	$5''$	$5'''$
1	$1a^+$	1	11	2772	2772	60480	20790	103950	16632	83160	99792
2	266	1	11	−1188	−1188	.	4950	24750	792	3960	4752
3	$760a^+$	1	11	792	792	−12960	2970	14850	−1188	−5940	−7128
4	$3344a^+$	1	11	72	72	2880	990	4950	1332	6660	7992
5	$8910a^+$	1	11	492	492	5760	1030	5150	−88	−440	−528
6	$16929a^+$	1	11	372	372	−2880	310	1550	632	3160	3792
7	70224	1	11	−288	−288	.	450	2250	−108	−540	−648
8	$267520a^+$	1	11	72	72	720	−90	−450	−18	−90	−108
9	$365750a^+$	1	11	−108	−108	.	−90	−450	72	360	432
10	$406296a^+$	1	11	72	72	−480	10	50	−68	−340	−408
11	$9405a^+$	1	−1	−168	168	.	1050	−1050	1428	2100	−3528
12	$653125a^+$	1	−1	−72	72	.	90	−90	180	468	−648
13	$1185030a^+$	1	−1	72	−72	.	250	−250	68	100	−168
14	$2031480a^+$	1	−1	12	−12	.	−50	50	208	−400	192
15	$2375000a^+$	1	−1	36	−36	.	−18	18	−144	144	.
16	$2407680a^+$	1	−1	−48	48	.	150	−150	−102	−150	252
17	$3878280a^+$	1	−1	−8	8	.	−150	150	−12	100	−88

φ	$6'$	$7'$	$8'$	$8''$	$9'$	$10'$	$10''$
1	369600	831600	1247400	2494800	3991680	362880	3991680
2	52800	−118800	−59400	−118800	.	17280	190080
3	13200	59400	.	.	−142560	6480	71280
4	−26400	3600	−3600	−7200	−25920	2880	31680
5	4800	10800	−6600	−13200	15360	−1920	−21120
6	4800	−3600	4200	8400	1920	−1920	−21120
7	−1200	−1800	3600	7200	.	−720	−7920
8	600	−1800	1800	3600	−6480	180	1980
9	960	2160	−1080	−2160	.	.	.
10	−1200	−600	−800	−1600	4320	80	880
11	.	.	25200	−25200	.	−10080	10080
12	.	.	−720	720	.	1440	−1440
13	.	.	−1200	1200	.	−480	480
14	.	.	400	−400	.	320	−320
15	.	.	1008	−1008	.	576	−576
16	.	.	.	.	.	−180	180
17	.	.	−400	400	.	−480	480

(13.2) Let still $G := HN.2$ and $H := S_{11}$, as well as $G' := HN$ and $H' := A_{11}$, and $\tilde{H} := S_{12}$. We have $r' = 19$.

A $H'|G'$ can be identified with $\Omega := H|G$, to determine the character table of $E_K^{1_{H'}^{G'}}$ we also use the explicit permutations for $\mathcal{G}$ obtained above. Sets of generators of $G' < G$ as well as of $A_{12} \cong \tilde{H}' := \tilde{H} \cap G' < G'$ and of $H' < \tilde{H}'$, given as words in $\mathcal{G}$, are available in [83]. Hence the same technique as used for the character table of E_K yields the character table of $E_K^{1_{H'}^{G'}}$. It is shown in Table 14, where $r_5 := \sqrt{5} \in \mathbb{R}$.

Alternatively, we could also apply the technique used in Section (12.4), for which we need to know which suborbits of Ω split. As we have $[H\colon H'] = 2$ as well as $r = 17$ and $r' = 19$, there are exactly two of the suborbits of the G-action on Ω which each split into two suborbits of the G'-action. To find out, which suborbits split, we also identify $\tilde{H}'|G'$ with $\tilde{\Omega}$, and compare the character tables of $E_K^{1_{\tilde{H}}^{G}}$, see Table 12, and $E_K^{1_{\tilde{H}'}^{G'}}$, which is contained in the database, see Section (11.1), and has originally been computed in [63]. We find that the suborbits $\tilde{\Omega}_3$ and $\tilde{\Omega}_9$ split, see Table 12. Hence we conclude that the suborbits $\Omega_{3'}$ and $\Omega_{9'}$ split, see Table 13. The relevant character values are obtained as in Section (12.4) and are indicated in Table 14 in bold type. In Table 14, rows and columns have been reordered and column indexing has been adjusted to exhibit the phenomena described in Section 5.

(13.3) Let $G := HN.2$ and $H := U_3(8).6$, as well as $G' := HN$ and $H' := U_3(8).3_1$. We have $r = 15$.

Let $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. We start with explicitly known matrices for the action of the elements of $\mathcal{G}$ on the absolutely irreducible $\mathbb{F}_2G$-module V of $\mathbb{F}_2$-dimension 760 available in [83]. The subgroup $H < G$ is a maximal subgroup of G, and $H' < G'$ is a maximal subgroup of G'. A set of generators $\mathcal{H}$ of H and a set of generators $\mathcal{G}'$ of G', both given as words in $\mathcal{G}$, is available in [83] as well. Using the MeatAxe, a set of generators of H', again as words in the set of generators $\mathcal{H}$ of H, can be found.

Using the MeatAxe, we find that $V_{H'}$ has a uniquely determined trivial $\mathbb{F}_2H'$-submodule. Hence, if we pick the vector $0 \neq v_{H'} \in V_{H'}$ in this submodule we conclude that there is a G-set isomorphism between the G-set $v_{H'} \cdot G \subseteq V$ and $\Omega := H|G$, where the latter can also be identified with $H'|G'$.

We apply the technique described in Section (10.3) for $U = H'$, where $U_1 < U$ is a cyclic subgroup of order 21. The $\mathbb{F}_2U_1$-epimorphic image V_1 is chosen to be isomorphic to $V_1 \cong 6a \oplus 6a$, where $6a$ is one of the irreducible $\mathbb{F}_2U_1$-modules of $\mathbb{F}_2$-dimension 6. We find the orbit counting numbers for the elements of $\mathcal{G}'$ with respect to $\Omega = \coprod_{j \in \mathcal{J}} \tilde{\Omega}_j$ first, using the notation of Section (9.8), and using the set of generators $\mathcal{H}$ of H yields the orbit counting numbers with respect to $\Omega = \coprod_{i \in \mathcal{I}} \Omega_i$.

Table 14: The character table for $G' := HN$ and $H' := A_{11}$.

φ	χ_φ	$1'$	$1''$	$2'$	$2''$	$3'_1$	$3'_2$	$4'$	$4''$	$5'$	$5''$	$5'''$
1	$1a^+$	1	11	2772	2772	30240	30240	20790	103950	16632	83160	99792
$2'$	$133a$	1	11	−1188	−1188	**−4320r$_5$**	**4320r$_5$**	4950	24750	792	3960	4752
$2''$	$133b$	1	11	−1188	−1188	**4320r$_5$**	**−4320r$_5$**	4950	24750	792	3960	4752
3	$760a^+$	1	11	792	792	−6480	−6480	2970	14850	−1188	−5940	−7128
4	$3344a^+$	1	11	72	72	1440	1440	990	4950	1332	6660	7992
5	$8910a^+$	1	11	492	492	2880	2880	1030	5150	−88	−440	−528
6	$16929a^+$	1	11	372	372	−1440	−1440	310	1550	632	3160	3792
$7'$	$35122a$	1	11	−288	−288	**−720r$_5$**	**720r$_5$**	450	2250	−108	−540	−648
$7''$	$35122b$	1	11	−288	−288	**720r$_5$**	**−720r$_5$**	450	2250	−108	−540	−648
8	$267520a^+$	1	11	72	72	360	360	−90	−450	−18	−90	−108
9	$365750a^+$	1	11	−108	−108	.	.	−90	−450	72	360	432
10	$406296a^+$	1	11	72	72	−240	−240	10	50	−68	−340	−408
11	$9405a^+$	1	−1	−168	168	.	.	1050	−1050	1428	2100	−3528
12	$653125a^+$	1	−1	−72	72	.	.	90	−90	180	468	−648
13	$1185030a^+$	1	−1	72	−72	.	.	250	−250	68	100	−168
14	$2031480a^+$	1	−1	12	−12	.	.	−50	50	208	−400	192
15	$2375000a^+$	1	−1	36	−36	.	.	−18	18	−144	144	.
16	$2407680a^+$	1	−1	−48	48	.	.	150	−150	−102	−150	252
17	$3878280a^+$	1	−1	−8	8	.	.	−150	150	−12	100	−88

φ	$6'$	$7'$	$8'$	$8''$	$9'_1$	$9'_2$	$10'$	$10''$
1	369600	831600	1247400	2494800	1995840	1995840	362880	3991680
$2'$	52800	−118800	−59400	−118800	**−95040r$_5$**	**95040r$_5$**	17280	190080
$2''$	52800	−118800	−59400	−118800	**95040r$_5$**	**−95040r$_5$**	17280	190080
3	13200	59400	.	.	−71280	−71280	6480	71280
4	−26400	3600	−3600	−7200	−12960	−12960	2880	31680
5	4800	10800	−6600	−13200	7680	7680	−1920	−21120
6	4800	−3600	4200	8400	960	960	−1920	−21120
$7'$	−1200	−1800	3600	7200	**2160r$_5$**	**−2160r$_5$**	−720	−7920
$7''$	−1200	−1800	3600	7200	**−2160r$_5$**	**2160r$_5$**	−720	−7920
8	600	−1800	1800	3600	−3240	−3240	180	1980
9	960	2160	−1080	−2160	.	.	.	.
10	−1200	−600	−800	−1600	2160	2160	80	880
11	.	.	25200	−25200	.	.	−10080	10080
12	.	.	−720	720	.	.	1440	−1440
13	.	.	−1200	1200	.	.	−480	480
14	.	.	400	−400	.	.	320	−320
15	.	.	1008	−1008	.	.	576	−576
16	.	.	.	.	.	.	−180	180
17	.	.	−400	400	.	.	−480	480

Using Remark (9.7), we obtain some of the structure constants matrices for E_K, such that using the technique described in Section (8.2) we obtain a splitting of $K^{1\times r}$ into eigenspaces of dimension 1. The character table of E_K is shown in Table 15.

(13.4) Let still $G := HN.2$ and $H := U_3(8).6$, as well as $G' := HN$ and $H' := U_3(8).3_1$. We have $r' = 19$.

The splitting of the suborbits Ω_i, for $i \in \mathcal{I}$, into the suborbits $\tilde{\Omega}_j$, for $j \in \mathcal{J}$, is known by [34], but can also be deduced from the results on orbit counting numbers in Section (13.3). The split suborbits are $\{5, 8, 12, 13\}$. As $[H : H'] = 2$, a split suborbit of G splits into two suborbits of G' of equal length.

Using the orbit counting matrices with respect to $\Omega = \coprod_{j\in\mathcal{J}} \tilde{\Omega}_j$ found for $\mathcal{G}'$ in Section (13.3), and the technique described in Section (8.2), we obtain a splitting of $K^{1\times r'}$ into 11 eigenspaces of dimension 1, and 4 eigenspaces of dimension 2, where K-bases $\{\psi_1, \psi_2, \}, \ldots, \{\psi_7, \psi_8\}$ of the latter are given in Table 16.

Using the character table of $E_K = E_K^{1_H^G}$ given in Table 15, we conclude that we have found $\{\varphi_1, \ldots, \varphi_4, \varphi_6, \varphi_7, \varphi_{10}, \varphi_{11}, \varphi_{13}, \ldots, \varphi_{15}\} \subseteq \mathrm{Irr}(E_K^{1_{H'}^{G'}})$, see Table 17. Furthermore, the Fitting correspondents of $\varphi_{5'}, \varphi_{5''}$ are $35112ab$, and we have $\varphi_{5'}, \varphi_{5''} \in \langle\psi_1, \psi_2\rangle_K$. Analogously, $\varphi_{8'}, \varphi_{8''}$ correspond to $374528ab$ and $\varphi_{8'}, \varphi_{8''} \in \langle\psi_3, \psi_4\rangle_K$, while $\varphi_{9'}, \varphi_{9''}$ correspond to $656250ab$ and $\varphi_{9'}, \varphi_{9''} \in \langle\psi_5, \psi_6\rangle_K$, and finally $\varphi_{12'}, \varphi_{12''}$ correspond to $1361920bc$ and $\varphi_{12'}, \varphi_{12''} \in \langle\psi_7, \psi_8\rangle_K$. Using GAP we find that $\chi_{\varphi_i}(G) \subseteq \mathbb{Q}(\sqrt{5}) \subseteq \mathbb{R}$, for $i \in \{5', 5'', 8', 8''\}$, while $\chi_{\varphi_i}(G) \subseteq \mathbb{Q}(\sqrt{-19}) \not\subseteq \mathbb{R}$, for $i \in \{9', 9''\}$, and $\chi_{\varphi_i}(G) \subseteq \mathbb{Q}(\sqrt{-10}) \not\subseteq \mathbb{R}$, for $i \in \{12', 12''\}$, where all of the latter irreducible characters are non-rational.

From this we obtain $\varphi_{5'}, \varphi_{5''}, \varphi_{8'}, \varphi_{8''}$, using the same technique as in Section (12.3). By Remark (3.21), the values of the characters $\varphi_{9'}, \varphi_{9''}, \varphi_{12'}, \varphi_{12''}$ on the Schur basis elements are not all real, and using the same technique as in Section (12.4) we conclude

$$\begin{aligned}
\varphi_{9'}(\alpha_{5'}^{1_{H'}^{G'}} + \alpha_{5'}^{1_{H'}^{G'}}) &= \varphi_{9'}(\alpha_{5'}^{1_{H'}^{G'}}) + \overline{\varphi_{9'}(\alpha_{5'}^{1_{H'}^{G'}})} = \varphi_9(\alpha_5^{1_H^G}) = 288, \\
\varphi_{12'}(\alpha_{5'}^{1_{H'}^{G'}} + \alpha_{5'}^{1_{H'}^{G'}}) &= \varphi_{12'}(\alpha_{5'}^{1_{H'}^{G'}}) + \overline{\varphi_{12'}(\alpha_{5'}^{1_{H'}^{G'}})} = \varphi_{12}(\alpha_5^{1_H^G}) = -108,
\end{aligned}$$

where $\overline{}$ is the involutory field automorphism of K as defined in Section 3.

As in Section (12.3) we let $\varphi_{9'} = \psi_5 + a\psi_6$ and $\varphi_{12'} = \psi_7 + b\psi_8$, for $a, b \in K$. The above equations already determine the real parts $\frac{a+\bar{a}}{2}$ and $\frac{b+\bar{b}}{2}$ of a and b, respectively. As we know the degrees $\chi_\varphi(1)$ of the Fitting correspondents of $\varphi_{9'}$ and $\varphi_{12'}$, the orthogonality relations, see Proposition (3.8), lead to quadratic equations for the imaginary parts $\frac{a-\bar{a}}{2i}$ and $\frac{b-\bar{b}}{2i}$ of a and b, respectively. This yields $\varphi_{9'}$ and $\varphi_{12'}$, as well as $\varphi_{9''} = \overline{\varphi_{9'}}$ and $\varphi_{12''} = \overline{\varphi_{12'}}$.

The character table of $E_K^{1_{H'}^{G'}}$ is shown in Table 17, where again we indicate the relevant character values in bold type, and where $r_5 := \sqrt{5} \in \mathbb{R}$ as well as

Table 15: The character table for $G := HN.2$ and $H := U_3(8).6$.

φ	χ_φ	1	2	3	4	5	6	7	8
1	$1a^+$	1	1539	14364	25536	51072	68096	131328	459648
2	$760a^+$	1	−81	3024	−4704	−2688	7616	1728	−24192
3	$3344a^+$	1	−261	1764	2436	672	896	5328	6048
4	$9405a^+$	1	99	924	2016	−2688	896	−3072	2688
5	70224	1	99	684	−144	−1248	−64	1728	−432
6	$267520a^+$	1	9	414	366	672	716	−72	−432
7	$270864a^+$	1	19	364	−644	672	−224	−1072	−672
8	749056	1	−81	−36	−144	72	−424	−72	−972
9	1312500	1	−45	−180	48	288	320	.	576
10	$1185030a^+$	1	19	44	176	192	−704	128	768
11	$1361920a^+$	1	54	9	126	252	116	153	−702
12	2723840	1	54	−81	−84	−108	56	−297	378
13	$1575936a^+$	1	−26	109	−194	−48	136	353	1218
14	$2407680a^+$	1	9	−66	−54	−48	−4	528	−672
15	$4561920a^+$	1	−26	29	86	−168	16	−247	−222

9	10	11	12	13	14	15
689472	787968	787968	1225728	1225728	5515776	5515776
54432	−67392	36288	−24192	56448	−108864	72576
15372	17568	24768	−51072	−17472	22176	−28224
14112	10368	−1152	16128	16128	−8064	−48384
−1008	−1152	4608	7488	−8832	−2304	576
−558	1908	−1692	408	−552	−7524	6336
1652	128	−672	448	−1792	5376	−3584
−648	1728	1728	648	1008	−3024	216
720	−576	576	1152	−1152	−576	−1152
592	−832	−192	128	128	−1344	896
−1053	−567	918	108	1098	1971	−2484
27	513	378	−432	−162	−1269	1026
−733	−127	−342	−212	1118	−129	−1124
702	468	−972	−72	−72	396	−144
−173	−367	−222	−292	−202	1191	596

Table 16: 2-dimensional eigenspaces for $G' := HN$ and $H' := U_3(8).3_1$.

		$1'$	$2'$	$3'$	$4'$	$5'$	$5''$	$6'$	$7'$	$8'$	$8''$
ψ_1	35112	1	99	684	−144	−624	−624	−64	1728	.	−432
ψ_2		.	.	.	.	.	.	.	.	3	−3
ψ_3	374528	1	−81	−36	−144	36	36	−424	−72	.	−972
ψ_4		.	.	.	.	.	.	.	.	1	−1
ψ_5	656250	1	−45	−180	48	.	288	320	.	288	288
ψ_6		.	.	.	.	1	−1	.	.	.	.
ψ_7	1361920	1	54	−81	−84	.	−108	56	−297	189	189
ψ_8		.	.	.	.	1	−1	.	.	.	.

$9'$	$10'$	$11'$	$12'$	$12''$	$13'$	$13''$	$14'$	$15'$
−1008	−1152	4608	3456	4032	−4416	−4416	−2304	576
.	.	.	−4	4	.	.	.	.
−648	1728	1728	1296	−648	504	504	−3024	216
.	.	.	2	−2	.	.	.	.
720	−576	576	576	576	576	−1728	−576	−1152
.	.	.	.	.	−8	8	.	.
27	513	378	−216	−216	81	−243	−1269	1026
.	.	.	.	.	3	−3	.	.

$i_{10} := i \cdot \sqrt{10} \in \mathbb{C}$ and $i_{19} := i \cdot \sqrt{19} \in \mathbb{C}$. The Fitting correspondence is determined in Section (11.5).

14 The Lyons group Ly

(14.1) Let $G := Ly$ and $H' := 3.McL$, as well as $H := 3.McL.2$. We have $r' = 8$ and $r = 5$.

Let $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. We start with explicitly known matrices for the action of the elements of $\mathcal{G}$ on the absolutely irreducible $\mathbb{F}_5G$-module V of $\mathbb{F}_5$-dimension 517 available in [83]. The subgroup $H < G$ is a maximal subgroup of G, and a set of generators of H, given as words in $\mathcal{G}$, is available in [83] as well. Using the MeatAxe, a set of generators $\mathcal{H}'$ of H', again as words in the set of generators of H, can be found.

Using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that $V_{H'} \cong 1a \oplus 210a \oplus 306a$ as $\mathbb{F}_5H'$-modules, where $1a$, $210a$ and $306a$ are the absolutely irreducible $\mathbb{F}_5H'$-modules of the respective dimensions, see [37]. Hence all $\mathbb{F}_5H'$-submodules of $V_{H'}$ are also invariant under the action of $\mathbb{F}_5H$ and using the MeatAxe we find that the trivial $\mathbb{F}_5H'$-module $1a$ extends to the non-trivial linear $\mathbb{F}_5H$-module $1a^-$. Hence, if we pick a vector $0 \neq v_{H'} \in 1a \leq V_{H'}$, we conclude that there is a G-set isomorphism between the G-orbit $v_{H'} \cdot G \subseteq V$ and $\Omega' := H'|G$.

