On a Theorem by Benson and Conway

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Abstract

We define some purely lattice theoretic translations of algebraic notions related to submodule lattices, leading to new structural features of modular lattices and to generalisations of the Benson-Conway Theorem.
Mathematics Subject Classification: 06C05.

The starting point of the present work is a paper by Benson and Conway [2], whose main result is the description of a modular lattice in terms of its join-irreducible elements and their mutual inclusions. This has been used in [6] as one of the ingredients of an algorithm to compute submodule lattices of finite-dimensional modules for algebras over fields. The consideration of submodule lattices showed how to translate certain algebraic notions into a purely lattice theoretic setting, and how to use the new notions to prove statements about arbitrary modular lattices. This might help to obtain a better structural understanding of modular lattices, and to learn which are the particular aspects distinguishing submodule lattices from general modular lattices.

The present work is organised as follows: In Section 1 we recall some background from lattice theory. In Section 2 we prove a generalisation and a refinement of the Benson-Conway Theorem. In Section 3 we introduce the new notion of blocks, which are a translation of a related algebraic notion, and apply the Benson-Conway Theorem to obtain a description of the centre of a modular lattice. In Section 4 we introduce the new notion of types, which are a translation of the related notion of the algebraic isomorphism type associated to each simple subquotient of a submodule lattice, and we prove a purely lattice theoretic version of the algebraic Jordan-Hölder Theorem. Moreover, we apply the notion of types to prove a further generalisation of the Benson-Conway Theorem. In Section 5 we finally consider submodule lattices, and discuss the relationship between lattice theoretic types and algebraic types, as well as the relationship between lattice theoretic blocks and algebraic blocks.

1 Modular lattices

We begin by fixing the setting we are working in, and then recall a few basic notions and facts from lattice theory, which we assume the reader to be familiar with. As general references see e. g. [1, Ch.2], [3], [4].

(1.1) Assumptions and notation. In the sequel let $\mathcal{M} \neq \emptyset$ be a lattice whose underlying partial order is denoted by $\leq$, and whose join and meet operations are denoted by $\vee$ and $\wedge$, respectively. We assume that $\mathcal{M}$ has a least element, being denoted by $0 \in \mathcal{M}$, but we do not in general assume that $\mathcal{M}$ has a greatest element, unless otherwise specified, in which case it is denoted by $1 \in \mathcal{M}$. Moreover, we assume that all chains in $\mathcal{M}$ between any two fixed elements $x < y \in \mathcal{M}$ are finite, and that $\mathcal{M}$ is modular, i. e. for all $x, y, z \in \mathcal{M}$ such that $z \leq x$ we have the modular law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = (x \wedge y) \vee z.$$

For $x \leq y \in \mathcal{M}$ we let $[x, y] := \{z \in \mathcal{M}; x \leq z \leq y\}$ denote the interval between x and y. For $x < y \in \mathcal{M}$ we write $x \lessdot y$, if for $z \in \mathcal{M}$ such that $x < z \leq y$ we already have $z = y$. In particular, the elements $0 \lessdot x \in \mathcal{M}$ are called atoms. Let $\mathcal{L} \subseteq \mathcal{M} \setminus \{0\}$ be the set of join-irreducible elements $z \in \mathcal{M}$, i. e. whenever there are $x, y \in \mathcal{M}$ such that $x \vee y = z$ then we already have $x = z$ or $y = z$. Note that we have $\mathcal{L} = \{z \in \mathcal{M}; |\{y \in \mathcal{M}; y \lessdot z\}| = 1\}$.

Let $l_{\mathcal{M}} \colon \mathcal{M} \to \mathbb{N}_0$ be the Jordan-Dedekind length function, defined by $l_{\mathcal{M}}(0) = 0$ and $l_{\mathcal{M}}(y) = l_{\mathcal{M}}(x) + 1$, whenever $x \lessdot y \in \mathcal{M}$. Moreover, each $x \in \mathcal{M}$ is the join of finitely many join-irreducible elements, and if $x = \bigvee_{i=1}^{r} x_i \in \mathcal{M}$ is irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$, then the number $r \in \mathbb{N}_0$ is independent of the particular choice of the irredundant representation, giving rise to the rank function $r_{\mathcal{M}} \colon \mathcal{M} \to \mathbb{N}_0$. Note that $r_{\mathcal{M}}(x) \leq l_{\mathcal{M}}(x)$ for $x \in \mathcal{M}$.