To use the strategy described in Section (10.3) efficiently, we proceed as described in Section (9.8). We choose a subgroup $U < H'$, and compute the orbit counting numbers with respect to $\Omega' = \coprod_{j \in \mathcal{J}} \tilde{\Omega}'_j$ first. From these the orbit counting numbers with respect to $\Omega' = \coprod_{i \in \mathcal{I}'} \Omega'_i$ are found. We choose $U := 3 \times M_{11} < 3.McL = H'$, which is a maximal subgroup of H'. A set of generators of U, given as words in $\mathcal{H}'$, is available in [83] as well. We have $|U| = 23760$ and using GAP we find $\langle 1_{H'}^G, 1_U^G \rangle_G = 837$, thus we have $|\mathcal{J}| = 837$, while $|\mathcal{I}'| = r' = 8$. Furthermore, we choose $U_1 < U$ to be a subgroup of order 11. Using the MeatAxe we find $V_{U_1} \cong 47 \cdot 1a \oplus 47 \cdot 5a \oplus 47 \cdot 5b$, where $1a$, $5a$ and $5b$ are the irreducible $\mathbb{F}_5U_1$-modules of the respective dimensions. As $\mathbb{F}_5U_1$-epimorphic image V_1 we choose one of the irreducible quotients $V_1 \cong 5a$.

We compute the orbit counting numbers for the elements in $\mathcal{G}$ and those in $\mathcal{H}'$, and applying Section (9.8) and Remark (9.7) we obtain two of the structure constants matrices P_{k_1} and P_{k_2}, for $k_1, k_2 \in \mathcal{I}'$. Using the technique described in Section (8.2), we obtain a splitting of $K^{1 \times r'}$, into 6 eigenspaces of dimension 1, and an eigenspace of dimension 2, where a K-basis $\{\psi_1, \psi_2\}$ of the latter is shown in Table 18. As the degrees of the characters in $\mathrm{Irr}_K^1(G)$ are pairwise different, we conclude that we have found $\{\varphi_1, \ldots, \varphi_5, \varphi_8\}$, while φ_6 and φ_7 are missing.

We could compute more of the structure constants matrices, until these yield only eigenspaces of dimension 1. But proceeding as in Section (12.3), we let $\varphi_6 = \psi_1 + a\psi_2$ and $\varphi_7 = \psi_1 + b\psi_2$, for $a, b \in K$. This yields $a \in \{\pm 1800\}$ and

Table 17: The character table for $G' := HN$ and $H' := U_3(8).3_1$.

φ	χ_φ	$1'$	$2'$	$3'$	$4'$	$5'$	$5''$	$6'$	$7'$	$8'$	$8''$
1	$1a$	1	1539	14364	25536	25536	25536	68096	131328	229824	229824
2	$760a$	1	−81	3024	−4704	−1344	−1344	7616	1728	−12096	−12096
3	$3344a$	1	−261	1764	2436	336	336	896	5328	3024	3024
4	$9405a$	1	99	924	2016	−1344	−1344	896	−3072	1344	1344
$5'$	$35112a$	1	99	684	−144	−624	−624	−64	1728	$\mathbf{-216+1800r_5}$	$\mathbf{-216-1800r_5}$
$5''$	$35112b$	1	99	684	−144	−624	−624	−64	1728	$\mathbf{-216-1800r_5}$	$\mathbf{-216+1800r_5}$
6	$267520a$	1	9	414	366	336	336	716	−72	−216	−216
7	$270864a$	1	19	364	−644	336	336	−224	−1072	−336	−336
$8'$	$374528a$	1	−81	−36	−144	36	36	−424	−72	$\mathbf{-486-450r_5}$	$\mathbf{-486+450r_5}$
$8''$	$374528b$	1	−81	−36	−144	36	36	−424	−72	$\mathbf{-486+450r_5}$	$\mathbf{-486-450r_5}$
$9'$	$656250a$	1	−45	−180	48	$\mathbf{144+48i_{19}}$	$\mathbf{144-48i_{19}}$	320	.	288	288
$9''$	$656250b$	1	−45	−180	48	$\mathbf{144-48i_{19}}$	$\mathbf{144+48i_{19}}$	320	.	288	288
10	$1185030a$	1	19	44	176	96	96	−704	128	384	384
11	$1361920a$	1	54	9	126	126	126	116	153	−351	−351
$12'$	$1361920b$	1	54	−81	−84	$\mathbf{-54+75i_{10}}$	$\mathbf{-54-75i_{10}}$	56	−297	189	189
$12''$	$1361920c$	1	54	−81	−84	$\mathbf{-54-75i_{10}}$	$\mathbf{-54+75i_{10}}$	56	−297	189	189
13	$1575936a$	1	−26	109	−194	−24	−24	136	353	609	609
14	$2407680a$	1	9	−66	−54	−24	−24	−4	528	−336	−336
15	$4561920a$	1	−26	29	86	−84	−84	16	−247	−111	−111

$9'$	$10'$	$11'$	$12'$	$12''$	$13'$	$13''$	$14'$	$15'$
689472	787968	787968	612864	612864	612864	612864	5515776	5515776
54432	−67392	36288	−12096	−12096	28224	28224	−108864	72576
15372	17568	24768	−25536	−25536	−8736	−8736	22176	−28224
14112	10368	−1152	8064	8064	8064	8064	−8064	−48384
−1008	−1152	4608	$\mathbf{3744-2400r_5}$	$\mathbf{3744+2400r_5}$	−4416	−4416	−2304	576
−1008	−1152	4608	$\mathbf{3744+2400r_5}$	$\mathbf{3744-2400r_5}$	−4416	−4416	−2304	576
−558	1908	−1692	204	204	−276	−276	−7524	6336
1652	128	−672	224	224	−896	−896	5376	−3584
−648	1728	1728	$\mathbf{324-900r_5}$	$\mathbf{324+900r_5}$	504	504	−3024	216
−648	1728	1728	$\mathbf{324+900r_5}$	$\mathbf{324-900r_5}$	504	504	−3024	216
720	−576	576	576	576	$\mathbf{-576-384i_{19}}$	$\mathbf{-576+384i_{19}}$	−576	−1152
720	−576	576	576	576	$\mathbf{-576+384i_{19}}$	$\mathbf{-576-384i_{19}}$	−576	−1152
592	−832	−192	64	64	64	64	−1344	896
−1053	−567	918	54	54	549	549	1971	−2484
27	513	378	−216	−216	$\mathbf{-81+225i_{10}}$	$\mathbf{-81-225i_{10}}$	−1269	1026
27	513	378	−216	−216	$\mathbf{-81-225i_{10}}$	$\mathbf{-81+225i_{10}}$	−1269	1026
−733	−127	−342	−106	−106	559	559	−129	−1124
702	468	−972	−36	−36	−36	−36	396	−144
−173	−367	−222	−146	−146	−101	−101	1191	596

$b \in \{\pm 675\}$. The orthogonality relations, see Proposition (3.8), imply $a \cdot b < 0$. This determines the character table of $E_K^{1_{H'}}$ up to a table automorphism of $\mathrm{Irr}(E_K^{1_{H'}})$, see Definition (8.5).

The character table of $E_K^{1_{H'}}$ is shown in Table 18. Rows and column have been reordered and column indexing has been adjusted to exhibit the phenomena described in Section 5, where the character table of $E_K = E_K^{1_H}$ is contained in the database, see Section (11.1).

(14.2) Remark. Let $C_{3A} \in \mathcal{C}l(G)$ denote the $3A$-conjugacy class of G, see [13]. Then G acts on C_{3A} by conjugation, and as $H' = C_G(3a)$, where $3a \in C_{3A}$ is a suitable representative of the $3A$-conjugacy class, the G-sets Ω' and C_{3A} are isomorphic. Using GAP, we compute the class multiplication coefficients

$$m_{3A,3A,C} := |\{(x,y) \in G \times G; x, y \in C_{3A}, xy = z_0 \in C\}| \in \mathbb{N}_0,$$

where $C \in \mathcal{C}l(G)$ and $z_0 \in C$ is a fixed element. We find $m_{3A,3A,C} \neq 0$ for the conjugacy classes $C \in \{C_{1A}, C_{3A}, C_{3B}, C_{4A}, C_{5B}, C_{6A}, C_{10A}, C_{15A}\}$. Hence we have a bijection between these conjugacy classes and the orbitals $\mathcal{O}'_i \subseteq \Omega' \times \Omega'$, for $i \in \mathcal{I}'$. Furthermore, the corresponding index parameters k'_i are given as $k'_C = \frac{|C| \cdot m_{3A,3A,C}}{|C_{3A}|}$. Using the character table of $E_K = E_K^{1_H}$, this determines the splitting of the suborbits Ω_i of Ω, for $i \in \mathcal{I}$, into those of Ω'. The split suborbits are $i \in \{1, 3, 5\}$.

C	k'_C	i
$1A$	1	1
$3A$	1	1
$3B$	30800	2
$4A$	534600	3
$5B$	7185024	5
$6A$	534600	3
$10A$	3742200	4
$15A$	7185024	5

15 The Thompson group Th

(15.1) Let $G := Th$ and $H := {}^3D_4(2).3$. We have $r = 11$.

Let $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. We start with explicitly known matrices for the action of the elements of $\mathcal{G}$ on the absolutely irreducible $\mathbb{F}_2G$-module of $\mathbb{F}_2$-dimension 248 available in [83]. Tensoring with $\mathbb{F}_4$ over $\mathbb{F}_2$ yields an $\mathbb{F}_4G$-module V.

The subgroup $H < G$ is a maximal subgroup of G, and a set of generators of H, given as words in $\mathcal{G}$, is available in [83] as well. Using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that V_H has exactly two H-invariant 1-dimensional $\mathbb{F}_4$-subspaces. We choose

Table 18: The character table for $G := Ly$ and $H' := 3.McL$.

φ	χ_φ	$1'$	$1''$	$2'$	$3'$	$3''$	$4'$	$5'$	$5''$
1	$1a$	1	1	30800	534600	534600	3742200	7185024	7185024
2	$45694a$	1	1	−650	10125	10125	−4050	−7776	−7776
3	$1534500a$	1	1	−20	−60	−60	5610	−2736	−2736
4	$3028266a$	1	1	350	125	125	−1050	224	224
5	$4997664a$	1	1	−200	−150	−150	−1050	774	774
6	$1152735a$	1	−1	.	1800	−1800	.	3024	−3024
7	$3073960a$	1	−1	.	−675	675	.	3024	−3024
8	$5379430a$	1	−1	.	.	.	.	−2376	2376
ψ_1		1	−1	.	.	.	.	3024	−3024
ψ_2		.	.	.	1	−1	.	.	.

one of them, $\langle v_H\rangle_{\mathbb{F}_4} \leq V$ say, and as $H < G$ is a maximal subgroup of G, we conclude that there is a G-set isomorphism between the G-orbit $\langle v_H\rangle_{\mathbb{F}_4} \cdot G$ of 1-dimensional $\mathbb{F}_4$-subspaces of V and $\Omega := H|G$.

To use the strategy described in Section (10.1) efficiently, we proceed as described in Section (9.8). We choose a subgroup $U < H$, and compute the orbit counting numbers with respect to $\Omega = \coprod_{j\in\mathcal{J}} \tilde{\Omega}_j$ first. From these the orbit counting numbers with respect to $\Omega = \coprod_{i\in\mathcal{I}} \Omega_i$ are found. We choose $U := 2_+^{1+8}{:}\, L_2(8){:}\, 3 = N_H(2a) < H$, where $C_{2A} \in \mathcal{C}l(H)$ denotes the $2A$-conjugacy class of H, and $2a \in C_{2A}$. The subgroup $U < H$ is a maximal subgroup of H. A set of generators of U, given as words in the set of generators of H, is found using a standard MeatAxe technique, exploiting the fact that the subgroup U is the centralizer in H of an element of order 2. We have $|U| = 774144$, and using GAP we find $\langle 1_H^G, 1_U^G\rangle_G = 241$, thus we have $|\mathcal{J}| = 241$, while $|\mathcal{I}| = r = 11$. Furthermore, we choose $U_1 := 9{:}\,6 < U$ to be a subgroup of order 54. Using the MeatAxe we find that V_{U_1} has an absolutely irreducible $\mathbb{F}_4U_1$-epimorphic image V_1 of $\mathbb{F}_4$-dimension 6. Hence U_1 acts faithfully on V_1.

We compute the orbit counting numbers for the elements in $\mathcal{G}$. Applying Section (9.8), Remark (9.7) and using the technique described in Section (8.2), we obtain a splitting of $K^{1\times r}$ into eigenspaces of dimension 1. The character table of E_K is shown in Table 19, where the index parameters have also been found in [51], see also [34].

(15.2) Let $G := Th$ and $H := 2^5.L_5(2)$. We have $r = 11$.

We apply the same strategy as described in Section (15.1). Let $\mathcal{G} \subseteq G$ be as in Section (15.1), and let V be the absolutely irreducible $\mathbb{F}_2G$-module of $\mathbb{F}_2$-dimension 248. Again, the subgroup $H < G$ is a maximal subgroup of G, and a set of generators of H, given as words in $\mathcal{G}$, is available in [83] as well. Using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that V_H has exactly one 5-dimensional absolutely irreducible $\mathbb{F}_2H$-submodule W. As $H < G$ is a maximal subgroup of G, we conclude that there is a G-set isomorphism between the G-orbit $W \cdot G$ of 5-dimensional $\mathbb{F}_2$-subspaces of V and $\Omega := H|G$.

We choose $U := (2 \times 2^4).L_4(2) < 2^5.(2^4{:}\, L_4(2)) < H$, hence U is a preimage of a Levi subgroup of a maximal, maximal parabolic subgroup of $L_5(2)$, with respect to the natural group epimorphism $H \to L_5(2)$. We have $|U| = 645120$, and using GAP we find $\langle 1_H^G, 1_U^G\rangle_G = 482$, thus we have $|\mathcal{J}| = 482$, while $|\mathcal{I}| = r = 11$. Applying a few standard MeatAxe techniques we find a set of generators of U as words in the set of generators of H.

We choose $U_1 := (7{:}\,3) \times 2 < A_7 < L_4(2) < U$ to be a subgroup of order 42; note that $L_4(2) \cong A_8$. Again applying a few standard MeatAxe techniques, we find a set of generators of U_1 as words on the set of generators of U. Furthermore, using the algorithms to compute submodule lattices described in [47] available

Table 19: The character table for $G := Th$ and $H := {}^3D_4(2).3$.

φ	χ_φ	1	2	3	4	5	6	7	8	9	10	11
1	$1a$	1	17199	45864	179712	1304576	2201472	5031936	8128512	8805888	11741184	105670656
2	$4123a$	1	−2457	6552	.	46592	.	−179712	−290304	.	419328	.
3	$30875a$	1	459	2340	−7776	−7840	−34992	76896	12960	101088	90144	−233280
4	$61256a$	1	1323	2772	−4320	16352	15120	6048	72576	−75600	56448	−90720
5	$2450240a$	1	−441	504	−288	4256	4032	7056	−6048	1008	−7056	−3024
6	$3376737a$	1	−117	612	1440	−928	−5040	3168	−864	−7200	288	8640
7	$4881384a$	1	403	492	960	112	2640	−1392	−144	6000	−2032	−7040
8	$11577384a$	1	99	−36	−288	−2224	1872	2736	−3888	−3312	1584	3456
9	$28861000a$	1	−117	180	−288	−928	144	−2016	1728	1008	−576	864
10	$40199250a$	1	99	−36	−288	800	−1152	−288	−864	−288	−1440	3456
11	$51684750a$	1	−45	−180	288	224	288	288	864	288	1440	−3456

in the MeatAxe, we find that V_{U_1} has an $\mathbb{F}_2U_1$-epimorphic image isomorphic to

$$V_0 := \bigoplus_{k=1}^{5} \left[\begin{array}{c} 3a \\ 3a \end{array} \right],$$

where the summands are pairwise isomorphic uniserial $\mathbb{F}_2U_1$-modules with composition series as indicated, and where $3a$ is one of the absolutely irreducible $\mathbb{F}_2U_1$-modules of $\mathbb{F}_2$-dimension 3. Hence we obtain $\mathbb{F}_2U_1$-epimorphisms $\hat{q}_k$, for $k \in \{1, \dots, 5\}$, given by concatenating the natural projection onto V_0 with either of the projections onto the indecomposable summands of V_0. We have $\dim_{\mathbb{F}_2} \mathrm{im}\,(\hat{q}_k) = 6$, for $k \in \{1, \dots, 5\}$.

We compute the orbit counting numbers for the elements in $\mathcal{G}$, applying Section (9.8), Remark (9.7) and using the technique described in Section (8.2), we obtain a splitting of $K^{1\times r}$ into 7 eigenspaces of dimension 1 and two eigenspaces of dimension 2, where K-bases $\{\psi_1, \psi_2\}$ and $\{\psi_3, \psi_4\}$ of the latter are shown in Table 20. Using the degrees of the characters in $\mathrm{Irr}^1_K(G)$ we conclude that we have found $\{\varphi_1, \varphi_2, \varphi_5, \varphi_6, \varphi_8, \varphi_{10}, \varphi_{11}\}$, while φ_3, φ_4, φ_7 and φ_9 are missing.

As $\chi_{\varphi_3}, \chi_{\varphi_4} \in \mathrm{Irr}^1_K(G)$ are a pair of complex conjugate characters, by Remark (3.21) this also holds for $\varphi_3, \varphi_4 \in \mathrm{Irr}_K(E_K)$. Hence, by Proposition (3.1), there is at least one pair of non-self-paired orbitals. Hence we conclude that there is exactly one such pair, namely the orbitals $\{4, 5\}$. Thus we have $\varphi_3, \varphi_4 \in \langle\psi_1, \psi_2\rangle_K$, and $\varphi_7, \varphi_9 \in \langle\psi_3, \psi_4\rangle_K$. As φ_7, φ_9 are real-valued, we obtain these characters using the technique described in (12.3). From $\varphi_3(\alpha_5) = \overline{\varphi_3(\alpha_4)}$ we find the real part $\frac{\varphi_3(\alpha_4)+\overline{\varphi_3(\alpha_4)}}{2} = -1008$ of $\varphi_3(\alpha_4)$. From this φ_3 and φ_4 are determined, using the technique described in Section (13.4).

The character table of E_K is shown in Table 20, where $i_6 := i \cdot \sqrt{6} \in \mathbb{C}$, and where the index parameters have also been found in [34].

16 The Janko group J_4

(16.1) Let $G := J_4$ and $H := 2^{11}\colon M_{24}$. We have $r = 7$.

The index parameters and the structure constants matrix for the smallest non-trivial suborbit Ω_2 with $k_2 = 15180$ have been computed in [35]. Using the technique described in Section (8.2), where this structure constants matrix is sufficient to get eigenspaces of dimension 1, we obtain the character table of E_K as given in Table 21, where $r_{33} := \sqrt{33} \in \mathbb{R}$.

(16.2) Let still $G := J_4$ and $H := 2^{11}\colon M_{24}$, as well as $H' := 2^{11}\colon M_{23}$. We have $r' = 10$. Let $\Omega := H|G$ and $\Omega' := H'|G$, hence we have $\frac{|\Omega'|}{|\Omega|} = [H\colon H'] = 24$.

Let $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. We start with explicitly known matrices for the action of the elements of $\mathcal{G}$ on the absolutely irreducible $\mathbb{F}_2G$-module V of $\mathbb{F}_2$-dimension 112 available in [83]. The subgroup

Table 20: The character table for $G := Th$ and $H := 2^5.L_5(2)$.

φ	χ_φ	1	2	3	4	5	6	7	8	9	10	11
1	$1a$	1	248	59520	2064384	2064384	2539520	6666240	35553280	63995904	63995904	106659840
2	$61256a$	1	59	2820	−31248	−31248	44720	−1680	202720	−16128	174384	−344400
5	$4881384a$	1	39	1000	2352	2352	5680	5600	−5600	−22848	−3696	15120
6	$11577384a$	1	23	120	1584	1584	1520	−8160	−7520	16704	5904	−11760
8	$30507008a$	1	14	−150	1557	1557	−1720	750	9850	−4383	−171	−7305
10	$72925515a$	1	−17	160	224	224	−480	1680	−2240	3584	2464	−5600
11	$91171899a$	1	−1	−240	−288	−288	800	−240	−2240	−4608	3744	3360
3	$1707264a$	1	−31	930	$-1008 + 3780i_6$	$-1008 - 3780i_6$	−4960	−13020	8680	−31248	15624	26040
4	$1707264b$	1	−31	930	$-1008 - 3780i_6$	$-1008 + 3780i_6$	−4960	−13020	8680	−31248	15624	26040
7	$28861000a$	1	−13	12	−144	−144	1520	−1680	6880	4608	−17424	6384
9	$40199250a$	1	23	120	−1440	−1440	−1504	912	−1472	4608	−3168	3360
ψ_1		1	−31	930	.	−2016	−4960	−13020	8680	−31248	15624	26040
ψ_2		.	.	.	1	−1	.	.	.	.	.	.
ψ_3		1	.	51	−612	−612	428	−744	3864	4608	−12276	5292
ψ_4		.	1	3	−36	−36	−84	72	−232	.	396	−84

Table 21: The character table for $G := J_4$ and $H := 2^{11}\colon M_{24}$.