Let $\mathcal{Z}(\mathcal{M})$ denote the centre of $\mathcal{M}$, i. e. the set of $x \in \mathcal{M}$ such that there is $x' \in \mathcal{M}$ such that $\mathcal{M} \cong [0, x] \times [0, x']$. Note that we have $\mathcal{Z}(\mathcal{M}) \neq \emptyset$ if and only if $\mathcal{M}$ has a greatest element. In this case, $\mathcal{Z}(\mathcal{M})$ is a finite set such that $\{0, 1\} \subseteq \mathcal{Z}(\mathcal{M})$. If $\mathcal{Z}(\mathcal{M}) = \{0, 1\}$ then $\mathcal{M}$ is called indecomposable, otherwise $\mathcal{M}$ is called decomposable. If $\{z_1, \ldots, z_d\}$ is the set of minimal elements of $\mathcal{Z}(\mathcal{M}) \setminus \{0\}$, then $\mathcal{M} \cong \prod_{i=1}^{d} [0, z_i]$ is the unique decomposition of $\mathcal{M}$ into nontrivial indecomposable intervals.

(1.2) Complemented lattices. Let $\mathcal{M}$ have a greatest element. Then $\mathcal{M}$ is called complemented, if for all $x \in \mathcal{M}$ there is a complement $y \in \mathcal{M}$, i. e. we have $x \vee y = 1$ and $x \wedge y = 0$.

Recall that by [4, Thm.4.3] the lattice $\mathcal{M}$ is complemented if and only if $1 \in \mathcal{M}$ is a join of atoms, which holds if and only if each element of $\mathcal{M}$ is a join of atoms, which in turn holds if and only if $\mathcal{L}$ consists entirely of atoms. We give another characterisation of complemented lattices, which probably is well-known, but for which we did not find an appropriate reference:

(1.3) Lemma. Let $\mathcal{M}$ have a greatest element. Then $\mathcal{M}$ is complemented if and only if $r_{\mathcal{M}}(1) = l_{\mathcal{M}}(1) \in \mathbb{N}_0$.

Proof. We proceed by induction on $r_{\mathcal{M}}(1) \in \mathbb{N}_0$, the case $r_{\mathcal{M}}(1) \leq 1$ being trivial, let $r = r_{\mathcal{M}}(1) \geq 2$. Let $1 = \bigvee_{i=1}^{r} x_i \in \mathcal{M}$ be irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$, and let $x := \bigvee_{i=1}^{r-1} x_i \in \mathcal{M}$ and $y := \bigvee_{i=2}^{r} x_i \in \mathcal{M}$.

If $\mathcal{M}$ is complemented, then by induction we have $l_{\mathcal{M}}(x) = r_{\mathcal{M}}(x) = r - 1$. Since $x_r \in \mathcal{M}$ is an atom we have $x \wedge x_r = 0$ and hence $r = l_{\mathcal{M}}(x) + l_{\mathcal{M}}(x_r) =$

$l_{\mathcal{M}}(x \vee x_r) + l_{\mathcal{M}}(x \wedge x_r) = l_{\mathcal{M}}(1)$. If conversely $r_{\mathcal{M}}(1) = l_{\mathcal{M}}(1)$, then we have $r-1 = r_{\mathcal{M}}(x) \leq l_{\mathcal{M}}(x) < r$, thus $r_{\mathcal{M}}(x) = l_{\mathcal{M}}(x) = r-1$, and hence by induction all the $x_i \in \mathcal{M}$, for $i \in \{1, \ldots, r-1\}$, are atoms. For $y \in \mathcal{M}$ we argue similarly, and hence $1 \in \mathcal{M}$ is a join of atoms. ♯

(1.4) Radicals. For $0 \neq x \in \mathcal{M}$ the element $x_* := \bigwedge\{y \in \mathcal{M}; y \lessdot x\} \in \mathcal{M}$ is called the radical of x; we let $0_* := 0$. The name 'radical' is motivated from the case of submodule lattices, where it coincides with the notion of the Jacobson radical of a module.