φ	χ_φ	1	2	3	4	5	6	7
1	$1a$	1	15180	28336	3400320	32643072	54405120	82575360
2	$889111a$	1	825	1166	14520	19008	10560	-46080
3	$1776888a$	1	517	-990	-1496	32560	-23936	-6656
4	$4290927a$	1	-253	.	7084	.	-28336	21504
5	$35411145a$	1	$66-17r_{33}$	$99+19r_{33}$	$-1166-154r_{33}$	$-1056+992r_{33}$	$-2552-328r_{33}$	$4608-512r_{33}$
6	$35411145b$	1	$66+17r_{33}$	$99-19r_{33}$	$-1166+154r_{33}$	$-1056-992r_{33}$	$-2552+328r_{33}$	$4608+512r_{33}$
7	$95288172a$	1	-55	-66	440	.	3520	-3840

$H < G$ is a maximal subgroup of G, and a set of generators of H, given as words in $\mathcal{G}$, is available in [83] as well.

Using the MeatAxe and the absolutely irreducible $\mathbb{F}_2H$-module $11a$, on which the normal 2-subgroup $2^{11} \trianglelefteq H$ hence acts trivially, by a random search we find a set of standard generators of $H/2^{11} \cong M_{24}$. Using V, it turns out that this set indeed generates a subgroup isomorphic to M_{24} in H. Furthermore, by a random search we find an element of H contained in the normal subgroup $2^{11} \trianglelefteq H$. As M_{24} acts non-trivially on the normal subgroup 2^{11}, the latter is an absolutely irreducible $\mathbb{F}_2M_{24}$-module. Altogether this yields a set of generators of H, which is a preimage of a set of standard generators of the epimorphic image $M_{24} \cong H/2^{11}$ of H. A set of generators of M_{23}, given as words in a set of standard generators of M_{24}, is available in [83] as well. Hence this can be used to find a set of generators $\mathcal{H}'$ of H' as words in the set of generators of H found above, and to find a set of standard generators of a subgroup $M_{23} = H' \cap M_{24} < H$. Note that M_{23} acts non-trivially on the normal subgroup $2^{11} \trianglelefteq H'$, hence the latter is an absolutely irreducible $\mathbb{F}_2M_{23}$-module, see [37].

Using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that

$$V_H \cong \begin{bmatrix} 1a \\ 11b \\ 44b \\ 44a \\ 11a \\ 1a \end{bmatrix},$$

a uniserial $\mathbb{F}_2H$-module with composition series as indicated, where the constituents are absolutely irreducible $\mathbb{F}_2H$-modules of the respective dimensions, see [37], and $11a/b$ and $44a/b$ are pairs of mutually contragredient $\mathbb{F}_2H$-modules. Furthermore, we find that $V_{H'}$ is a uniserial $\mathbb{F}_2H'$-module, where the $\mathbb{F}_2H$-constituents of V_H restrict to pairwise non-isomorphic absolutely irreducible $\mathbb{F}_2H'$-modules. Let $V' \leq V_H$ be the uniquely determined $\mathbb{F}_2H$-submodule of $\mathbb{F}_2$-dimension 12, being isomorphic to $V' \cong \begin{bmatrix} 11a \\ 1a \end{bmatrix}$ as $\mathbb{F}_2H$-modules. Hence the G-orbit $V' \cdot G$ of 12-dimensional $\mathbb{F}_2$-subspaces of V is as a G-set isomorphic to Ω. While enumerating the G-orbit $V' \cdot G$, we collect a set $\{g_i \in G; i \in \mathcal{I}\}$ of representatives of the right cosets $H|G$ of H in G, as words in the set $\mathcal{G}$. From that we find the G-action on Ω' as follows.

We use the strategy which has also been used in Section (13.1). Let $\Xi := H'|H$ be the set of right cosets of H' in H. Let $\{h_j; j \in \{1, \ldots, [H\colon H']\}\}$ be a set of representatives of the right cosets $H'|H$ of H' in H, where $h_1 := 1$. Hence we obtain a set of representatives $\{h_j g_i; j \in \{1, \ldots, [H\colon H']\}, i \in \mathcal{I}\}$ of the right cosets $H'|G$ of H in G. Let $\pi_\Omega\colon G \to \mathcal{S}_n$ as well as $\pi_\Xi\colon H \to \mathcal{S}_{[H:H']}$ and $\pi_{\Omega'}\colon G \to \mathcal{S}_{n'}$ denote the group homomorphisms defined by the action of G on Ω, by the action of H on Ξ, and by the action of G on Ω', respectively.

For $g \in G$ as well as $i \in \mathcal{I}$ and $j \in \{1, \ldots, [H{:}H']\}$, let $i' := i \cdot \pi_\Omega(g)$ and $j' := j \cdot \pi_\Xi(g_i \cdot g \cdot g_{i'}^{-1})$. Hence we have $h_j g_i \cdot g = h \cdot h_{j'} g_{i'}$, for some $h \in H$. Thus $\pi_{\Omega'}(g)$ can be determined from $\pi_\Omega(g)$ and π_Ξ, where we have to determine $\pi_\Xi(g_i \cdot g \cdot \tilde{g}_{i'}^{-1})$ explicitly. This is achieved as follows.

Let $V'^* := \mathrm{Hom}_{\mathbb{F}_2 H}(V', \mathbb{F}_2)$ denote the $\mathbb{F}_2 H$-module contragredient to V'. Hence we have $V'^* \cong \begin{bmatrix} 1a \\ 11b \end{bmatrix}$ as $\mathbb{F}_2 H$-modules. It turns out that there is $v^* \in V'^*$ such that the H-orbit $v^* \cdot H \subseteq V'^*$ is as an H-set isomorphic to the exterior square $\Xi \wedge \Xi$ of Ξ, hence we have $|\Xi \wedge \Xi| = 276$. Using the H'-action on $\Xi \wedge \Xi$, the elements of $v^* \cdot H \subseteq V'^*$ can be identified with the subsets of cardinality 2 of Ξ. Given $g_i \cdot g \cdot g_{i'}^{-1} \in H = \mathrm{Stab}_G(V')$, using the MeatAxe, we compute matrices representing its action on V' and on V'^*. From that its action on $\Xi \wedge \Xi$ is found, and using the identification with subsets of cardinality 2 of Ξ, the permutation $\pi_\Xi(g_i \cdot g \cdot \tilde{g}_{i'}^{-1}) \in \mathcal{S}_{[H:H']}$ can be determined.

To use the strategy described in Section (10.1) efficiently, we proceed as described in Section (9.8). We choose $U_1 := L_2(11) < U := M_{22} < M_{23} = H' \cap M_{24} < H$, where a set of standard generators of M_{22}, given as words in a set of standard generators of M_{23}, and a set of standard generators of $L_2(11)$, given as words in a set of standard generators of M_{22}, are available in [83]. We have $|U| = 443520$, and using GAP we find $\langle 1_H^G, 1_U^G \rangle_G = 582$ and $\langle 1_{H'}^G, 1_U^G \rangle_G = 9609$. Furthermore, using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that

$$V_{U_1} \cong \begin{bmatrix} 10a \\ 1a \end{bmatrix} \oplus \begin{bmatrix} 1a \\ 10a \end{bmatrix} \oplus \bigoplus_{i=1}^{2} 1a \oplus \bigoplus_{i=1}^{4} 10b \oplus \bigoplus_{i=1}^{2} 24a,$$

as $\mathbb{F}_2 U_1$-modules, where the constituents $1a$ and $10a$ are absolutely irreducible $\mathbb{F}_2 U_1$-modules of the respective dimensions, and $10b$ and $24a$ are irreducible $\mathbb{F}_2 U_1$-modules having splitting field $\mathbb{F}_4$, see [37]. As $\mathbb{F}_2 U_1$-epimorphic image V_1 we choose an $\mathbb{F}_2 U_1$-direct summand of V_{U_1} isomorphic to $\begin{bmatrix} 10a \\ 1a \end{bmatrix} \oplus 10b$, together with the corresponding $\mathbb{F}_2 U_1$-projection. Hence we have $\dim_{\mathbb{F}_2} V_1 = 21$.

Using the technique described in Sections (10.3) and (9.8) we compute the orbit counting numbers for the elements in $\mathcal{G}$ and $\mathcal{H}'$. Using Remark (9.7) and the technique described in Section (8.2), it turns out that that the resulting structure constants matrices are sufficient to obtain a splitting of $K^{1 \times r'}$ into eigenspaces of dimension 1. The character table of $E_K^{1_{H'}}$ is given in Table 22. Rows and column have been reordered and column indexing has been adjusted to exhibit the phenomena described in Section 5, see Example (5.14), where the character table of E_K is given in Table 21, and $r_{33} := \sqrt{33} \in \mathbb{R}$.

17 The Baby Monster B

(17.1) Let $G := B$ and $H := 2.^2E_6(2).2$, as well as $H' := 2.^2E_6(2)$ and $\lambda' = 1$, hence we have $\mathrm{Irr}_K^1(H) = \{1, 1^-\}$, see Remark (5.15). We have $r = 5$

Table 22: The character table for $G := J_4$ and $H' := 2^{11}\colon M_{23}$.

φ	χ_φ	$1'$	$1''$	$2'$	$2''$	$3'$	$4'$
1	$1a$	1	23	121440	242880	680064	81607680
2	$889111a$	1	23	6600	13200	27984	348480
3	$1776888a$	1	23	4136	8272	-23760	-35904
4	$4290927a$	1	23	-2024	-4048	.	170016
5	$35411145a$	1	23	$528 - 136r_{33}$	$1056 - 272r_{33}$	$2376 + 456r_{33}$	$-27984 - 3696r_{33}$
6	$35411145b$	1	23	$528 + 136r_{33}$	$1056 + 272r_{33}$	$2376 - 456r_{33}$	$-27984 + 3696r_{33}$
7	$95288172a$	1	23	-440	-880	-1584	10560
8	$460559498a$	1	-1	528	-528	.	.
9	$493456605a$	1	-1	352	-352	.	.
10	$1184295852a$	1	-1	-352	352	.	.
11	$1842237992a$	1	-1	.	.	.	.

$5'$	$5''$	$6'$	$7'$	$7''$
130572288	652861440	1305722880	82575360	1899233280
76032	380160	253440	-46080	-1059840
130240	651200	-574464	-6656	-153088
.	.	-680064	21504	494592
$-4224 + 3968r_{33}$	$-21120 + 19840r_{33}$	$-61248 - 7872r_{33}$	$4608 - 512r_{33}$	$105984 - 11776r_{33}$
$-4224 - 3968r_{33}$	$-21120 - 19840r_{33}$	$-61248 + 7872r_{33}$	$4608 + 512r_{33}$	$105984 + 11776r_{33}$
.	.	84480	-3840	-88320
16896	-16896	.	-12288	12288
.	.	.	21504	-21504
9856	-9856	.	1792	-1792
-10560	10560	.	-3840	3840

and the split suborbits are $\mathcal{I}_{1^-} = \{1, 2, 4\}$, as is shown in [29], where also the character tables of E_K and $E_K^{1^-}$ are given. Using Remark (5.15), from this the character table of $E_K^{1_{H'}}$ can be determined. The character tables of E_K and $E_K^{1^-}$ as well as $E_K^{1_{H'}}$ are given in Table 23.

(17.2) Let $G := B$ and $H := 2^{1+22}.Co_2$. We have $r = 10$.

The index parameters k_i, for $i \in \mathcal{I}$ have been determined in [34], but no explicit proof is given there. Unfortunately, the values for the index parameters given there do not sum up to $n = [G\colon H]$. Hence we compute the index parameters anew, by applying the same strategy as in Remark (14.2).

Let $C_{2B} \in \mathcal{C}l(G)$ denote the $2B$-conjugacy class of G, see [13]. Then G acts on C_{2B} by conjugation, and as $H = C_G(2b)$, where $2b \in C_{2B}$ is a suitable representative of the $2B$-conjugacy class, the G-sets Ω and C_{2B} are isomorphic. For $C \in \mathcal{C}l(G)$ let $(C_{2B})_C := \{g \in C_{2B}; (2b) \cdot g \in C\} \subseteq C_{2B}$, which are unions of H-orbits. Letting $k_C := |(C_{2B})_C| \in \mathbb{N}_0$, we have $k_C = \frac{|C| \cdot m_{2B,2B,C}}{|C_{2B}|}$, where $m_{2B,2B,C} \in \mathbb{N}_0$ is the corresponding class multiplication coefficient. Using GAP we compute the class multiplication coefficients $m_{2B,2B,C} \in \mathbb{N}_0$ and find $k_C \neq 0$ for the conjugacy classes

$$C \in \{C_{1A}, C_{2B}, C_{2D}, C_{3A}, C_{4B}, C_{4E}, C_{4G}, C_{5A}, C_{6C}\},$$

and the cardinalities k_C as given in Table 24.

As we have $r = 10$, but only find 9 conjugacy classes $C \in \mathcal{C}l(G)$ such that $k_C \neq 0$, we conclude that precisely one of the corresponding subsets $(C_{2B})_C \subseteq C_{2B}$ consists of two H-orbits, while the others consist of one H-orbit. As k_{2B} is the only of these cardinalities which is not a divisor of $|H|$, we conclude that $(C_{2B})_{2B}$ splits. The lengths of the two suborbits contained in $(C_{2B})_{2B}$ are determined in Section (17.4). Sorting the suborbits with respect to increasing lengths gives the indexing with $i \in \mathcal{I}$ also indicated in Table 24.

After all, it turns out that in [34] the value of $k_7 = k_{4G}$ is falsely stated as 4 700 602 368, obviously a misprint.

(17.3) Let $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. We start with explicitly known matrices for the action of the elements of $\mathcal{G}$ on the absolutely irreducible $\mathbb{F}_2G$-module V of $\mathbb{F}_2$-dimension 4370 available in [83]. The subgroup $H < G$ is a maximal subgroup of G, and a set of generators of H, given as words in $\mathcal{G}$, is available in [83] as well. Using a random search and the MeatAxe, we find a set of generators $\mathcal{H}$ of H being a preimage of a set of standard generators of Co_2 with respect to the natural group epimorphism $H \to Co_2$. Using the MeatAxe we find that V_H has a uniquely determined trivial submodule $1a \leq V_H$, and if we pick $0 \neq v_H \in 1a \leq V$ we conclude that the G-orbit $v_H \cdot G \subseteq V$ is as a G-set isomorphic to Ω.

Table 23: The character tables for $G := B$ and $H := 2.^2E_6(2).2$, where $\lambda = 1$ and $\lambda = 1^-$, as well as for $H' := 2.^2E_6(2)$.

φ	χ_φ	1	2	3	4	5
1	$1a$	1	3968055	23113728	2370830336	11174042880
2	$96255a$	1	228735	−709632	14483456	−14002560
3	$9458750a$	1	50895	133056	124928	−308880
4	$4275362520a$	1	1935	−4032	−31744	33840
5	$9287037474a$	1	−945	1728	14336	−15120

φ	χ_φ	1	2	4
1	$4371a$	1	566865	84672512
2	$63532485a$	1	28665	−114688
3	$13508418144a$	1	−135	512

φ	χ_φ	$1'$	$1''$	$2'$	$2''$	$3'$	$4'$	$4''$	$5'$
1	$1a$	1	1	3968055	3968055	46227456	2370830336	2370830336	22348085760
2	$96255a$	1	1	228735	228735	−1419264	14483456	14483456	−28005120
3	$9458750a$	1	1	50895	50895	266112	124928	124928	−617760
4	$4275362520a$	1	1	1935	1935	−8064	−31744	−31744	67680
5	$9287037474a$	1	1	−945	−945	3456	14336	14336	−30240
6	$4371a$	1	−1	566865	−566865	.	84672512	−84672512	.
7	$63532485a$	1	−1	28665	−28665	.	−114688	114688	.
8	$13508418144a$	1	−1	−135	135	.	512	−512	.

Table 24: Conjugacy classes and suborbits.

i	C	k_C	splits into	$\dim_{\mathbb{F}_2} \mathrm{Fix}_V(\cdot)$
1	$1A$	1		
2, 3	$2B$	7 379 550	93 150 + 7 286 400	2322
4	$2D$	262 310 400		2202
6	$3A$	9 646 899 200		
5	$4B$	4 196 966 400		1256
8	$4E$	537 211 699 200		1114
7	$4G$	470 060 236 800		1166
9	$5A$	4 000 762 036 224		
10	$6C$	6 685 301 145 600		

We apply the strategy described in Section (10.6), and we choose the following chain of subgroups

$$\begin{array}{rcl} G = B & > & U_4 := U := H = 2^{1+22}.Co_2 \quad > \quad 2^{1+22}.M_{23} \\ & > & U_3 := 2^{11}.M_{22} \\ & > & U_2 := 2.M_{22} \\ & > & U_1 := L_2(11), \end{array}$$

where we have the following group orders

$$\begin{array}{rcrcl} |B| & = & 4\,154\,781\,481\,226\,426\,191\,177\,580\,544\,000\,000 & \sim & 4 \cdot 10^{33}, \\ |2^{1+22}.Co_2| & = & 354\,883\,595\,661\,213\,696\,000 & \sim & 4 \cdot 10^{20}, \\ |2^{11}.M_{22}| & = & 908\,328\,960 & \sim & 9 \cdot 10^{8}, \\ |2.M_{22}| & = & 887\,040 & \sim & 9 \cdot 10^{5}, \\ |L_2(11)| & = & 660 & \sim & 7 \cdot 10^{2}. \end{array}$$

Words in the set of standard generators of Co_2 giving a set of standard generators of the maximal subgroup $M_{23} < Co_2$ are available in [83]. We apply these to the set of generators $\mathcal{H}$ of H, which indeed yields a set of generators of the maximal subgroup $2^{1+22}.M_{23} < H$, as an analysis using the MeatAxe shows. Furthermore, words in the set of standard generators of M_{23} giving a set of standard generators of the maximal subgroup $M_{22} < M_{23}$ are also available in [83]. An application of these yields a subgroup $2^{1+22}.M_{22} < H$, as the MeatAxe shows.

Let $2^{1+22} \cong N \trianglelefteq H$ be the maximal normal 2-subgroup of H, which is an extraspecial group, such that Co_2 acts absolutely irreducibly on the $\mathbb{F}_2$-vector space $N/Z(N)$ of $\mathbb{F}_2$-dimension 22, see [13, 37]. The MeatAxe shows that the $\mathbb{F}_2 M_{22}$-module $(N/Z(N))_{M_{22}}$, for the subgroup $M_{22} < M_{23} < Co_2$, has the structure

$$(N/Z(N))_{M_{22}} \cong \begin{bmatrix} 1a \\ 10b \\ 10a \\ 1a \end{bmatrix},$$

a uniserial $\mathbb{F}_2 M_{22}$-module with composition series as indicated, where the constituents are absolutely irreducible $\mathbb{F}_2 M_{22}$-modules of the respective dimensions, and $10a, b$ are a pair of mutually contragredient $\mathbb{F}_2 M_{22}$-modules, see [37].

Going over to an $\mathbb{F}_2 H$-epimorphic image W of V_H of an $\mathbb{F}_2$-dimension small enough to do random searches using the MeatAxe quickly and on which $Z(H) = Z(N)$ acts trivially, we proceed as follows. The group acting on $W_{2^{1+22}.M_{22}}$ is isomorphic to $2^{22}.M_{22}$, and by a random search we find an element of $N/Z(N) \cong 2^{22}$ which under the conjugation action of M_{22} generates the $\mathbb{F}_2 M_{22}$-submodule of $(N/Z(N))_{M_{22}}$ of $\mathbb{F}_2$-dimension 11. By a random search using the MeatAxe, where we modify the given generators of $2^{22}.M_{22}$ by multiplying with elements of this $\mathbb{F}_2 M_{22}$-submodule, we find a set of generators of a subgroup $2^{11}.M_{22}$, as words in the given generators of $2^{1+22}.M_{22}$. Applying these to $2^{1+22}.M_{22}$ acting on V_H, we obtain a subgroup $2 \times 2^{11}.M_{22}$, as the MeatAxe shows, and we straightforwardly find a subgroup $2^{11}.M_{22} < 2 \times 2^{11}.M_{22}$ in there.