We recall a few properties of radicals: For $y \leq x \in \mathcal{M}$ such that $y \vee x_* = x$ we already have $y = x$. Using the characterisation of complemented lattices in (1.2) we conclude that for $x \in \mathcal{M}$ we have $x_* = \bigwedge\{y \in \mathcal{M}; [y, x] \text{ complemented}\}$. In particular we have $\mathcal{L} = \{x \in \mathcal{M}; x_* \lessdot x\}$. The following statements are probably also well-known, but again we did not find an appropriate reference:

(1.5) Lemma. a) For $x \in \mathcal{M}$ we have $r_{\mathcal{M}}(x) = r_{[x_*, x]}(x)$.
b) For $x, y \in \mathcal{M}$ we have $(x \vee y)_* = x_* \vee y_*$.

Proof. a) Let $x = \bigvee_{i=1}^{r} x_i$ be irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$, and $r = r_{\mathcal{M}}(x)$. Hence we have $x_i \not\leq x_*$, and since $[x_*, x_i \vee x_*] \cong [x_i \wedge x_*, x_i]$ is complemented, we have $x_* \lessdot x_i \vee x_*$. Assume that $x = \bigvee_{i \in \mathcal{I}}(x_i \vee x_*)$, for some $\mathcal{I} \subset \{1, \ldots, r\}$, then we also have $x = \bigvee_{i \in \mathcal{I}} x_i$, a contradiction. Thus $x = \bigvee_{i=1}^{r}(x_i \vee x_*)$ is irredundant in $[x_*, x]$, hence $r = r_{[x_*, x]}(x)$.

b) Let $x = \bigvee_{i=1}^{r} x_i$ and $y = \bigvee_{j=1}^{s} y_j$ be irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$, and $y_j \in \mathcal{L}$ for all $j \in \{1, \ldots, s\}$. As $[x_* \vee y_*, x_i \vee (x_* \vee y_*)] \cong [x_i \wedge (x_* \vee y_*), x_i]$ is complemented, we have either $x_* \vee y_* = x_i \vee (x_* \vee y_*)$ or $x_* \vee y_* \lessdot x_i \vee (x_* \vee y_*)$. A similar statement holds for the y_j, and hence $x \vee y$ is a join of atoms of $[x_* \vee y_*, x \vee y]$. Thus we have $(x \vee y)_* \leq x_* \vee y_*$.

Conversely, assume that we have $x_* \not\leq (x \vee y)_*$. Then $[(x \vee y)_*, x \vee (x \vee y)_*] \cong [(x \vee y)_* \wedge x, x]$ is not complemented. Since $(x \vee y)_* \leq x \vee (x \vee y)_* \leq x \vee y$ this a contradiction. Hence we have $x_* \leq (x \vee y)_*$ and similarly $y_* \leq (x \vee y)_*$. ♯

This leads to the definition of certain subsets of modular lattices, which will turn out to convey interesting structural features. Later on we give a finer structural partition of these sets, see (4.9).

(1.6) Definition. For $r \in \mathbb{N}_0$ let

$$\mathcal{M}_r := \{x \in \mathcal{M}; r_{\mathcal{M}}(x) = r\} = \{x \in \mathcal{M}; r_{[x_*, x]}(x) = r\} \subseteq \mathcal{M}$$

as well as $\mathcal{L}_r := \{x \in \mathcal{M}_r; [x_*, x] \text{ indecomposable}\} \subseteq \mathcal{M}_r$.

Hence in particular we have $\mathcal{M}_0 = \mathcal{L}_0 = \{0\}$ and $\mathcal{M}_1 = \mathcal{L}_1 = \mathcal{L}$.

2 The Benson-Conway Theorem

The Benson-Conway Theorem, originally stated and proved in [2], gives a description of a modular lattice $\mathcal{M}$ in terms of certain subsets of $\mathcal{L}$, called complete subsets, in whose definition the set $\mathcal{L}_2$ plays a central role. Our approach begins with having a closer look at the set $\mathcal{L}_2$, which first leads to the notion of dotted-lines and subsequently to the notion of complete subsets. Our definition of dotted-lines is slightly more structural than the one in [2], but we proceed to show equivalence of both definitions. We prove a slightly more general version of the Benson-Conway Theorem, inasmuch lattices are not assumed to have a greatest element; still our proof is similar to the original one. We proceed to prove a refinement, and in (4.10) we give a further generalisation.