We already know that the normal subgroup $2^{11} \trianglelefteq 2^{11}.M_{22}$ as an $\mathbb{F}_2 M_{22}$-module is uniserial having the trivial $\mathbb{F}_2 M_{22}$-module $1a$ as its socle. Using the above strategy again, we find a subgroup $2.M_{22} < 2^{11}.M_{22}$, which is a non-split central extension of M_{22} by a cyclic group of order 2. As the set of generators we have obtained is a preimage of a set of standard generators of M_{22}, we use the words giving a maximal subgroup $L_2(11) < M_{22}$ available in [83], to find a subgroup $2 \times L_2(11) < 2.M_{22}$ and straightforwardly a subgroup $L_2(11) < 2 \times L_2(11) < 2.M_{22}$ in there.

To specify the $\mathbb{F}_2 U_i$-modules V_i and the maps $\hat{q}_{i+1,i}\colon (V_{i+1})_{U_i} \to V_i$ of $\mathbb{F}_2 U_i$-modules, for $i \in \{1, \ldots, 3\}$, as in Section (10.5), we proceed as follows. Let $V_4 := V_H = (4370a)_H$. Using the programs to determine socle series described in [49] available in the MeatAxe, we compute a few layers of the socle series of the $\mathbb{F}_2 U_3$-module $V_{U_3}^*$ contragredient to V_{U_3}, which amounts to computing a few layers of the radical series of V_{U_3}. Going over to $V_{U_3}/\mathrm{rad}^5(V_{U_3})$, by a random search using the MeatAxe we look for a suitable $\mathbb{F}_2 U_3$-epimorphic image. The most restrictive of the conditions required for an application of the strategy described in Section (10.6) turns out to be the one, that the regular transitive U_i-sets Ξ'_i, for $i \in \{1, \ldots, 3\}$, are assumed to be realizable as a regular U_i-orbit of vectors in one of the quotient modules $(V_j)_{U_i}$, for $i \leq j \in \{1, \ldots, 4\}$, see Section (10.5). Using the MeatAxe and a random search, we find a suitable quotient module V_3 of $V_{U_3}/\mathrm{rad}^5(V_{U_3})$ of $\mathbb{F}_2$-dimension 78, and let $\hat{q}_{4,3}\colon (V_4)_{U_3} = V_{U_3} \to V_3$ denote the corresponding natural $\mathbb{F}_2 U_3$-epimorphism. Furthermore, using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find a suitable quotient module V_2 of $(V_3)_{U_2}$ of $\mathbb{F}_2$-dimension 31 with corresponding natural $\mathbb{F}_2 U_2$-epimorphism $\hat{q}_{3,2}\colon (V_3)_{U_2} \to V_2$, and a suitable quotient module V_1 of $(V_2)_{U_1}$ of $\mathbb{F}_2$-dimension 21 with corresponding natural $\mathbb{F}_2U_1$-epimorphism $\hat{q}_{2,1}\colon (V_2)_{U_1} \to V_1$. Hence the chosen quotient modules have the following $\mathbb{F}_2$-dimensions.

i	U_i	$\dim_{\mathbb{F}_2} V_i$
	B	4370
4	$2^{1+22}.Co_2$	4370
3	$2^{11}.M_{22}$	78
2	$2.M_{22}$	31
1	$L_2(11)$	21

(17.4) Let $\omega_1 = v_H \in V$. To find representatives $\omega_i \in \Omega_i \subseteq v_H \cdot G \subseteq V$ and elements $g_i \in G$ such that $\omega_i = \omega_1 g_i$, for $1 \neq i \in \mathcal{I}$, we use the G-set C_{2B} isomorphic to Ω, see Section (17.2). By a random search using the MeatAxe we compute the action on V of few elements $g \in G$, given as words in $\mathcal{G}$, and check to which conjugacy class of G the commutator $[(2b), g] := (2b) \cdot (g^{-1} \cdot (2b) \cdot g) \in G$ belongs. This is done by computing the order of $[(2b), g] \in G$ and $\dim_{\mathbb{F}_2} \mathrm{Fix}_V([(2b), g])$, where the $\mathbb{F}_2$-dimensions of the fixed spaces of representatives of the relevant conjugacy classes of G are as given in Table 24. This yields representatives of the suborbits $i \in \{1, 4, 6, \ldots, 10\}$. Summing up the k_i for $i \in \{1, 4, 6, \ldots, 10\}$ and dividing by $|\Omega|$, we obtain a fraction of ~ 0.9996. Hence it seems to be rather improbable to find further suborbits using a random search.

To proceed we concentrate on Ω_2. If we had indeed $k_2 = 93150$, we would be tempted to conjecture that there is an element $2b' \in N \trianglelefteq H$ such that $2b' \in C_{2B}$ and $(2b) \cdot (2b') \in C_{2B}$ as well as $C_H(2b') = 2^{1+21}.(2^{10}\colon M_{22}\colon 2)$, where $2^{10}\colon M_{22}\colon 2 < Co_2$ is a maximal subgroup and $C_H(2b') \cap N = 2^{1+21}$. Words in the set of standard generators of Co_2 giving a set of standard generators of the maximal subgroup $2^{10}\colon M_{22}\colon 2 < Co_2$ are available in [83], and the MeatAxe indeed shows that the $\mathbb{F}_2(2^{10}\colon M_{22}\colon 2)$-module $(N/Z(N))_{2^{10}:M_{22}:2}$ is uniserial with structure

$$(N/Z(N))_{2^{10}:M_{22}:2} \cong \begin{bmatrix} 1a \\ 10b \\ 10a \\ 1a \end{bmatrix},$$

using the notation from Section (17.3). Applying these words to the set of generators $\mathcal{H}$ of H, an analysis using the MeatAxe indeed yields a set of generators of a subgroup $2^{1+21}.(2^{10}\colon M_{22}\colon 2) < H$, where the normal subgroup $2^{1+21} < 2^{1+21}.(2^{10}\colon M_{22}\colon 2)$ necessarily is a preimage of the $\mathbb{F}_2(2^{10}\colon M_{22}\colon 2)$-submodule of $\mathbb{F}_2$-dimension 21 with respect to the natural group epimorphism $N \to N/Z(N)$. Computing $\mathrm{Fix}_V(2^{1+21}.(2^{10}\colon M_{22}\colon 2))$ we find a fixed vector $\omega_2 \in \Omega$ of $2^{1+21}.(2^{10}\colon M_{22}\colon 2)$, different from ω_1, and as $[H : (2^{1+21}.(2^{10}\colon M_{22}\colon 2))] = 93150$ we have thus proved that $k_2 = 93150$ and $k_3 = 7\,286\,400$, see Section (17.2). By applying the strategy described in Section (10.6), we enumerate a substantial part of suborbit Ω_9, say, and by checking randomly a few elements in the G-orbit of $\omega_2 \in \Omega$ we find an element in Ω_9, and thus an element $g_2 \in G$, as a word in the set of generators $\mathcal{G}$, such that $\omega_1 g_2 = \omega_2$. Furthermore, it is straightforward to enumerate Ω_2 completely by a standard breadth-first orbit algorithm.

We are also tempted to conjecture that the set $\Omega_2 \cdot g_2 \subseteq \Omega$ contains elements of Ω_3 and Ω_5, which are the two suborbits of which we not yet have representatives. This indeed turns out to be true, by checking a few elements of $\Omega_2 \cdot g_2$ using the same strategy as was used above for the longer suborbits.

(17.5) We are now able, by applying the strategy described in Section (10.6), to enumerate substantial parts of the suborbits $i \in \{5, \ldots, 10\}$. A problem arises for the suborbits $i \in \{3, 4\}$, since it turns out that in these cases the order $|\mathrm{Stab}_{U_3}(\hat{q}_{4,3}(\omega_i))|$ is large, which contradicts the assumptions made in Section (10.5) and causes the programs to become ineffective; we circumvent this.

We determine the structure constants matrix $P_2 = [p_{i,2,k}; i, k \in \mathcal{I}] \in \mathbb{Z}^{r \times r}$ for the smallest non-trivial suborbit Ω_2, with $k_2 = 93150$. For $i, k \in \mathcal{I}$, by Proposition (9.6) we have

$$p_{i,2,k} = \frac{k_i}{k_k} \cdot c_{2,k}(g_i) = \frac{k_i}{k_k} \cdot |\Omega_2 g_i \cap \Omega_k|,$$

where the $c_{2,k}(g_i) \in \mathbb{N}_0$ are the orbit counting numbers with respect to $\Omega = \coprod_{i \in \mathcal{I}} \Omega_i$, see Definition (9.4). Hence the remaining task is to apply successively the elements $g_i \in G$, for $i \in \mathcal{I}$, to all elements of $\Omega_2 \subseteq \Omega \subseteq V$ explicitly, and find the cardinalities $|\Omega_2 g_i \cap \Omega_k| \in \mathbb{N}_0$, for $k \in \mathcal{I}$.

Given $\omega \in \Omega_2 g_i$ and $k \in \mathcal{I}$, we have to check whether $\omega \in \Omega_k$ holds. For $k \notin \{3, 4\}$, as we have enumerated only a part of Ω_k explicitly, again it is not sufficient to check $\omega \in \Omega_2 g_i \subseteq V$ itself, but a few other elements of $\omega \cdot H \subseteq V$ have to be checked as well. Still, this method only allows to prove membership, but not to disprove it. Hence, in a first run over $k \in \mathcal{I}$ we only test very few elements of $\omega \cdot H \subseteq V$, at most 5 say, for membership in Ω_k. If $\omega \in \Omega_2 g_i$ cannot be proven to belong to a particular suborbit, we start a second run over $k \in \mathcal{I}$, where we now test some more elements of $\omega \cdot H \subseteq V$, at most 1000 say.

We could repeat this until all of $\Omega_2 g_i$ is treated. But actually after the second run, only a very few elements have not been proven to belong to a particular suborbit. Hence we have found lower bounds for the $c_{2,k}(g_i) \in \mathbb{N}_0$, where by Remark (9.7) we have $\sum_{k \in \mathcal{I}} c_{2,k}(g_i) = k_i$, for $i \in \mathcal{I}$. Furthermore, we have the following numerical conditions on the $c_{2,k}(g_i) \in \mathbb{N}_0$. As all the index parameters k_i, for $i \in \mathcal{I}$, are pairwise different, we conclude that all suborbits are self-paired. Hence by Proposition (3.17) we have, for $i, k \in \mathcal{I}$,

$$c_{2,k}(g_i) = \frac{k_k}{k_i} \cdot p_{i,2,k} = \frac{k_k}{k_i} \cdot p_{i,2^*,k} = \frac{k_k}{k_i} \cdot \frac{k_i}{k_k} \cdot p_{k,2,i} = p_{k,2,i} = \frac{k_k}{k_i} \cdot c_{2,i}(g_k) \in \mathbb{Z},$$

which hence is an integrality condition. In particular, we have $c_{2,k}(g_i) = 0$ if and only if $c_{2,i}(g_k) = 0$. It turns out that these conditions are sufficient to find all the numbers $c_{2,k}(g_i) \in \mathbb{N}_0$, for $i, k \in \mathcal{I}$, where $(i,k) \notin \{(3,3),(3,4),(4,3),(4,4)\}$. Using these numerical conditions, there are only finitely many possibilities for the matrix entries $c_{2,k}(g_i) \in \mathbb{N}_0$, for $i, k \in \{3,4\}$, left. It turns out that the number of candidate matrices is small enough to check the following additional necessary condition for all of them.

By Proposition (1.19) the structure constants matrix $P_2 \in \mathbb{Z}^{r \times r}$ is diagonalisable over an algebraic closure of $\mathbb{Q}$. As all characters in $\mathrm{Irr}^1_K(G)$ are rational-valued, by Propositions (3.10) and (3.20) we have $\Phi \in \mathbb{Z}^{r \times r}$. Thus the characteristic polynomial of P_2 splits into linear factors over the rationals. It turns out that the latter condition is fulfilled by precisely one of the candidate structure constants matrices obtained by Remark (9.7) from the above candidate orbit counting matrices. This determines the structure constants matrix $P_2 \in \mathbb{Z}^{r \times r}$ as is shown in Table 25.

Using the technique described in Section (8.2), and the matrix P_2 we obtain a splitting of $K^{1 \times r}$ into 8 eigenspaces of dimension 1 and an eigenspace of dimension 2. Using the degrees of the characters in $\mathrm{Irr}^1_K(G)$, see Section (8.1), we conclude that we have found the characters $\{\varphi_1, \varphi_3, \varphi_5, \ldots, \varphi_{10}\}$, while φ_2 and φ_4 are missing. These are found using the technique described in Section (12.3). The character table of E_K is shown in Table 26.

(17.6) Let $G := B$ and $H := Fi_{23}$, which is a maximal subgroup of G. We have $r = 23$.

First of all we construct an FG-module, for a suitable finite field F, containing a vector being H-invariant, but not G-invariant. Let $4370a$ be the absolutely irreducible $\mathbb{F}_2G$-module of $\mathbb{F}_2$-dimension 4370. Representing matrices for a set of standard generators $\{a, b\} \subseteq G$ in the sense of [81] are available in [83]; the elements a and b have order 2 and 3, respectively. Words in the set of standard generators giving a set of standard generators of H are also available in [83]. The $\mathbb{F}_2H$-module $(4370a)_H$ turns out to have the constituents $782a$ and $3588a$, where the latter are absolutely irreducible $\mathbb{F}_2H$-modules of the respective dimensions; hence $4370a$ would not serve our purposes.

Let $R \subset K$ and F be as in Section (2.10). Let $\hat{V} \in \mathbf{mod}_R\text{-}RG$ be an R-free RG-module such that $\hat{V} \otimes_R K$ is an irreducible KG-module of K-dimension 4371. By [13], $\hat{V} \otimes_R K$ is absolutely irreducible and uniquely determined up to equivalence. All the character values of $\hat{V} \otimes_R K$ are rational integers. As $\hat{V} \otimes_R K$ occurs as a constituent of multiplicity 1 in a rational representation of G, namely the permutation representation $1^G_{{}^2E_6(2)}$, see Section (17.1), we by [18, La.IV.9.1] conclude that the rational Schur index of $\hat{V} \otimes_R K$ is equal to 1. Hence for our constructive purposes we may choose $K := \mathbb{Q}$ and, as we construct a module in characteristic 2, let $R := \mathbb{Z}_{(2)}$, the localisation of $\mathbb{Z} \subseteq \mathbb{Q}$ at the prime ideal $(2) \lhd \mathbb{Z}$, hence we have $F = \mathbb{F}_2$.

As the 2-modular reduction $V := \tilde{\hat{V}} \in \mathbf{mod}\text{-}\mathbb{F}_2G$ of $\hat{V}$ has the $\mathbb{F}_2G$-module $4370a$ and the trivial $\mathbb{F}_2G$-module $1a$ as its constituents, we conclude by [39,

Table 25: Structure constants matrix P_2 for $G := B$ and $H := 2^{1+22}.Co_2$.

i	k_i	1	2	3	4	5	6	7	8	9	10
1	1	.	1	.	.	.	.	.	.	.	.
2	93150	93150	925	63	15	1	.	.	.	.	.
3	7286400	.	4928	63	120	42	.	1	.	.	.
4	262310400	.	42240	4320	1815	420	.	30	15	.	.
5	4196966400	.	45056	24192	6720	1807	891	272	120	.	27
6	9646899200	.	.	.	.	2048	891	512	.	100	36
7	470060236800	.	.	64512	53760	30464	24948	10287	5040	3850	3060
8	537211699200	.	.	.	30720	15360	.	5760	3495	4125	4320
9	4000762036224	.	.	.	.	.	41472	32768	30720	31175	32256
10	6685301145600	.	.	.	.	43008	24948	43520	53760	53900	53451

Table 26: The character table for $G := B$ and $H := 2^{1+22}.Co_2$.

φ	χ_φ	1	2	3	4	5	6
1	$1a$	1	93150	7286400	262310400	4196966400	9646899200
2	$96255a$	1	−2025	772200	−5702400	42768000	290816000
3	$9458750a$	1	10287	215424	3777840	25974432	35514368
4	$347643114a$	1	−2025	99000	356400	−5702400	8806400
5	$4275362520a$	1	495	48960	−334800	1631520	2769920
6	$9287037474a$	1	3375	28800	356400	1015200	−870400
7	$536105794455a$	1	1095	1560	7200	−113280	81920
8	$635966233056a$	1	−425	9400	−3600	−57600	−115200
9	$4375623425250a$	1	135	−360	−12960	17280	−40960
10	$6145833622500a$	1	−153	−936	8640	1152	32768

7	8	9	10
470060236800	537211699200	4000762036224	6685301145600
−2714342400	5474304000	8833204224	−11921817600
607533696	100362240	−42467328	−730920960
.	45619200	−191102976	141926400
−9636480	−12441600	−2359296	20321280
−6652800	4147200	−14155776	16128000
107520	−921600	2555904	−1720320
358400	−76800	1409024	−1523200
138240	414720	−884736	368640
−129024	−207360	294912	.

Cor.I.17.5] that $\hat{V}$ can be chosen such that

$$V \cong \begin{bmatrix} 1a \\ 4370a \end{bmatrix}$$

is a uniserial $\mathbb{F}_2G$-module with composition series as indicated. Furthermore, by [13], we have $(\hat{V} \otimes_R K)_H \cong \widehat{1a} \oplus \widehat{782a} \oplus \widehat{3588a}$ as KH-modules, where the latter are absolutely irreducible KH-modules of the respective dimensions. Hence we conclude that the 2-modular reductions $\widetilde{\widehat{782a}}$ and $\widetilde{\widehat{3588a}}$ are irreducible $\mathbb{F}_2H$-modules, where we have $\widetilde{\widehat{782a}} \cong 782a$ and $\widetilde{\widehat{3588a}} \cong 3588a$. As V, and hence V_H as well, are 2-modular reductions of R-free modules, we conclude by [39, I.17.3] that, as $\mathbb{F}_2H$-modules,

$$V_H = (\tilde{\hat{V}})_H = \widetilde{(\hat{V}_H)} \cong 1a \oplus 782a \oplus 3588a.$$

Let $0 \neq v_H \in V_H$ such that $1a = \langle v_H \rangle_{\mathbb{F}_2} \leq V_H$. Hence the G-orbit $v_H \cdot G \subseteq V$ is isomorphic to $\Omega := H|G$ as G-sets.

We construct the $\mathbb{F}_2G$-module V explicitly, using the $\mathbb{F}_2G$-module $4370a$ and a variant of the randomised technique to compute an upper bound on the dimension $\dim_{\mathbb{F}_2} \mathrm{Ext}^1_{\mathbb{F}_2G}(1a, 4370a)$ described in [46], of which we have a new GAP implementation, using the fast arithmetic for vectors over finite fields. We use the interpretation of $\mathrm{Ext}^1_{\mathbb{F}_2G}(1a, 4370a)$ as group cohomology

$$\mathrm{Ext}^1_{\mathbb{F}_2G}(1a, 4370a) \cong H^1_{\mathbb{F}_2}(G, 4370a) := Z^1_{\mathbb{F}_2}(G, 4370a)/B^1_{\mathbb{F}_2}(G, 4370a),$$

where $Z^1 := Z^1_{\mathbb{F}_2}(G, 4370a)$ and $B^1 := B^1_{\mathbb{F}_2}(G, 4370a) \leq Z^1$ are the group of 1-cocyles and 1-coboundaries of G with values in $4370a$, respectively. Let $Z^1_b := \{\zeta \in Z^1; \zeta(b) = 0\} \leq Z^1$ and $B^1_b := Z^1_b \cap B^1 \leq B^1$, where $b \in G$ is the standard generator of G of order 3. Using the restriction map

$$\mathrm{res}_{G,\langle b \rangle}: H^1_{\mathbb{F}_2}(G, 4370a) \to H^1_{\mathbb{F}_2}(\langle b \rangle, (4370a)_{\langle b \rangle})$$

to the cyclic subgroup $\langle b \rangle < G$, see [3, Ch.3.6], as well as the semisimplicity of the group algebra $\mathbb{F}_2\langle b \rangle$, we obtain

$$H^1_{\mathbb{F}_2}(G, 4370a) = Z^1_{\mathbb{F}_2}(G, 4370a)/B^1_{\mathbb{F}_2}(G, 4370a) \cong Z^1_b/B^1_b.$$

The elements of Z^1 are maps from G to $4370a$ fulfilling the cocyle relations, hence $\zeta \in Z^1_b$ is determined if $\zeta(a)$ is known, where $a \in G$ is the standard generator of G of order 2. Hence we have a $\mathbb{F}_2$-linear embedding $\nu_a: Z^1_b \to V: \zeta \mapsto \zeta(a)$. If $w(A, B)$ is an abstract word in the letters $\{A, B\}$, such that $w(a, b) = 1 \in G$, then using the cocyle and coboundary relations this translates into $\mathbb{F}_2$-linear equations to be fulfilled by the elements of $\nu_a(Z^1_b)$ and $\nu_a(B^1_b)$. We choose some abstract words as above, where we simply use the orders of some elements in G, and finally end up with $\mathbb{F}_2$-subspaces $\nu_a(B^1_b) \leq \nu_a(Z^1_b) \leq V$ such that $\dim \nu_a(B^1_b) + 1 = \dim \nu_a(Z^1_b)$. As we already know that there is a non-split extension of $1a$ with $4370a$, by [3, Cor.2.5.4] we have $\mathrm{Ext}^1_{\mathbb{F}_2G}(1a, 4370a) \neq \{0\}$. Hence we have shown that $\dim_{F_2} \mathrm{Ext}^1_{\mathbb{F}_2G}(1a, 4370a) = 1$.

Using the interpretation in [3, Prop.3.7.2] an element in $\nu_a(Z^1_b)\backslash\nu_a(B^1_b)$ describes the matrix entries in a representing matrix for the action of $a \in G$ on a non-split extension V of $1a$ with $4370a$. Furthermore, as $|\mathrm{Ext}^1_{\mathbb{F}_2G}(1a, 4370a) \setminus \{0\}| = 1$ the $\mathbb{F}_2G$-module V is uniquely determined up to isomorphism.

(17.7) To apply the strategy described in Section (10.6), we choose the following chain of subgroups

$$G = B > U_4 := U := H = Fi_{23} > U_3 := S_8(2) > U_2 := 2^{10}\colon A_8 > U_1 := A_7,$$

where we have the following group orders

$$\begin{array}{rcrcl} |B| & = & 4\,154\,781\,481\,226\,426\,191\,177\,580\,544\,000\,000 & \sim & 4 \cdot 10^{33}, \\ |Fi_{23}| & = & 4\,089\,470\,473\,293\,004\,800 & \sim & 4 \cdot 10^{18}, \\ |S_8(2)| & = & 47\,377\,612\,800 & \sim & 4 \cdot 10^{10}, \\ |2^{10}\colon A_8| & = & 20\,643\,840 & \sim & 2 \cdot 10^{7}, \\ |A_7| & = & 2\,520 & \sim & 2 \cdot 10^{3}. \end{array}$$

Words in the set of standard generators of H giving a set of non-standard generators of the maximal subgroup $S_8(2) < H$, are available in [83]. Using standard MeatAxe techniques, using the constituents of $V_{S_8(2)}$, we derive a suitable small faithful permutation representation of $S_8(2)$. Then running through some randomly chosen elements of $S_8(2)$, we find a set of standard generators in the sense of [81].

The subgroup $2^{10}\colon A_8 < S_8(2)$ is a maximal subgroup of index 2295. To find a set of generators of $2^{10}\colon A_8$, we first compute the uniquely determined transitive permutation representation of $S_8(2)$ on 2295 points, again using standard MeatAxe techniques and the constituents of $V_{S_8(2)}$. From this, using the Schreier-Sims algorithm, a set of generators of the point stabilizer $2^{10}\colon A_8$ is found. Running through some randomly chosen elements of $2^{10}\colon A_8$, we find a set of generators of a complement A_8 of the normal subgroup $2^{10} \trianglelefteq 2^{10}\colon A_8$, and finally a set of generators of $A_7 < A_8$.

We specify the $\mathbb{F}_2U_i$-modules V_i and the maps $\hat{q}_{i+1,i}\colon (V_{i+1})_{U_i} \to V_i$ of $\mathbb{F}_2U_i$-modules, for $i \in \{1, \ldots, 3\}$, as in Section (10.5). Let $V_4 := 782a$ be as in Section (17.6), and let $\hat{q}\colon V_H \to V_4$ be the natural $\mathbb{F}_2H$-projection of V_H onto its $\mathbb{F}_2H$-direct summand isomorphic to V_4. Using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that $(V_4)_{U_3}$ has a uniquely determined $\mathbb{F}_2U_3$-quotient module isomorphic to

$$V_3 \cong \left[\begin{array}{c} 16a \\ 26a \end{array} \right],$$

a uniserial $\mathbb{F}_2U_3$-module with composition series as indicated, where the constituents are the absolutely irreducible $\mathbb{F}_2U_3$-modules of the respective dimensions, see [37]. Let $\hat{q}_{4,3}\colon (V_4)_{U_3} \to V_3$ be the natural $\mathbb{F}_2U_3$-epimorphism. Analysis of the $\mathbb{F}_2U_2$-module $(V_3)_{U_2}$ shows that $(V_3)_{U_2}$ has a uniquely determined $\mathbb{F}_2U_2$-submodule of $\mathbb{F}_2$-dimension 11. The $\mathbb{F}_2U_2$-quotient module V_2 with respect to this submodule has the structure

$$V_2 \cong \begin{bmatrix} 1a \\ 4a \\ 6a \oplus 6a \\ 14a \end{bmatrix},$$

where the diagram indicates the radical and socle series, and the constituents are absolutely irreducible $\mathbb{F}_2U_2$-modules of the respective dimensions, see [37]. Let $\hat{q}_{3,2}\colon (V_3)_{U_2} \to V_2$ be the natural $\mathbb{F}_2U_2$-epimorphism. Finally, the $\mathbb{F}_2U_1$-module $(V_2)_{U_1}$ turns out to have a uniquely determined $\mathbb{F}_2U_1$-quotient module isomorphic to

$$V_1 \cong 4a \oplus 14a,$$

where the constituents are absolutely irreducible $\mathbb{F}_2U_1$-modules of the respective dimensions, see [37], obtained as the restrictions of the absolutely irreducible $\mathbb{F}_2U_2$-modules of these dimensions. Let $\hat{q}_{2,1}\colon (V_2)_{U_1} \to V_1$ be the natural $\mathbb{F}_2U_1$-epimorphism. Hence the chosen quotient modules have the following $\mathbb{F}_2$-dimensions.

i	U_i	$\dim_{\mathbb{F}_2} V_i$
	B	4371
4	Fi_{23}	782
3	$S_8(2)$	42
2	$2^{10}\colon A_8$	31
1	A_7	18

(17.8) We do not describe the partition $\Omega = \coprod_{i \in \mathcal{I}} \Omega_i$ into the G-suborbits Ω_i directly, but instead find the H-orbits $\hat{q}(\Omega_i) \subseteq V_4$, for $i \in \mathcal{I}$. This is done using the strategy described in Section (10.6), applied to the group $H = U = U_4$ and the chain of subgroups $U_3 > U_2 > U_1$. In turn, to describe an H-orbit $\hat{q}(\Omega_i) = \tilde{\omega}_i \cdot H \subseteq V_4$, for $\tilde{\omega}_i := \hat{q}(\omega_i) \in \hat{q}(\Omega_i)$, we do not enumerate $\tilde{\omega}_i \cdot H$ completely, but while enumerating $\tilde{\omega}_i \cdot H$ use a randomised Schreier-Sims technique to find subgroups of $\mathrm{Stab}_H(\tilde{\omega}_i)$. To do this, we use the smallest faithful permutation representation of H on 31671 points, which in terms of a set of standard generators of H is available in [83].

We terminate the enumeration of $\tilde{\omega}_i \cdot H$ if the product of the number of elements of $\tilde{\omega}_i \cdot H$ found and the order of the subgroup of $\mathrm{Stab}_H(\tilde{\omega}_i)$ found exceeds $\frac{|H|}{2}$. Then we know the orbit length $|\tilde{\omega}_i \cdot H|$ and have even obtained $\mathrm{Stab}_H(\tilde{\omega}_i)$ explicitly as a permutation group, where we additionally find a set of generators of $\mathrm{Stab}_H(\tilde{\omega}_i)$ as words in the set of standard generators of H. Hence we may compute $\omega_i \cdot \mathrm{Stab}_H(\tilde{\omega}_i) \subseteq \Omega \subseteq V$, provided $|\omega_i \cdot \mathrm{Stab}_H(\tilde{\omega}_i)|$ is small enough to do so. In this case, as we have

$$|\mathrm{Stab}_H(\omega_i)| \cdot |\omega_i \cdot \mathrm{Stab}_H(\tilde{\omega}_i)| = |\mathrm{Stab}_H(\tilde{\omega}_i)|,$$

we are able to find $|\Omega_i| = \frac{|H|}{|\mathrm{Stab}_{H(\omega_i)}|}$, and obtain $H_i = \mathrm{Stab}_H(\omega_i)$ explicitly as a permutation group. Hence, using the algorithms dealing with permutation groups available in GAP, we are also able to find the structure of the subgroup $H_i \leq H$. The cases for which $|\omega_i \cdot \mathrm{Stab}_H(\tilde{\omega}_i)|$ is too large to proceed as just described, have to be dealt with separately, which is commented on below.

It turns out that there are suborbits Ω_i, for $i \in \mathcal{I}$, apart from the trivial suborbit Ω_1, for which $\hat{q}(\omega_i) = \{0\} \subseteq V_4$ and hence $\mathrm{Stab}_H(\tilde{\omega}_i) = H$ holds. To find $H_i = \mathrm{Stab}_H(\omega_i)$ for these $i \in \mathcal{I}$, these suborbits have to be dealt with separately, which also is commented on below. Furthermore, it might happen that $\hat{q}(\Omega_i) = \hat{q}(\Omega_j)$, for $i \neq j \in \mathcal{I}$. In this case we would have to distinguish Ω_i and Ω_j by other means. But it turns out that, apart from the cases where $\hat{q}(\omega_i) = \{0\}$, even the orbit lengths $|\hat{q}(\Omega_i)|$, for $i \in \mathcal{I}$, are pairwise different.

To find some of the representatives $\omega_i \in \Omega$, for $i \in \mathcal{I}$, we begin with $\omega_1 = v_H \in \Omega \subseteq V$, apply a few random elements of G, and for the elements $\omega \in \Omega$ thus obtained enumerate $\hat{q}(\omega) \cdot H \subseteq V_4$, as was described above. This random search yields 14 of the suborbits Ω_i, namely for $i \in \{1, 7, 11, 13, \ldots, 23\}$, see Table 27, where the suborbits Ω_i are sorted with respect to increasing index parameters $k_i = |\Omega_i|$. Summing up the k_i for $i \in \{1, 7, 11, 13, \ldots, 23\}$, and dividing by $|\Omega|$, we obtain a fraction of ~ 0.998. Hence it seems to be rather improbable to find further suborbits using a random search. To proceed, using the facts we already know, we are tempted to look for candidate subgroups $\tilde{H} \leq H$ which might occur as stabilizers $\mathrm{Stab}_H(\omega_i) = H_i$. Indeed, the author has been hinted to the right guesses for the remaining 9 subgroups $\mathrm{Stab}_H(\omega_i) \leq H$, namely for $i \in \{2, \ldots, 6, 8, \ldots, 10, 12\}$, by [82].

Given a candidate $\tilde{H} \leq H$, we apply the usual strategy of combining sets of generators of subgroups given in [83] with standard MeatAxe techniques to find a set of generators of $\tilde{H}$ as words in the set of standard generators of H. Using the MeatAxe, we find the $\mathbb{F}_2$-subspace $\mathrm{Fix}_V(\tilde{H}) \leq V$, and for each $0 \neq v \in \mathrm{Fix}_V(\tilde{H})$ we proceed as follows. We compute a few elements $v' \in v \cdot G \subseteq V$, and check whether $\hat{q}(v') \in V_4$ is an element of an H-orbit $\hat{q}(\Omega_i) \subseteq V_4$, for some $i \in \mathcal{I}$, encountered earlier. As we have enumerated only a part of $\hat{q}(\Omega_i)$ explicitly, it is not sufficient to check $\hat{q}(v')$ itself, but depending on the proportion of elements of $\hat{q}(\Omega_i)$ enumerated explicitly, we check a few other elements of $\hat{q}(v') \cdot H \subseteq V_4$ as well. If we succeed in proving $\hat{q}(v') \in \hat{q}(\Omega_i)$, then the technique described in Section (10.6) also yields an element of $h \in H$, given as a word in the set of standard generators of H, mapping $\hat{q}(v')$ to $\tilde{\omega}_i \in \hat{q}(\Omega_i)$. It is then checked whether $v'h = \omega_i \in \Omega$ holds, which proves that indeed $v \in \Omega$. Thus the cases $i \in \{2, 5, 10\}$ are dealt with straightforwardly. We briefly comment on the other cases, including those for which $|\omega_i \cdot \mathrm{Stab}_H(\tilde{\omega}_i)|$ is large.

a) We have $\hat{q}(\omega_i) = 0 \in V_4$ for $i \in \{3, 4\}$, and, by construction, $\mathrm{Stab}_H(\omega_3) \geq S_8(2)$ and $\mathrm{Stab}_H(\omega_4) \geq 2^{11}.M_{23}$. As both candidate subgroups $S_8(2) < H$ and $2^{11}.M_{23} < H$ are maximal subgroups, we conclude that equality holds.

b) For $i = 8$ we have $2 \times {}^2F_4(2)' < N_H(2a) \cong 2.Fi_{22} < H$, where for $1 \neq$

Table 27: Suborbits of $G := B$ and $H := Fi_{23}$.

i	$k_i = \|\Omega_i\|$	$\|H_i\|$	H_i	$\mathrm{Stab}_H(\tilde{\omega}_i)$	$\frac{\|\mathrm{Stab}_H(\tilde{\omega}_i)\|}{\|H_i\|}$	split
1	1	4 089 470 473 293 004 800	Fi_{23}			+
2	412 896	9 904 359 628 800	$O_8^+(3){:}\,2_2$			
3	86 316 516	47 377 612 800	$S_8(2)$	Fi_{23}	86 316 516	+
4	195 747 435	20 891 566 080	$2^{11}.M_{23}$	Fi_{23}	195 747 435	+
5	8 537 488 128	479 001 600	S_{12}			
6	23 478 092 352	174 182 400	$O_8^+(2)$			+
7	33 816 182 400	120 932 352	$3_+^{1+8}.2_-^{1+6}.2_-^{1+2}.S_3$	$3_+^{1+8}.2_-^{1+6}.2_-^{1+2}.3^2.2$	3	
8	113 778 447 552	35 942 400	$2 \times {}^2F_4(2)'$	$2.Fi_{22}$	3 592 512	+
9	160 533 964 800	25 474 176	$S_3 \times G_2(3)$	$S_3 \times O_7(3)$	1 080	
10	504 245 392 560	8 110 080	$2^{10}.M_{11}$	$2^{11}.M_{11}$	2	+
11	1 044 084 577 536	3 916 800	$S_4(4){:}\,4$			
12	1 152 560 897 280	3 548 160	$(2 \times 2.M_{22}).2$			
13	1 584 771 233 760	2 580 480	$2^7.A_8$			
14	5 282 570 779 200	774 144	$2^7.U_3(3)$	$2^7.U_3(3).2$	2	+
15	7 888 639 030 272	518 400	$(A_6 \times A_6){:}\,2^2$			+
16	12 678 169 870 080	322 560	$2^2.L_3(4).2^2$			
17	21 514 470 082 560	190 080	$2 \times M_{12}$			
18	43 028 940 165 120	95 040	M_{12}			+
19	50 712 679 480 320	80 640	$2.L_3(4).2_2$			+
20	133 120 783 635 840	30 720	$2^4.2^4.A_5.2$			
21	190 172 548 051 200	21 504	$2^6{:}\,L_3(2){:}\,2$			
22	262 954 634 342 400	15 552	$3^4.2^{1+4}.S_3$			
23	283 991 005 089 792	14 400	$(A_5 \times A_5){:}\,2^2$			+

$2a \in Z(2 \times {}^2F_4(2)')$ we have $2a \in C_{2A} \in \mathcal{C}l(H)$, where the latter in turn denotes the $2A$-conjugacy class of H. Using the ordinary character tables of $2 \times {}^2F_4(2)'$ and of all the maximal subgroups of H, as well as the programs using ordinary character tables to find candidates for the natural maps between the conjugacy classes of a candidate subgroup and those of a given group, available in GAP, we find that $2.Fi_{22} \cong N_H(2a) < H$ is the only maximal subgroup of H containing $2 \times {}^2F_4(2)'$. Furthermore, $2 \times {}^2F_4(2)' < N_H(2a)$ in turn is a maximal subgroup. As by construction $\mathrm{Stab}_H(\omega_8) \geq 2 \times {}^2F_4(2)'$, we only have to check that $\mathrm{Stab}_H(\omega_8) < N_H(2a)$, and that $\hat{q}(\omega_8) \neq 0 \in V_4$ as well as $\mathrm{Stab}_H(\hat{q}(\omega_8)) \geq N_H(2a)$ holds.

c) For $i = 9$ we have $S_3 \times G_2(3) < N_H(3a) \cong S_3 \times O_7(3) < H$, where $3a \in S_3 \trianglelefteq S_3 \times G_2(3)$ is an element of order 3, which turns out to be an element of $C_{3A} \in \mathcal{C}l(H)$, where the latter in turn denotes the $3A$-conjugacy class of H. It turns out that $S_3 \times O_7(3)$ contains two conjugacy classes of subgroups isomorphic to $S_3 \times G_2(3)$, and indeed exactly one of them yields a fixed vector in V belonging to Ω, different from $\omega_1 = v_H \in V$. Proceeding as in the case $i = 7$, for the correct subgroup $S_3 \times G_2(3)$, we find that $S_3 \times O_7(3) \cong N_H(3a) < H$ is the only maximal subgroup of H containing $S_3 \times G_2(3)$. Furthermore, $S_3 \times O_7(3) < N_H(3a)$ in turn is a maximal subgroup. As by construction $\mathrm{Stab}_H(\omega_9) \geq S_3 \times G_2(3)$, we only have to check that $\mathrm{Stab}_H(\omega_9) < N_H(3a)$, and that $\hat{q}(\omega_9) \neq 0 \in V_4$ as well as $\mathrm{Stab}_H(\hat{q}(\omega_9)) \geq N_H(3a)$ holds.

d) For $i = 12$ we have $(2 \times 2.M_{22}).2 < N_H(2b) \cong 2^2.U_6(2).2 < H$, where for $1 \neq 2b \in Z((2 \times 2.M_{22}).2)$ we have $2b \in C_{2B} \in \mathcal{C}l(H)$, where the latter in turn denotes the $2B$-conjugacy class of H. It turns out that $2^2.U_6(2).2$ contains three conjugacy classes of subgroups isomorphic to $(2 \times 2.M_{22}).2$, and indeed exactly one of them yields a fixed vector in V belonging to Ω, different from $\omega_1 = v_H \in V$.

e) For the last remaining case $i = 6$ we may assume that all the other 22 suborbits have already been found. We find that H has exactly three conjugacy classes of maximal subgroups which contain a subgroup isomorphic to $O_8^+(2)$, namely subgroups isomorphic to $S_8(2)$, to $O_8^+(3){:}\,S_3$ and to $2.Fi_{22}$, respectively. It turns out that a subgroup $O_8^+(2) < S_8(2)$ yields fixed vectors in V belonging to Ω, different from $\omega_1 = v_H \in \Omega$ and $\omega_3 \in \Omega$.

(17.9) For later use, see Section (17.11), we collect the following facts about some of the groups H_i, using the programs dealing with permutation groups available in GAP.
a) For $i = 4$ the subgroup $M_{23} < 2^{11}.M_{23} = H_4$ acts irreducibly on the elementary abelian normal subgroup $2^{11} \trianglelefteq 2^{11}.M_{23}$, hence $2^{11}.M_{23}$ is a perfect group.
b) For $i = 10$ the subgroup $M_{11} < 2^{10}.M_{11} = H_{10}$ acts irreducibly on the elementary abelian normal subgroup $2^{10} \trianglelefteq 2^{10}.M_{11}$, hence $2^{10}.M_{11}$ is a perfect group.
c) For $i = 13$ the group $H_{13} = 2^7.A_8$ is not 2-perfect, since the normal sub-

group $2^7 \trianglelefteq 2^7.A_8$ is elementary abelian, but as an $\mathbb{F}_2A_8$-module is isomorphic to $\left[\begin{array}{c} 1a \\ 6a \end{array}\right]$, a uniserial $\mathbb{F}_2A_8$-module with composition series as indicated, where the constituents are absolutely irreducible $\mathbb{F}_2A_8$-modules of the respective dimensions, see [37].
d) For $i = 14$ the group $H_{14} = 2^7.U_3(3)$ is a perfect group, since the normal subgroup $2^7 \trianglelefteq 2^7.U_3(3)$ is elementary abelian, and as an $\mathbb{F}_2U_3(3)$-module is isomorphic to $\left[\begin{array}{c} 6a \\ 1a \end{array}\right]$, a uniserial $\mathbb{F}_2U_3(3)$-module with composition series as indicated, where the constituents are absolutely irreducible $\mathbb{F}_2U_3(3)$-modules of the respective dimensions, see [37].