(2.1) The set $\mathcal{L}_2$. For an element $z \in \mathcal{M}$ we have $r_{\mathcal{M}}(z) = 2$, if and only if there are $x_1, x_2 \in \mathcal{L}$ such that $x_1 \not\leq x_2 \not\leq x_1$ and $z = x_1 \vee x_2$. In this case we have $l_{[z_*,z]}(z) = 2$. Hence the elements of $[z_*, z]$ are its least element z_*, its greatest element z, and further elements $z_* \lessdot z_i \lessdot z$, for $i \in \mathcal{I}_z$, where $\mathcal{I}_z$ is a suitable index set. In particular, for $z_i := x_i \vee z_*$ we have $z_* \lessdot z_i \lessdot z$ and $z_1 \neq z_2$, hence $|\mathcal{I}_z| \geq 2$.

If $|\mathcal{I}_z| = 2$, then we have $[z_*, z] \cong [z_*, z_1] \times [z_*, z_2]$, hence $[z_*, z]$ is decomposable. Conversely, let $[z_*, z] \cong [z_*, z'] \times [z_*, z'']$ be a nontrivial decomposition. Since $l_{[z_*,z]}(z) = 2$, we have $l_{[z_*,z]}(z') = l_{[z_*,z]}(z'') = 1$, hence $z_* \lessdot z', z''$, and thus $|\mathcal{I}_z| = 2$. Hence $[z_*, z]$ is indecomposable if and only if $|\mathcal{I}_z| \geq 3$. Thus in conclusion we have $\mathcal{L}_2 = \{z \in \mathcal{M}_2; |\mathcal{I}_z| \geq 3\}$.

(2.2) Definition. Let $z \in \mathcal{L}_2$, and let $\mathcal{I}_z$ as in (2.1). A set $\mathcal{D} = \{x_i \in \mathcal{L}; i \in \mathcal{I}_z\}$, such that $x_i \vee z_* = z_i$ for all $i \in \mathcal{I}_z$, is called a **dotted-line** for z.

Note that dotted-lines always exist: Given $z \in \mathcal{L}_2$, each z_i is the join of the join-irreducible elements contained in z_i, and since $z_* \lessdot z_i$ we may choose $x_i \in \mathcal{L}$ such that $x_i \leq z_i$ and $x_i \not\leq z_*$. Dotted-lines are characterised as follows, where (2.3)b) actually is the original definition given in [2].

(2.3) Proposition. a) Let $z \in \mathcal{L}_2$ and let $\mathcal{D} = \{x_i \in \mathcal{L}; i \in \mathcal{I}_z\}$ be a dotted-line for z. Then we have $x_i \not\leq x_j$ and $x_i \vee x_j = z$, for all $i \neq j \in \mathcal{I}_z$, and $\mathcal{D} \subseteq \mathcal{L}$ is maximal having this property. In particular, we have $|\mathcal{D}| = |\mathcal{I}_z| \geq 3$ and $\bigvee \mathcal{D} = z$ is well-defined.
b) Let $\mathcal{D} = \{x_i \in \mathcal{L}; i \in \mathcal{I}\}$ be a subset of $\mathcal{L}$, for an index set $\mathcal{I}$, such that $|\mathcal{D}| = |\mathcal{I}| \geq 3$ and $x_i \vee x_j \in \mathcal{M}$ is independent of the choice of $i \neq j \in \mathcal{I}$, and such that $\mathcal{D} \subseteq \mathcal{L}$ is maximal having this property. Then $z := \bigvee \mathcal{D} \in \mathcal{L}_2 \subseteq \mathcal{M}$ is well-defined and $\mathcal{D}$ is a dotted-line for z.

Proof. a) For all $i \in \mathcal{I}_z$ let $z_i = x_i \vee z_* \lessdot z$. As $z_i \not\leq z_j$, for all $i \neq j \in \mathcal{I}_z$, we also have $x_i \not\leq x_j$. Since $z = z_i \vee z_j = x_i \vee x_j \vee z_*$ we have $x_i \vee x_j = z$. Finally, assume that there is $x_0 \in \mathcal{L} \setminus \mathcal{D}$ such that $x_0 \vee x_i = z$ for all $i \in \mathcal{I}_z$, and let