(17.10) We compute the structure constants matrix $P_2 = [p_{i,2,k}; i, k \in \mathcal{I}] \in \mathbb{Z}^{r\times r}$ for the smallest non-trivial suborbit Ω_2, with $k_2 = 412896$, using the strategy described in Section (17.5). Hence again the remaining task is to enumerate $\Omega_2 \subseteq V$ explicitly, to apply successively the elements $g_i \in G$, for $i \in \mathcal{I}$, to all elements of $\Omega_2 \subseteq \Omega \subseteq V$, and to find the cardinalities $c_{2,k}(g_i) = |\Omega_2 g_i \cap \Omega_k| \in \mathbb{N}_0$ by checking for membership in Ω_k, for $k \in \mathcal{I}$. As we have not enumerated the G-suborbits directly, but the H-orbits $\hat{q}(\Omega_i) \subseteq V_4$, for $i \in \mathcal{I}$, instead, see Section (17.8), the membership test is done by checking whether $\hat{q}(\omega) \in \hat{q}(\Omega_k)$ holds, for $\omega \in \Omega_2 g_i$ and $k \notin \{3, 4\}$. As we have enumerated only a part of $\hat{q}(\Omega_k)$ explicitly, again it is not sufficient to check $\hat{q}(\omega)$ itself, but a few other elements of $\hat{q}(\omega) \cdot H \subseteq V$ have to be checked as well. For the exceptional cases $k \in \{3, 4\}$ we cannot check at all whether $\hat{q}(\omega) \in \hat{q}(\Omega_k)$ holds. But it turns out that the numerical conditions given in Section (17.5) are sufficient to find all the matrix entries $c_{2,k}(g_i) \in \mathbb{N}_0$, for $i, k \in \mathcal{I}$, in particular those for $k \in \{3, 4\}$.

The structure constants matrix $P_2 \in \mathbb{Z}^{r\times r}$ can be determined using Remark (9.7), it is shown in Tables 28 and 29. Using the technique described in Section (8.2), the structure constants matrix P_2 turns out to be sufficient to obtain a splitting of $K^{1\times r}$ into eigenspaces of dimension 1. The character table of E_K is given in Tables 30, 31 and 32.

(17.11) Let $G := 2.B$ and $H' := Fi_{23}$ as well as $H := Z(G) \times H' \cong 2 \times Fi_{23}$. The assumptions of Remark (5.15) are fulfilled. We have $r = 23$ and $r' = 34$, where $\Omega := H|G$ and $\Omega' := H'|G$. We determine the 11 split and 12 non-split suborbits.

Let $i \in \mathcal{I}$ such that Ω_i is a non-split suborbit. Hence by Remark (5.15) we have $[(H' \cap H^{g_i}) : (H' \cap H'^{g_i})] = 2$, and thus we have $H' \cap H^{g_i} \leq H'$ but $H' \cap H^{g_i} \not\leq H'^{g_i}$. Furthermore, by Corollary (5.5) we have $[H' : (H' \cap H^{g_i})] = k_i = [H : H_i]$ anyway, and thus $[H_i : (H' \cap H^{g_i})] = [H : H'] = 2$. Hence $H' \cap H^{g_i}$ is a normal subgroup in H_i of index 2, and in turn $H' \cap H^{g_i}$ has $H' \cap H'^{g_i}$ as a normal subgroup of index 2.

The structure of the subgroups $H_i \leq H$ is indicated in Table 27, see also Section (17.9), where the subgroups $H_i \leq H$ considered here are split central extensions

Table 28: Structure constants matrix P_2 for $G := B$ and $H := Fi_{23}$.

i	k_i	1	2	3	4	5	6	7	8	9	10
1	1	.	1	.	.	.	.	.	.	.	.
2	412896	412896	2	136	.	.	1	4	.	.	.
3	86316516	.	28431	.	.	462	1	.	.	.	.
4	195747435	.	.	.	.	.	135	.	.	.	.
5	8537488128	.	.	45696	.	.	.	3888	.	.	1056
6	23478092352	.	56862	272	16192	.	136	.	.	.	.
7	33816182400	.	327600	.	.	15400	.	8	.	364	.
8	113778447552	.	.	.	.	.	.	.	3200	1134	.
9	160533964800	.	.	.	.	.	.	1728	1600	728	.
10	504245392560	.	.	.	.	62370	.	.	.	.	.
11	1044084577536	.	.	.	.	.	12096	.	.	.	.
12	1152560897280	.	.	.	129536	.	.	.	.	.	1760
13	1584771233760	.	.	275400	8096	.	16335	78732	.	.	33440
14	5282570779200	.	.	.	.	.	16200	.	.	2106	.
15	7888639030272	.	.	91392	.	924	79296	23328	.	.	37312
16	12678169870080	.	.	.	.	178200	.	.	.	37908	.
17	21514470082560	.	.	.	.	.	.	139968	12480	.	101376
18	43028940165120	.	.	.	.	.	.	.	24960	58968	.
19	50712679480320	.	.	.	259072	124740	.	.	.	.	2112
20	133120783635840	.	.	.	.	.	226800	157464	.	.	135168
21	190172548051200	.	.	.	.	.	16200	.	280800	75816	10560
22	262954634342400	.	.	.	.	30800	33600	7776	.	235872	.
23	283991005089792	.	.	.	.	.	12096	.	89856	.	90112

Table 29: Structure constants matrix P_2 for $G := B$ and $H := Fi_{23}$, continued.

11	12	13	14	15	16	17	18	19	20	21	22	23
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	15	.	1	.	.	.	.	.	.	.	.
.	22	1	.	.	.	.	.	1	.	.	.	.
.	.	.	.	1	120	.	.	21	.	.	1	.
272	.	242	72	236	.	.	.	.	40	2	3	1
.	.	1680	.	100	.	220	.	.	40	.	1	.
.	.	.	.	.	.	66	66	.	.	168	.	36
.	.	.	64	.	480	.	220	.	.	64	144	.
.	770	10640	.	2385	.	2376	.	21	512	28	.	160
1360	1232	.	.	36	112	.	1980	700	.	672	486	176
1360	.	.	4320	1575	1400	.	.	211	496	128	567	600
.	.	30	2376	.	9632	.	396	3420	40	30	945	175
.	19800	7920	128	1350	.	6270	990	2370	2560	844	1512	3300
272	10780	.	2016	626	15120	792	3696	12866	480	1008	4596	2546
1360	15400	77056	.	24300	240	29700	396	420	13056	3088	1350	5400
.	.	.	25536	2160	50400	440	6996	28560	1792	3136	13824	6360
81600	.	10752	8064	20160	1344	13992	21032	3360	24064	30016	11232	14760
34000	9284	109440	22752	82710	1680	67320	3960	5542	41664	16016	9828	24110
.	57288	3360	64512	8100	137088	11088	74448	109368	23672	38976	76707	45600
122400	21120	3600	30384	24300	46320	27720	132660	60060	55680	108608	81972	64800
122400	129360	156800	75264	153200	28000	168960	68640	50960	151520	113344	81640	118600
47872	147840	31360	177408	91656	120960	83952	97416	135016	97280	96768	128088	126272

Table 30: The character table for $G := B$ and $H := Fi_{23}$.

φ	χ_φ	1	2	3	4	5	6	7	8
1	1	1	412896	86316516	195747435	8537488128	23478092352	33816182400	113778447552
2	4371	1	−137632	18115812	−10472085	−1159411968	1449264960	3757353600	1404672192
3	96255	1	82016	8890596	5701995	457037568	327742272	1297296000	−1788671808
4	9458750	1	41888	3232548	−43605	123026688	57841344	314160000	183218112
5	63532485	1	−32032	2275812	414315	−77223168	−2312640	179625600	−32332608
6	347643114	1	10208	704484	1589355	10679040	46398528	−9609600	57081024
7	356054375	1	−17248	900900	−1508949	−20097792	43902144	32672640	−21155904
8	4221380670	1	−3232	324324	103275	−2453760	15121728	−12297600	−15494976
9	4275362520	1	14816	725796	−43605	16743168	−7316928	31920000	14841792
10	9287037474	1	6896	132516	699435	736128	11096352	4502400	−38864448
11	13508418144	1	−11632	475812	111915	−9283968	−491040	17673600	7584192
12	108348770530	1	7328	246564	−43605	3421440	1729728	4502400	−11866176
13	309720864375	1	−1120	89892	−181845	−172800	3172032	−3638400	6934464
14	635966233056	1	3408	69284	147755	295040	2450528	−169600	6681024
15	1095935366250	1	−4576	126756	2475	−1324800	−949824	1061760	−254016
16	6145833622500	1	2864	51876	−26325	316800	−507744	309120	1197504
17	6619124890560	1	1088	39204	25515	138240	−300672	−1065600	−1498176
18	12927978301875	1	−2128	19620	−40149	67968	706464	186240	−627264
19	38348970335820	1	−1232	15524	37675	19840	−69472	−233600	−576
20	89626740328125	1	944	1188	15147	−79488	61344	63360	36288
21	211069033500000	1	560	1188	−12501	−51840	12960	−68736	−129600
22	284415522641250	1	−16	−5724	8235	17280	50976	78720	−46656
23	364635285437500	1	−400	−1116	−5589	26496	−71136	−7296	119232

Table 31: The character table for $G := B$ and $H := Fi_{23}$, continued.

9	10	11	12	13	14	15	16
160533964800	504245392560	1044084577536	1152560897280	1584771233760	5282570779200	7888639030272	12678169870080
−5945702400	39426594480	−21483221760	−4743048960	−110868769440	65216923200	−292171815936	573908924160
−511948800	12027702960	−9527341824	6966984960	30484602720	28447848000	58091185152	118446831360
258508800	1991288880	1252323072	−1021697280	4906012320	−3514104000	3727696896	12802648320
35481600	1084693680	550851840	−432034560	−2400567840	1235995200	−300174336	4718165760
−167270400	224426160	533820672	271607040	−9741600	916660800	2067158016	−1656357120
63866880	185985072	−186810624	778242816	−259829856	−2109032640	−1909619712	−643458816
74188800	87499440	−219034368	−142145280	29121120	499867200	−274627584	−544631040
4147200	110118960	−61012224	62588160	198033120	197640000	−366363648	5218560
20044800	−21727440	115105536	171953280	32315760	217339200	−118153728	122446080
−18662400	32946480	−61205760	−22584960	−74323440	−10756800	200600064	−34179840
−6912000	5609520	−1790208	−28857600	−1265760	−80222400	35030016	−96802560
−6912000	12798000	19554048	−7568640	3745440	−43200	−48356352	−17729280
5913600	−1900240	−8656128	8992640	−2385200	−15211200	36246016	7220480
1935360	−841680	6983424	3168000	10755360	2721600	1741824	−31921920
691200	−2857680	2467584	−777600	−4879440	5417280	−5515776	518400
−460800	2430000	−1928448	3732480	−3810240	648000	5308416	933120
−414720	−2332368	−1292544	−307584	−943056	2928960	787968	6269184
76800	−292560	472832	−1668480	588720	−1924800	−2025984	4348160
−709632	−452304	−850176	134784	854064	938304	−1866240	518400
248832	73008	200448	−335232	518832	−720576	898560	1237248
138240	114480	532224	−293760	−481680	25920	−262656	−1416960
−82944	86832	−352512	508032	−42768	191808	290304	−311040

Table 32: The character table for $G := B$ and $H := Fi_{23}$, continued.

17	18	19	20	21	22	23
21514470082560	43028940165120	50712679480320	133120783635840	190172548051200	262954634342400	283991005089792
−796832225280	531221483520	1460859079680	−2739110774400	−782603078400	3246353510400	−1168687263744
158430504960	−222361251840	239651343360	190079809920	−857327328000	28598169600	218194808832
10166446080	20332892160	7936220160	8210885760	47791814400	−25333862400	−90188550144
−4534548480	−8511713280	−1053803520	12753417600	10828857600	−17953689600	3908653056
−679311360	1892782080	3994721280	−5895711360	1568160000	−10005811200	6838013952
1675634688	1177473024	3238050816	−155675520	−44478720	−6826659840	4981616640
−5806080	592220160	722856960	813214080	−13996800	−1025740800	−578285568
−75479040	−452874240	−1233239040	−1778474880	666144000	148377600	2518290432
−322237440	661893120	−489991680	959091840	−1020988800	174182400	−479582208
269982720	836075520	−664312320	−183254400	−1004918400	593510400	125024256
−145152000	−11612160	83082240	268168320	−170553600	212889600	−59609088
18524160	−16035840	−61793280	98133120	−116640000	190771200	−74649600
−39797760	−41656320	−22725120	16717440	9264000	80076800	−41576448
−5806080	−58060800	36449280	−18264960	41644800	94187520	−83349504
14515200	11612160	15137280	9797760	−15085440	−21934080	−10450944
14100480	−9953280	−9953280	−18195840	27993600	27648000	−35831808
7216128	−6967296	−2225664	−16744320	22654080	−8663040	−276480
−1582080	5468160	−919040	−1537920	−7036800	−17100800	23365632
−746496	−1658880	2198016	3825792	6065280	−4534272	−3815424
−1410048	995328	−1893888	−4053888	−1316736	−2764800	8570880
2903040	−1658880	−69120	−3058560	51840	6082560	−2709504
−1741824	995328	705024	4572288	−1026432	−700416	−3151872

of the subgroups given in Table 27 by the central subgroup $Z(G)$ of order 2. Hence we conclude that for $i \in \{1,3,4,6,10,14,18\}$ the subgroup $H_i \leq H$ has only 2-perfect subgroups of index 2, and thus these are 7 of the split suborbits, as is indicated in the last column of Table 27. Note that by Section (17.9) the above condition on the subgroup structure does not hold for H_{13}.

We have a closer look at the embedding of the subgroups H_i into G. As all characters in $\mathrm{Irr}_K^{1_{H'}}(G)$ are rational-valued, by Propositions (3.20) and (3.1) and the orthogonality relations, see Proposition (3.8) we conclude that all suborbits of Ω' are self-paired. Hence we may without loss of generality choose the set of representatives $\{g_i; i \in \mathcal{I}\}$ of the H-H-double cosets in G such that $g_i^2 \in H'$. We still assume that $i \in \mathcal{I}$ such that Ω_i is a non-split suborbit. Let $\tilde{H}' \leq H'$ be a subgroup such that $H' \cap H^{g_i} \leq \tilde{H}'$, and let $\tilde{H} := Z(G) \times \tilde{H}' \leq H$. Hence we have $H_i \leq \tilde{H}$. Since $H' \cap H^{g_i} \not\leq H'^{g_i}$, we also have $H' \cap H^{g_i} \not\leq \tilde{H}'^{g_i}$, but we have $H' \cap H^{g_i} \leq H_i = H_i^{g_i} \leq \tilde{H}^{g_i}$.

Let $f_{\tilde{H}}: \mathcal{C}l(\tilde{H}) \to \mathcal{C}l(G)$ denote the natural map between the conjugacy classes of $\tilde{H}$ and those of G, and let $f_{\tilde{H}',\tilde{H}}: \mathcal{C}l(\tilde{H}') \to \mathcal{C}l(\tilde{H})$ denote the natural map between the conjugacy classes of $\tilde{H}'$ and those of $\tilde{H}$. Let $f_{g_i}: \mathcal{C}l(\tilde{H}) \to \mathcal{C}l(\tilde{H}^{g_i})$ be the natural bijection between the conjugacy classes of $\tilde{H}$ and those of $\tilde{H}^{g_i}$, induced by conjugation with $g_i \in G$; its restriction to $\mathcal{C}l(\tilde{H}')$ also is denoted by f_{g_i}. Hence the natural map between the conjugacy classes of $\tilde{H}^{g_i}$ and those of G is $f_{\tilde{H}} \circ f_{g_i}^{-1}: \mathcal{C}l(\tilde{H}^{g_i}) \to \mathcal{C}l(G)$, and the natural map between the conjugacy classes of $\tilde{H}'^{g_i}$ and those of $\tilde{H}^{g_i}$ is $f_{g_i} \circ f_{\tilde{H}',\tilde{H}} \circ f_{g_i}^{-1}: \mathcal{C}l(\tilde{H}'^{g_i}) \to \mathcal{C}l(\tilde{H}^{g_i})$.

For the natural maps

$$f': \mathcal{C}l(H' \cap H^{g_i}) \to \mathcal{C}l(\tilde{H}') \quad \text{and} \quad f'': \mathcal{C}l(H' \cap H^{g_i}) \to \mathcal{C}l(\tilde{H}^{g_i})$$

we hence have

$$f''(\mathcal{C}l(H' \cap H^{g_i})) \not\subseteq f_{g_i} \circ f_{\tilde{H}',\tilde{H}} \circ f_{g_i}^{-1}(\mathcal{C}l(\tilde{H}'^{g_i})) \text{ and } f_{\tilde{H}} \circ f_{\tilde{H}',\tilde{H}} \circ f' = f_{\tilde{H}} \circ f_{g_i}^{-1} \circ f''.$$

We use the programs using ordinary character tables to find candidates for the natural maps between the conjugacy classes of a candidate subgroup and those of a given group available in GAP, to check whether such maps f' and f'' exist for the index 2 subgroups H'' of the groups H_i not yet dealt with. Given H'', we compute the candidates for the natural map $\mathcal{C}l(H'') \to \mathcal{C}l(\tilde{H})$, and check whether we can find candidate maps f_1 and f_2, such that f_1 factors through some $f': \mathcal{C}l(H'') \to \mathcal{C}l(\tilde{H}')$ as $f_1 = f_{\tilde{H}',\tilde{H}} \circ f'$, where f' and $f'' := f_{g_i} \circ f_2: \mathcal{C}l(H'') \to \mathcal{C}l(\tilde{H}^{g_i})$ fulfil the above conditions, which amount to

$$f_2(\mathcal{C}l(H'')) \not\subseteq f_{\tilde{H}',\tilde{H}}(\mathcal{C}l(\tilde{H}')) \quad \text{and} \quad f_{\tilde{H}} \circ f_1 = f_{\tilde{H}} \circ f_2.$$

We specify $\tilde{H}' := H'$, hence $\tilde{H} = H$. The ordinary character tables of G as well as H and H' are available in GAP. It turns out that there are 4 candidates for the natural map $f_H: \mathcal{C}l(H) \to \mathcal{C}l(G)$, which are exactly one orbit under the action of the group of table automorphisms of G, hence we may choose one of them, and keep it fixed. The map $f_{H',H}: \mathcal{C}l(H') \to \mathcal{C}l(H)$ is uniquely determined.

Let $i = 8$, hence we have $H_8 = 2^2 \times {}^2F_4(2)'$. Thus all its index 2 subgroups are isomorphic to $2 \times {}^2F_4(2)'$, whose ordinary character table is available in GAP. It turns out that no pair of maps $f_{1,2}: \mathcal{C}l(2 \times {}^2F_4(2)') \to \mathcal{C}l(H)$ fulfilling the above conditions exists.

Let $i = 15$, hence we have $H_{15} = 2 \times (A_6 \times A_6){:}\,2^2$. As we are looking for maps f_1 factoring through $f_{H',H}$, we may restrict ourselves to the direct factor $(A_6 \times A_6){:}\,2^2$ of index 2. Its ordinary character table can be determined using the Dixon-Schneider algorithm available in GAP. It turns out that no pair of maps $f_{1,2}: \mathcal{C}l((A_6 \times A_6){:}\,2^2) \to \mathcal{C}l(H)$ fulfilling the above conditions exists.

Let $i = 19$, hence we have $H_{19} = 2 \times 2.L_3(4).2_2$. Again we may restrict ourselves to the direct factor $2.L_3(4).2_2$ of index 2, whose ordinary character table is available in GAP. It turns out that no pair of maps $f_{1,2}: \mathcal{C}l(2.L_3(4).2_2) \to \mathcal{C}l(H)$ fulfilling the above conditions exists.

Let $i = 23$, hence we have $H_{23} = 2 \times (A_5 \times A_5){:}\,2^2$. Again we may restrict ourselves to the direct factor $(A_5 \times A_5){:}\,2^2$ of index 2, whose ordinary character table can be determined using the Dixon-Schneider algorithm available in GAP. But it turns out that it would be too time-consuming to compute the candidates for the natural map $\mathcal{C}l((A_5 \times A_5){:}\,2^2) \to \mathcal{C}l(H)$. Using GAP and the permutation representations of $(A_5 \times A_5){:}\,2^2 \cong H_{23}/Z(G)$ and of $(A_6 \times A_6){:}\,2^2 \cong H_{15}/Z(G)$ as subgroups of $Fi_{23} = H/Z(G)$, we find that $(A_5 \times A_5){:}\,2^2$ is Fi_{23}-conjugate to a subgroup of $(A_6 \times A_6){:}\,2^2$. Hence we may assume $(A_5 \times A_5){:}\,2^2 < (A_6 \times A_6){:}\,2^2 < H'$. Thus we specify $\tilde{H}' := (A_6 \times A_6){:}\,2^2$. We use the candidates for the natural map $\mathcal{C}l(\tilde{H}') \to \mathcal{C}l(H)$ already found above, and as we keep $f_H: \mathcal{C}l(H) \to \mathcal{C}l(G)$ fixed, we find that the natural map $\mathcal{C}l(\tilde{H}) \to \mathcal{C}l(G)$ is uniquely determined. Finally, it turns out that no pair of maps $f_{1,2}: \mathcal{C}l((A_5 \times A_5){:}\,2^2) \to \mathcal{C}l(\tilde{H})$ fulfilling the above conditions exists.

Hence we have found the remaining 4 split suborbits to be $i \in \{8, 15, 19, 23\}$, as is indicated in Table 27.

(17.12) Unfortunately, up to now it has not been possible to compute the character table of $E_K^{1_{H'}}$, apart from the relations between the character values given in Remark (5.15). There are serious obstacles we are faced with.

It turns out that no suitable faithful representation of G is available to be used for a computational approach analogous to the one which has been used for $\Omega = H|G$, see Section (17.6). Furthermore, only the second smallest non-trivial suborbit $i = 3$, where $k_3 = 86\,316\,516$, is a split suborbit. To find the character table of $E_K^{1_{H'}}$, by Remark (5.15), we have to determine $\mathrm{Irr}_K(E_K^{1^-})$, where $\mathcal{I}_{1^-} \subset \mathcal{I}$ is the set of split suborbits. Applying a technique as was used in Section (17.10) would imply to run explicitly through the $k_3 = 86\,316\,516$ elements of Ω_3, instead of the $k_2 = 412896$ elements of Ω_2.

We mention some more indirect ideas, which might be helpful, but still have to be elaborated further.

a) We could try to use a technique which was used in [29] for the groups B and $2.^2E_6(2).2$ as well as $2.^2E_6(2)$. For the present case this involves finding the 1^--weights $\zeta \in \{\pm 1\}$ of the triangles in $\mathcal{T}_{ijk} \subseteq \Omega \times \Omega \times \Omega$, for $j, k \in \mathcal{I}_{1^-}$, and some fixed $i \in \mathcal{I}_{1^-}$, see Definition (1.15). This can be reduced to the sets $\Omega_{ijk}^{1^-,\zeta} \subseteq \Omega_i$, see Remark (1.16), which in turn are unions of H_k-orbits. We are tempted to choose $i = 3$, the smallest non-trivial split suborbit, but still we are faced with $k_3 = 86\,316\,516$ elements, and the sets $\Omega_{3,j,k}^{1^-,\zeta} \subseteq \Omega_3$ in most cases seem to be far away from being single H_k-orbits.

b) By Remark (3.24), the matrix $\Gamma_1 \in \mathbb{Z}^{\mathcal{I} \times |\mathcal{C}l(G)|}$ can be determined, see Definition (3.19). To find the character table of $E_K^{1^-}$, it is sufficient, by Proposition (3.20), to find the matrix $\Gamma_{1^-} \in \mathbb{Z}^{\mathcal{I}_{1^-} \times |\mathcal{C}l(G)|}$. The matrix entries of Γ_1 and Γ_{1^-} as well as $\Gamma_{1_{H'}} \in \mathbb{Z}^{\mathcal{I}' \times |\mathcal{C}l(G)|}$ are related, where the relations can be made precise. Furthermore, the rows of Γ_{1^-} form an orthogonal K-basis of the K-subspace of $K^{\mathcal{I}_{1^-} \times |\mathcal{C}l(G)|}$ they span, where the latter can also be described in terms of values of the characters in $\mathrm{Irr}_K^{1^-}(G)$, see Remark (3.24).

18 The Thompson-Smith lattice

In Section 18 we give another application of the technique described in Section (10.6), to a problem related to the still open question of determining the minimum of the so-called Thompson-Smith lattice. We begin by fixing the setting and stating the problem we solve computationally.

(18.1) Let $G := Th$ and $\mathcal{G} \subseteq G$ be a set of standard generators of G in the sense of [81]. Let V be the absolutely irreducible, even, unimodular $\mathbb{Z}G$-lattice of $\mathbb{Z}$-rank 248, the so-called *Thompson-Smith lattice*. Matrices for the action of the elements of $\mathcal{G}$ and the Gram matrix of the scalar product $\langle \cdot, \cdot \rangle_V$ on V are known, see [62]. Let the *minimum* of V be defined as $\min V := \min\{\langle v, v\rangle_V; 0 \neq v \in V\}$. By [40] we have $\min V \geq 10$.

Let $H := 3 \times G_2(3) < N_G(H) = (3 \times G_2(3)){:}2 < G$, where $N_G(H) < G$ is a maximal subgroup. It turns out that $\mathrm{Fix}_H(V) = \langle v_H \rangle_{\mathbb{Z}}$ for some $v_H \in V$, while $v_H \cdot N_G(H) = \{\pm v_H\}$. As $N_G(H) < G$ is a maximal subgroup, there is a G-set isomorphism between the G-orbit $v_H \cdot G \subseteq V$ and $\Omega := H|G$, where $n = |\Omega| = 7\,124\,544\,000$, and using GAP we find $|\mathcal{I}| = r = 778$. Note that the G-orbit $v_H \cdot G \subseteq V$ is a *symmetric* orbit, hence we have $-v \in v_H \cdot G$ whenever $v \in v_H \cdot G$.

It turns out that $\langle v_H, v_H \rangle_V = 12$. Hence we have $\min V \in \{10, 12\}$. It is conjectured and still an open problem that $\min V = 12$. Related to this problem, it has been conjectured [61] that

$$\{\langle v, v_H \rangle_V; v \in v_H \cdot G \subseteq V, v \neq \pm v_H\} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6\}.$$

We prove the latter conjecture, see Table 33.

(18.2) Let $\tilde{V}$ be the absolutely irreducible $\mathbb{F}_3G$-module obtained from V by 3-modular reduction. Using the MeatAxe, we find that $\tilde{V}_H$ has a uniquely determined trivial $\mathbb{F}_3H$-submodule. We pick one of the vectors $0 \neq \tilde{v}_H \in V_H$ in this submodule, and it turns out that $\tilde{v}_H \cdot N_G(H) = \{\pm\tilde{v}_H\}$. Hence we conclude that there is a G-set isomorphism between the G-orbit $\tilde{v}_H \cdot G \subseteq \tilde{V}$ and Ω.

We enumerate the G-orbit $\Omega \cong \tilde{v}_H \cdot G$ piecewise, H-orbit by H-orbit, using the technique described in Section (10.6). As G acts by lattice automorphisms on V, and $H = \mathrm{Stab}_G(v_H)$, we have $\langle v, v_H\rangle_V = \langle v \cdot h, v_H\rangle_V$ for $v \in V$ and $h \in H$. Hence the *level sets*

$$(v_H \cdot G)_c := \{v \in v_H \cdot G \subseteq V; \langle v, v_H\rangle_V = c\},$$

for $c \in \{-12, \dots, 12\}$, are unions of H-orbits. Hence to find out for which of these levels we have $(v_H \cdot G)_c \neq \emptyset$, we only have to compute $\langle v_i, v_H\rangle_V$ for a set of representatives $v_i \in V$, for $i \in \mathcal{I}$, for the H-orbits in $v_H \cdot G \cong \Omega$, where we have $|\mathcal{I}| = r = 778$. This even yields further information, namely how the level sets decompose into H-orbits.

We choose the chain of subgroups

$$U_1 := U_3(3){:}\,2 < U_2 := G_2(3) < U_3 = U := H = 3 \times G_2(3),$$

where $U_1 < U_2$ is a maximal subgroup. A set of generators of $N_G(H) = (3 \times G_2(3)){:}\,2$, given as words in $\mathcal{G}$, is available in [83]. Using the MeatAxe, we find a set of generators of H as well as a set of standard generators of U_2, in the sense of [81], as words in $\mathcal{G}$, and again a set of generators of U_1 as words in the set of standard generators of U_2 is available in [83].

Using the 3-modular Brauer character table of $G_2(3){:}\,2$, see [37], and GAP we find that $\tilde{V}_{N_G(H)}$ has the following constituents with multiplicities: $1a$, $2 \cdot 1b$, $7 \cdot 14a$, $49a$, $2 \cdot 49b$, where these are absolutely irreducible $\mathbb{F}_3N_G(H)$-modules of the indicated $\mathbb{F}_3$-dimensions, and the ordering is as in the 3-modular Brauer character table of $G_2(3){:}\,2$. Using the algorithms to compute submodule lattices described in [47] available in the MeatAxe, we find that $\tilde{V}_{N_G(H)}$ has a unique epimorphic image $\tilde{V}_2$ of $\mathbb{F}_3$-dimension 63. We have $\tilde{V}_2 \cong \begin{bmatrix} 14a \\ 49b \end{bmatrix}$ as $\mathbb{F}_3N_G(H)$-modules, a uniserial $\mathbb{F}_3N_G(H)$-module where the diagram indicates the radical and socle series of $\tilde{V}_2$. Furthermore, we have $(\tilde{V}_2)_{U_2} \cong \begin{bmatrix} 7a \oplus 7b \\ 49a \end{bmatrix}$ as $\mathbb{F}_3U_2$-modules, where again the constituents are absolutely irreducible $\mathbb{F}_3U_2$-modules, and the the diagram indicates the radical and socle series. Using the 3-modular Brauer character table of $U_1 = U_3(3){:}\,2$, see [37], and GAP we find that $(\tilde{V}_2)_{U_1}$ has the following absolutely irreducible constituents: $1b$, $6a$, $7a$, $7b$, $12a$, $30a$, all with multiplicity 1. Again using the MeatAxe we find that $(\tilde{V}_2)_{U_1}$ has a unique epimorphic image $\tilde{V}_1 \cong 6a \oplus 7b$ as $\mathbb{F}_3U_1$-modules, hence $\tilde{V}_1$ has $\mathbb{F}_3$-dimension 13.

Table 33: Level sets and H-orbits.

c	$\|(v_H \cdot G)_c\|$	$\|\mathcal{I}_c\|$
0	3 712 825 584	380
1	1 587 081 600	158
2	117 615 888	32
3	1 106 560	4
4	53 703	2
5	.	.
6	1 456	2
7	.	.
8	.	.
9	.	.
10	.	.
11	.	.
12	1	1

c	H-orbit lengths
6	$728, 728$
4	$9477, 44226$
3	$5824, 157248, 471744, 471744$

(18.3) We enumerate the orbit $\tilde{v}_H \cdot G$, using the technique described in Section (10.6), and find the H-orbits in $\tilde{v}_H \cdot G$, where there are $r = 778$ of them. Additionally, for each such H-orbit $(\tilde{v}_H \cdot G)_i \cong \Omega_i$, for $i \in \mathcal{I}$, we compute an element $g_i \in G$, as a word in the set of generators $\mathcal{G}$, mapping $\tilde{v}_H$ to an element of $(\tilde{v}_H \cdot G)_i$. As we have $\tilde{v}_H \cdot G \cong \Omega \cong v_H \cdot G$ as G-sets, we apply the $g_i \in G$, for $i \in \mathcal{I}$, to v_H, and collect the data on the scalar products $\langle v_H g_i, v_H \rangle_V$ and the suborbit lengths $|v_H g_i \cdot H| = |\Omega_i| = |\tilde{v}_H g_i \cdot H|$.

The result is shown in Table 33, where for each level $c \in \{0, \ldots, 12\}$ we give the cardinality $|(v_H \cdot G)_c|$ and the number $|\mathcal{I}_c|$ of H-orbits comprising the level set $(v_H \cdot G)_c$. For $c \in \{-12, \ldots, -1\}$ we have $|(v_H \cdot G)_c| = |(v_H \cdot G)_{-c}|$ and $|\mathcal{I}_c| = |\mathcal{I}_{-c}|$. In particular, the non-empty level sets are as stated in the conjecture in Section (18.1). For given $c \in \{-12, \ldots, 12\}$ the lengths of the H-orbits comprising the level set $(v_H \cdot G)_c$ are also known. This seems to be particularly interesting for $c \in \{3, 4, 6\}$, where the lengths of the H-orbits comprising these level sets are also given in Table 33.

19 The Harada-Norton group HN in characteristic 3

In Section 19 we present by example a new technique to use condensation results to determine decomposition numbers for finite groups. Historically, finding decomposition numbers was the very problem condensation techniques have been invented for, see [77]. Since then, these techniques have been applied by various people, see for example [26, 38, 57, 59].

Keeping the notation of Section 9, and letting $\lambda = 1$ and $\epsilon = \epsilon_1 \in A := FG$, we consider $V := \epsilon A$, which is a projective A-module. Hence the criterion in Section (9.12) is applicable as well as Proposition (9.13), see Remark (9.14). The information thus obtained is used to find projective indecomposable characters. We need some preparations first.

(19.1) Let $G := HN$ and $p := 3$. Let K, R and F be as in Section (2.10), where K is a splitting field of KG and F is a splitting field of FG of characteristic $p = 3$. Let $5^{1+4}{:}\,2^{1+4}.5.4 \cong H := N_G(5b) < G$, where $5b \in C_{5B} \in \mathcal{C}l(G)$, where the latter in turn denotes the $5B$-conjugacy class of G; the ordinary character table of H is available in GAP.

Using GAP we find the following data on the 3-modular blocks B_i of G, where $d_{B_i} \in \mathbb{N}_0$ denotes the defect of B_i, while $k_{B_i} := |\mathrm{Irr}_K(B_i)| \in \mathbb{N}$ and $l_{B_i} := |\mathrm{Irr}_F(B_i)| = |\mathrm{IBr}_F(B_i)| \in \mathbb{N}$ denote the number of irreducible ordinary and irreducible 3-modular Brauer characters of B_i, respectively. The last column corresponds to the union of the blocks of defect 0.

i	1	2	3	
d_{B_i}	6	2	1	0
k_{B_i}	33	9	3	9
l_{B_i}	20	7	2	9

Nothing has to be done for block B_3 of defect 1. We partly analyse block B_2 of defect 2. This is part of a full analysis of block B_2 and the principal block B_1 currently being work in progress [31]. We assume the reader familiar with the notion of *basic sets*, see [30].

Using GAP we find that the set

$$\mathcal{B}S := \{8910a, 16929a, 270864a, 1185030a, 1354320a, 1575936a, 4561920a\}$$

of irreducible ordinary characters in B_2 is a basic set of Brauer characters in B_2, as is indicated by the underlined entries in the first column in Table 34. We also give there a basic set $\mathcal{P}S := \{\Psi_1, \ldots, \Psi_7\} \subseteq \mathbb{Z}\mathrm{Irr}_K(B_2)$ of projective characters in $\mathbb{Z}\mathrm{Irr}_K(B_2)$, decomposed into the irreducible ordinary characters B_2, and indicate the origin of the Ψ_i, for $i \in \{1, \ldots, 7\}$. The characters $\{69255a, 1066527a, 3878280a\} \subseteq \mathrm{Irr}_K(G)$ are ordinary characters of defect 0. Since $|H|$ is not divisible by 3, all the irreducible ordinary characters in $\mathrm{Irr}_K(H)$ are projective characters, where $1^-, \lambda \in \mathrm{Irr}_K(H)$ denote linear characters of order 2 and 4, respectively, and $5b \in \mathrm{Irr}_K(H)$ is one of the rational-valued characters of degree 5. It turns out that $\langle \frac{1}{2} \cdot (5b)^G, \chi \rangle \in \mathbb{Z}$, for all $\chi \in \mathcal{B}S$, hence the B_2-component of $\frac{1}{2} \cdot (5b)^G$ is a projective character in B_2.

Thus Ψ_7 is a projective indecomposable character. We consider the possible projective summands of Ψ_6. These are sums of a multiple of Ψ_7 and the characters

Table 34: Basic set $\mathcal{P}S$ of projective characters in B_2 for $G := HN$ and $p := 3$.

i	χ_i	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	1^G	Ψ_6^1	Ψ_6^2	Ψ_6^3
<u>8</u>	$8910a$	1	.	.	.	.	.	.	2	.	.	.
<u>10</u>	$16929a$	.	1	.	.	.	.	.	1	.	.	.
<u>19</u>	$270864a$	.	.	1	.	.	.	.	.	.	.	.
<u>32</u>	$1185030a$	1	1	.	1	3	1	.	4	1	.	.
<u>33</u>	$1354320a$	2	2	.	2	2	1	.	2	.	1	.
<u>37</u>	$1575936a$	1	2	1	1	2	1	.	.	.	.	1
43	$2784375a$	2	2	1	3	5	2	.	3	1	1	.
<u>49</u>	$4561920a$	1	2	2	6	9	4	1	2	a	b	c
50	$4809375a$	1	3	3	5	9	4	1	3	a	$b-1$	$c+1$
	1^G in $\mathcal{P}S$	2	1	.	.	5	-14	9				

Ψ_i	origin
1	$(1^-)^G$
2	$1066527a \cdot 133b$
3	λ^G
4	$\frac{1}{2} \cdot (5b)^G$
5	$3878280a \cdot 133a$
6	$69255a \cdot 3344a$
7	$69255a \cdot 760a$

Table 35: Basic set $\mathcal{P}S'$ of projective characters in B_2.

i	χ_i	Ψ'_1	Ψ'_2	Ψ'_3	Ψ'_4	Ψ'_5	Ψ'_6	Ψ'_7
$\underline{8}$	$8910a$	1	.	.	.	.	.	.
$\underline{10}$	$16929a$	.	1	.	.	.	.	.
$\underline{19}$	$270864a$	.	.	1	.	.	.	.
$\underline{32}$	$1185030a$	1	1	.	1	1	.	.
$\underline{33}$	$1354320a$	1	.	.	2	.	1	.
$\underline{37}$	$1575936a$	.	.	1	1	.	1	.
43	$2784375a$	1	.	1	3	1	1	.
$\underline{49}$	$4561920a$	1	2	2	6	3	1	1
50	$4809375a$	1	3	3	5	3	1	1
	1^G in $\mathcal{P}S'$	2	1	.	.	1	.	-5
	$(1^-)^G$ in $\mathcal{P}S'$	1	.	.	.	.	1	-1

$\Psi_6^{1,2,3}$ shown in Table 34, where $a + b + c \leq 4$. Using the decomposition of the B_2-component of the projective character 1^G, where $1 \in \mathrm{Irr}_K(H)$ is the trivial character, into the basic set $\mathcal{P}S$ of projective characters, as is also shown in Table 34, we conclude that the projective indecomposable summand of Ψ_6 containing Ψ_6^1 is a 3-fold summand of Ψ_5. Hence Ψ_6^1 is a projective indecomposable character, and we have $a \in \{0, \ldots, 3\}$. Furthermore, both the projective indecomposable summands of Ψ_6 containing Ψ_6^2 and Ψ_6^3, respectively, are summands of Ψ_1. Hence we have $b+c \leq 1$. From that we conclude that both $\Psi_1 - \Psi_6^2 - \Psi_6^3$ and $\Psi_2 - 2 \cdot \Psi_6^2 - 2 \cdot \Psi_6^3$ are projective characters. Hence we obtain the basic set $\mathcal{P}S'$ of projective characters as shown in Table 35, where $\Psi'_i = \Psi_i$, for $i \in \{1, 2, 3, 4, 7\}$, as well as $\Psi'_5 := \Psi_5 - 2 \cdot \Psi_6 + 2 \cdot \Psi_7$ and $\Psi'_6 := 3 \cdot \Psi_6 - \Psi_5 - 2 \cdot \Psi_7$, while $\Psi'_1 := \Psi_1 - \Psi'_6 + \Psi_7$ and $\Psi'_2 := \Psi_2 - 2 \cdot \Psi'_6 + 2 \cdot \Psi_7$.

In Table 35 we also show the decomposition of the B_2-component of the projective characters 1^G and $(1^-)^G$ into the basic set $\mathcal{P}S'$ of projective characters. From this it follows that $\Psi''_5 := \Psi'_5 - \Psi'_7$ is a projective character, and we obtain the basic set $\mathcal{P}S''$ of projective characters as shown in Table 36, where $\Psi''_i = \Psi'_i$, for $i \in \{1, 2, 3, 4, 6, 7\}$. In Table 36 we also show the decomposition of the B_2-components of the projective characters 1^G and $(1^-)^G$ into the basic set $\mathcal{P}S''$ of projective characters.

(19.2) We are prepared to apply the technique described in Section (10.3) to $\Omega := H|G$ and $U := H$, yielding the action of $\epsilon\mathbb{F}_3G\epsilon$ on $\mathbb{F}_3\Omega\epsilon$, where the latter $\epsilon\mathbb{F}_3G\epsilon$-module is isomorphic to the regular $\epsilon\mathbb{F}_3G\epsilon$-module $\epsilon\mathbb{F}_3G\epsilon$.

Using the decomposition of 1^G into $\mathrm{Irr}_K(G)$ shown in Table 37, where the distribution of $\mathrm{Irr}_K(G)$ into the blocks B_1, B_2, B_3 and the characters of defect 0 is indicated as well, we obtain $\langle 1^G, 1^G \rangle_G = 127$, while $\langle 1^G \cdot \epsilon_{B_1}, 1^G \rangle_G = 62$ as

Table 36: Basic set $\mathcal{PS}''$ of projective characters in B_2.

i	χ_i	Ψ''_1	Ψ''_2	Ψ''_3	Ψ''_4	Ψ''_5	Ψ''_6	Ψ''_7	1^G
$\underline{8}$	$8910a$	1	.	.	.	.	.	.	2
$\underline{10}$	$16929a$	.	1	.	.	.	.	.	1
$\underline{19}$	$270864a$	.	.	1	.	.	.	.	.
$\underline{32}$	$1185030a$	1	1	.	1	1	.	.	4
$\underline{33}$	$1354320a$	1	.	.	2	.	1	.	2
$\underline{37}$	$1575936a$	.	.	1	1	.	1	.	.
43	$2784375a$	1	.	1	3	1	1	.	3
$\underline{49}$	$4561920a$	1	2	2	6	2	1	1	2
50	$4809375a$	1	3	3	5	2	1	1	3
	1^G in $\mathcal{PS}''$	2	1	.	.	1	.	-4	
	$(1^-)^G$ in $\mathcal{PS}''$	1	.	.	.	.	1	-1	

well as $\langle 1^G \cdot \epsilon_{B_2}, 1^G\rangle_G = 47$ and $\langle 1^G \cdot \epsilon_{B_3}, 1^G\rangle_G = 2$, where $\epsilon_{B_i} \in KG$ denote the central block idempotents of KG, for $i \in \{1, 2, 3\}$, see Remark (9.2).

We choose

$$V \cong \left[\begin{array}{c} 1a \\ 132a \end{array}\right],$$

a uniserial $\mathbb{F}_4G$-module with composition series as indicated, where the constituents are absolutely irreducible $\mathbb{F}_4G$-modules of the respective dimensions, which is the 2-modular reduction of an absolutely irreducible $\mathbb{Q}(\sqrt{5})G$-module. Representing matrices for the action of a set of standard generators of G, in the sense of [81], are available in [83]. Furthermore, a generating set of H given as words in the set of standard generators of G is also available there. We find $V_H \cong 1a \oplus 32a \oplus 100a$ as $\mathbb{F}_4H$-modules, where the summands are absolutely irreducible $\mathbb{F}_4H$-modules of the respective dimensions. Choosing $0 \neq v \in 1a \leq V_H$, as $H < G$ is a maximal subgroup, we obtain that Ω is as a G-set isomorphic to the G-orbit $\langle v\rangle_{\mathbb{F}_4} \cdot G$ of 1-dimensional $\mathbb{F}_4$-subspaces of V. Furthermore we choose $C_{25} \cong U_1 \leq H$. As U_1 is a cyclic group, the centrally primitive idempotents of $\mathbb{F}_4U_1$ are straightforwardly determined. This yields the decomposition of the semisimple $\mathbb{F}_4U_1$-module V_{U_1} into its $\mathbb{F}_4U_1$-isotypic components. A standard MeatAxe technique then allows to find an irreducible $\mathbb{F}_4U_1$-epimorphic image V_1 of V_{U_1} of $\mathbb{F}_4$-dimension 10.

This yields the orbit counting numbers with respect to $\Omega = \coprod_{i \in \mathcal{I}} \Omega_i$. By Proposition (9.5) we obtain representing matrices for the action of a few randomly chosen elements $\{\epsilon\tilde{g}_k\epsilon \in \epsilon\mathbb{F}_3G\epsilon; k \in \{1, 2, \ldots\}\}$, on $\mathbb{F}_3\Omega\epsilon$, where the above set is chosen such the criterion in Section (9.12) is fulfilled. Using the MeatAxe we find the constituents of the $\epsilon\mathbb{F}_3G\epsilon$-module $\mathbb{F}_3\Omega\epsilon$, their multiplicities, and the $\mathbb{F}_3$-dimensions of the endomorphism algebras of the simple $\epsilon\mathbb{F}_3G\epsilon$-modules as follows, see Remark (9.14).

Table 37: Characters 1^G and $(1^-)^G$ decomposed into $\mathrm{Irr}_K(G)$.

i	χ_i	1	1^-
1	$1a$	1	.
2	$133a$	.	1
3	$133b$	.	1
4	$760a$	.	.
5	$3344a$	.	.
6	$8778a$	.	.
7	$8778b$	.	.
9	$9405a$	.	2
11	$35112a$	.	.
12	$35112b$	.	.
13	$65835a$	1	.
14	$65835b$	1	.
17	$214016a$	1	.
18	$267520a$	1	1
20	$365750a$	2	3
21	$374528a$	.	.
22	$374528b$	.	.
24	$653125a$	2	1
25	$656250a$	1	.
26	$656250b$	1	.
27	$718200a$	1	2
28	$718200b$	1	2
29	$1053360a$	.	.
34	$1361920a$	1	1
35	$1361920b$	.	.
36	$1361920c$	.	.
40	$2375000a$	3	3
41	$2407680a$	2	2
42	$2661120a$	4	4
45	$3200000a$	1	2
46	$3424256a$	3	3
48	$4156250a$	1	.
54	$5878125a$	2	2
8	$8910a$	2	1
10	$16929a$	1	.
19	$270864a$	.	.
32	$1185030a$	4	1
33	$1354320a$	2	2
37	$1575936a$	.	1
43	$2784375a$	3	2
49	$4561920a$	2	1
50	$4809375a$	3	1
23	$406296a$	.	.
38	$1625184a$	1	1
39	$2031480a$	1	1
15	$69255a$	1	.
16	$69255b$	1	.
30	$1066527a$	.	1
31	$1066527b$	.	1
44	$2985984a$	2	2
47	$3878280a$	2	4
51	$5103000a$	1	1
52	$5103000b$	1	1
53	$5332635a$	2	4

$1a$	$1b$	$1c$	$1d$	$1e$	$1f$	$1g$	$1h$	$2a$	$2b$	$2c$	$2d$	$2e$
12	8	5	1	2	22	40	1	2	2	2	1	11
1	1	1	1	1	1	1	1	1	1	1	2	1

Let $\chi \in \mathrm{Irr}_K(G)$ be a character of defect 0, and let $\widehat{S_\chi}$ be an R-free RG-module affording χ. Then the 3-modular reduction $\widetilde{\widehat{S_\chi}}$ of $\widehat{S_\chi}$ is a projective simple FG-module. If $\widetilde{\widehat{S_\chi}} \cdot \epsilon \neq \{0\}$, then by Propositions (6.7) and (6.15) the $\epsilon FG\epsilon$-module $\widetilde{\widehat{S_\chi}} \cdot \epsilon$ is a projective simple $\epsilon FG\epsilon$-module. Using Table 37 we conclude by Proposition (9.13) that in this sense $\{69255a, 69255b\}$ correspond to $\{1d, 1h\}$, that $\{2985984a, 3878280a, 5332635a\}$ correspond to $\{2a, 2b, 2c\}$, and that $\{5103000a, 5103000b\}$ correspond to $2d$, where the 3-modular character field of the characters $5103000a, b$ is $\mathbb{F}_9$, yielding a projective simple $\mathbb{F}_3G$-module S such that $\mathrm{End}_{\mathbb{F}_3G}(S) \cong \mathbb{F}_9$. By Corollary (6.12) we conclude that $S\epsilon$ is a projective simple $\epsilon\mathbb{F}_3G\epsilon$-module such that $\mathrm{End}_{\epsilon\mathbb{F}_3G\epsilon}(S\epsilon) \cong \mathbb{F}_9$.

As the block B_3 is of defect 1, using the Brauer-Dade theory of blocks of cyclic defect, see [18, Ch.VII], we find from Table 37 that the B_3-component of the $\mathbb{F}_3G$-module 1^G is a projective indecomposable $\mathbb{F}_3G$-module P. As $\langle 1^G \cdot \epsilon_{B_3}, 1^G \rangle_G = 2$, using Propositions (6.19) and (9.13) we conclude that P corresponds to $1e$.

Using Table 36 we find that the B_2-component of the $\mathbb{F}_3G$-module 1^G has at least three different projective indecomposable $\mathbb{F}_3G$-summands, where at least one of them occurs with multiplicity 2. Again by Proposition (9.13) we conclude that $2e$ corresponds to a projective indecomposable $\mathbb{F}_3G$-module in B_2. From $\langle 1^G \cdot \epsilon_{B_2}, 1^G \rangle_G = 47$ we conclude that $\{1a, 1b, 1c\}$ also correspond to projective indecomposable $\mathbb{F}_3G$-modules in B_2, while $\{1f, 1g\}$ correspond to projective indecomposable $\mathbb{F}_3G$-modules in B_1.

Using Proposition (6.19) again, we find $\langle \Psi_1'', 1^G \rangle_G = 16$ and $\langle \Psi_2'', 1^G \rangle_G = 18$ as well as $\langle \Psi_5'', 1^G \rangle_G = 17$ and $\langle \Psi_7'', 1^G \rangle_G = 5$. As was shown in Section (19.1), the character Ψ_7'' is projective indecomposable, and the character $\Psi_5'' - a \cdot \Psi_7''$ is a projective indecomposable character for some $a \in \{0, 1, 2\}$. Hence by Proposition (9.13) we conclude that the projective indecomposable $\mathbb{F}_3G$-module affording Ψ_7'' corresponds to $1c$, and that $\Psi_5''' := \Psi_5'' - \Psi_7''$ is a projective indecomposable character afforded by a projective indecomposable $\mathbb{F}_3G$-module corresponding to $1a$. Hence from Table 36 we obtain the projective characters $\Psi_1''' := \Psi_1'' - \Psi_7''$ and $\Psi_2'' - \Psi_7''$. As Ψ_1''' occurs with multiplicity 2 in 1^G and $\langle \Psi_1''', 1^G \rangle_G = 11$, we conclude that Ψ_1''' is a projective indecomposable character, being afforded by a projective indecomposable $\mathbb{F}_3G$-module corresponding to $2e$. As $\langle \Psi_2'' - \Psi_7'', 1^G \rangle_G = 13$, the character $\Psi_2'' - \Psi_7''$ is not a projective indecomposable character. A consideration of the possible projective summands of $\Psi_2'' - \Psi_7''$ shows that $\Psi_2''' := \Psi_2'' - 2 \cdot \Psi_7''$ is a projective indecomposable character, being afforded by a projective indecomposable $\mathbb{F}_3G$-module corresponding to $1b$. Hence we obtain the basic set $\mathcal{PS}'''$ of projective characters as shown in Table

Table 38: Basic set $\mathcal{P}S'''$ of projective characters in B_2.

i	χ_i	Ψ'''_1	Ψ'''_2	Ψ'''_3	Ψ'''_4	Ψ'''_5	Ψ'''_6	Ψ'''_7	1	1^-
$\underline{8}$	$8910a$	1	.	.	.	.	.	.	2	1
$\underline{10}$	$16929a$	.	1	.	.	.	.	.	1	.
$\underline{19}$	$270864a$	.	.	1	.	.	.	.	.	.
$\underline{32}$	$1185030a$	1	1	.	1	1	.	.	4	1
$\underline{33}$	$1354320a$	1	.	.	2	.	1	.	2	2
$\underline{37}$	$1575936a$	.	.	1	1	.	1	.	.	1
43	$2784375a$	1	.	1	3	1	1	.	3	2
$\underline{49}$	$4561920a$	.	.	2	6	1	1	1	2	1
50	$4809375a$	.	1	3	5	1	1	1	3	1
	1^G in $\mathcal{P}S'''$	2	1	.	.	1	.	1		
	$(1^-)^G$ in $\mathcal{P}S'''$	1	.	.	.	.	1	.		

38, where $\Psi'''_i = \Psi''_i$, for $i \in \{3, 4, 6, 7\}$.

(19.3) Let $H' < H$ be the unique subgroup of index 2. We apply the technique described in Section (10.3) to $\Omega := H'|G$ and $U := H'$, yielding the action of $\epsilon'\mathbb{F}_3G\epsilon'$ on $\mathbb{F}_3\Omega'\epsilon'$, where $\epsilon' \in \mathbb{F}_3H' \subseteq \mathbb{F}_3G$ is the centrally primitive idempotent belonging to $1_{H'}$. Using the decomposition of $1_{H'}^G = 1^G + (1^-)^G$ into $\mathrm{Irr}_K(G)$ shown in Table 37, we obtain $\langle 1_{H'}^G, 1_{H'}^G\rangle_G = 460$, as well as $\langle 1_{H'}^G \cdot \epsilon_{B_1}, 1_{H'}^G\rangle_G = 250$ and $\langle 1_{H'}^G \cdot \epsilon_{B_2}, 1_{H'}^G\rangle_G = 102$ and $\langle 1_{H'}^G \cdot \epsilon_{B_3}, 1_{H'}^G\rangle_G = 8$.

We choose as V one of the absolutely irreducible $\mathbb{F}_9G$-modules of $\mathbb{F}_9$-dimension 133; it is the 3-modular reduction of an absolutely irreducible $\mathbb{Q}(\sqrt{5})G$-module. Representing matrices for the action of a set of standard generators of G on V are available in [83]. A generating set of H' as words in the generating set of H is found by a standard application of the MeatAxe. Using this we find $V_{H'} \cong 1a \oplus 32a \oplus 100a$ as $\mathbb{F}_9H'$-modules, where the constituents are absolutely irreducible $\mathbb{F}_9H'$-modules of the respective dimensions. Furthermore, all the $\mathbb{F}_9H'$-submodules of $V_{H'}$ are invariant under the action of H on V, where $1a$ extends to a linear $\mathbb{F}_9$-representation of $\mathbb{F}_9H$ of order 2. Hence choosing $0 \neq v \in 1a \leq V_{H'}$ we obtain that Ω' is as a G-set isomorphic to the G-orbit $\cong v \cdot G \subset V$. Furthermore, by a random search, we choose $5^2\colon D_8 \cong U_1 \leq H'$. A standard MeatAxe technique yields an epimorphic image V_1 of the semisimple $\mathbb{F}_9U_1$-module $V_{U'}$ of $\mathbb{F}_9$-dimension 4.

Proceeding as in Section (19.2), we find the constituents of the $\epsilon'\mathbb{F}_3G\epsilon'$-module $\mathbb{F}_3\Omega'\epsilon'$, their multiplicities and the $\mathbb{F}_3$-dimensions of the endomorphism algebras of the simple $\epsilon'\mathbb{F}_3G\epsilon'$-modules as follows.

$1a$	$1b$	$1c$	$1d$	$1e$	$1f$	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$4a$	$4b$	$6a$	$6b$
17	10	10	1	74	1	1	4	7	42	46	17	2	4	6	6
1	1	1	1	1	1	2	1	1	2	1	1	2	1	1	1

Using Table 37, and proceeding as in Section (19.2), we conclude that the constituents $\{1d, 1f, 2a, 4a, 4b, 6a, 6b\}$ correspond to the irreducible ordinary characters of defect 0 occurring in $1_{H'}^G$. Furthermore, the B_3-component of the $\mathbb{F}_3G$-module $1_{H'}^G$ is the direct sum $P \oplus P$, where P is the projective indecomposable $\mathbb{F}_3G$-module as in Section (19.2). Hence $2b$ corresponds to P. As all irreducible ordinary characters in $\mathrm{Irr}_K(B_2)$ are rational-valued, we conclude that $\mathbb{F}_3$ is the character field of all irreducible characters in $\mathrm{Irr}_F(B_2)$, hence $\mathbb{F}_3$ is a splitting field for all simple FG-modules affording a character in $\mathrm{Irr}_F(B_2)$. From this and Table 38 we conclude that all projective indecomposable $\epsilon'\mathbb{F}_3G\epsilon'$-submodules of $\mathbb{F}_3\Omega'\epsilon' \cdot \epsilon_{B_2}$ have an $\mathbb{F}_3$-dimension at most 17. Hence $\{1a, 1b, 1c, 2c, 3a\}$ are the constituents of $\mathbb{F}_3\Omega'\epsilon'$ corresponding to projective indecomposable $\mathbb{F}_3G$-summands of $1_{H'}^G \cdot \epsilon_{B_2}$.

We have already shown that the characters Ψ_1''' and Ψ_2''' as well as Ψ_5''' and Ψ_7''' are projective indecomposable, see Table 38. As $\langle \Psi_1''', 1_{H'}^G \rangle_G = 17$ and $\langle \Psi_2''', 1_{H'}^G \rangle_G = 10$ as well as $\langle \Psi_5''', 1_{H'}^G \rangle_G = 17$ and $\langle \Psi_6''', 1_{H'}^G \rangle_G = 17$, while $\langle \Psi_7''', 1_{H'}^G \rangle_G = 7$, we conclude that $3a$ corresponds to a projective indecomposable $\mathbb{F}_3G$-module affording Ψ_1''', one of $\{1b, 1c\}$ corresponds to Ψ_2''', while $1a$ corresponds to Ψ_5''', and $2c$ corresponds to Ψ_7'''. Hence we conclude that $\Psi_6'''' := \Psi_6''' - \Psi_7'''$ is a projective indecomposable character, being afforded by a projective indecomposable $\mathbb{F}_3G$-module corresponding to the other one of $\{1b, 1c\}$.

Note that we could determine which of $\{1b, 1c\}$ corresponds to Ψ_2''' and which to Ψ_6'''' by an analysis of the submodule structure of the $\epsilon'\mathbb{F}_3G\epsilon'$-modules $\mathbb{F}_3\Omega'\epsilon'$, using the algorithms to compute submodule lattices described in [47] available in the MeatAxe. Anyway, we obtain the basic set $\mathcal{PS}''''$ of projective characters as shown in Table 39, where $\Psi_i'''' = \Psi_i'''$, for $i \in \{1, 2, 3, 4, 5, 7\}$.

Hence for block B_2 it remains to find the projective indecomposable summands of Ψ_3'''' and Ψ_4''''. This requires different tools as well and will be done elsewhere, together with an analysis of the principal block B_1 [31].

IV References

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Table 39: Basic set $\mathcal{P}S''''$ of projective characters in B_2.

i	χ_i	Ψ_1''''	Ψ_2''''	Ψ_3''''	Ψ_4''''	Ψ_5''''	Ψ_6''''	Ψ_7''''
$\underline{8}$	$8910a$	1	.	.	.	.	.	.
$\underline{10}$	$16929a$	.	1	.	.	.	.	.
$\underline{19}$	$270864a$	.	.	1	.	.	.	.
$\underline{32}$	$1185030a$	1	1	.	1	1	.	.
$\underline{33}$	$1354320a$	1	.	.	2	.	1	.
$\underline{37}$	$1575936a$	.	.	1	1	.	1	.
43	$2784375a$	1	.	1	3	1	1	.
$\underline{49}$	$4561920a$	.	.	2	6	1	.	1
50	$4809375a$	.	1	3	5	1	.	1

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