$z_0 := x_0 \vee z_*$. Hence we have $z_* \lessdot z_0 \lessdot z$, and there is $i \in \mathcal{I}_z$ such that $z_0 = z_i$, thus $x_0 \vee z_* = x_i \vee z_*$. Hence $z = x_0 \vee x_i \vee z_* = x_0 \vee z_* = x_0$, a contradiction, proving the maximality property.
b) Assume that we have $x_i \leq x_j$, for some $i \neq j \in \mathcal{I}$, and let $k \in \mathcal{I}$ such that $i \neq k \neq j$. Hence we have $x_i \vee x_k = x_i \vee x_j = x_j \in \mathcal{L}$, a contradiction. Thus we have $x_i \not\leq x_j$ for all $i \neq j \in \mathcal{I}$. Hence for $z := x_i \vee x_j = \bigvee \mathcal{D}$ we have $r_{\mathcal{M}}(z) = l_{[z_*,z]}(z) = 2$. Let $z_i := x_i \vee z_*$ for all $i \in \mathcal{I}$, hence we have $z_* \lessdot z_i \lessdot z$. Assume that $z_i = z_j$ for some $i \neq j \in \mathcal{I}$, then we have $z = x_i \vee x_j \vee z_* = z_i \vee z_j = z_i$, a contradiction. Thus we have $\mathcal{I} \subseteq \mathcal{I}_z$. Assume that there is $k \in \mathcal{I}_z \setminus \mathcal{I}$, and choose $x_k \in \mathcal{L}$ such that $z_k = x_k \vee z_*$. Hence for all $i \in \mathcal{I}$ we have $x_i \vee x_k \vee z_* = z_i \vee z_k = z$ and thus $x_i \vee x_k = z$, contradicting the maximality property. Hence we have $\mathcal{I} = \mathcal{I}_z$ and $\mathcal{D}$ is a dotted-line for z. ♯

(2.4) Definition. Let $\mathcal{X} \subseteq \mathcal{L}$ be an ideal, i. e. whenever we have $x \in \mathcal{X}$ and $y \in \mathcal{L}$ such that $y \leq x$, we also have $y \in \mathcal{X}$. The ideal $\mathcal{X} \subseteq \mathcal{L}$ is called **complete**, if $\mathcal{X}$ is bounded in $\mathcal{M}$, i. e. there is $z \in \mathcal{M}$ such that $x \leq z$ for all $x \in \mathcal{X}$, and if for each dotted-line $\mathcal{D} \subseteq \mathcal{L}$ such that $|\mathcal{D} \cap \mathcal{X}| \geq 2$ we already have $\mathcal{D} \subseteq \mathcal{X}$.

Let $\mathcal{M}(\mathcal{L})$ be the partially ordered set of complete ideals of $\mathcal{L}$, where the partial order is given by set theoretic inclusion. Hence $\mathcal{M}(\mathcal{L})$ is closed under taking intersections, and thus becomes a lattice by letting $\mathcal{X} \wedge \mathcal{X}' := \mathcal{X} \cap \mathcal{X}'$ and

$$\mathcal{X} \vee \mathcal{X}' := \bigcap \{\mathcal{Y} \in \mathcal{M}(\mathcal{L}); \mathcal{X} \cup \mathcal{X}' \subseteq \mathcal{Y}\} \in \mathcal{M}(\mathcal{L}).$$

(2.5) Theorem: Benson-Conway.
The following maps are a pair of mutually inverse isomorphisms of lattices:

$$\beta\colon \mathcal{M} \to \mathcal{M}(\mathcal{L})\colon x \mapsto \{y \in \mathcal{L}; y \leq x\} \quad \text{and} \quad \beta^{-1}\colon \mathcal{M}(\mathcal{L}) \to \mathcal{M}\colon \mathcal{X} \mapsto \bigvee \mathcal{X}.$$

Proof. The maps β and β^{-1} are well-defined and order-preserving, and we have $\beta^{-1} \circ \beta = \mathrm{id}_{\mathcal{M}}$. Hence we have to show that $\beta \circ \beta^{-1} = \mathrm{id}_{\mathcal{M}(\mathcal{L})}$ also holds. Assume to the contrary that there are $\mathcal{X} \in \mathcal{M}(\mathcal{L})$ and $y \in \mathcal{L} \setminus \mathcal{X}$ such that $y \leq \bigvee \mathcal{X}$. Let $n_y := \min\{|\mathcal{Y}|; \mathcal{Y} \subseteq \mathcal{X}, y \leq \bigvee \mathcal{Y}\}$; note that $n_y \in \mathbb{N}$ is well-defined. As $\mathcal{X}$ is an ideal we have $n_y \geq 2$, and we may choose y such that n_y is minimal. Let $\mathcal{Y} = \{y_1, \ldots, y_{n_y}\} \subseteq \mathcal{X} \subseteq \mathcal{L}$, and we may choose $\mathcal{Y}$ such that $y \vee y_1$ is minimal.

By modularity we have

$$y \vee y_1 = (y \vee y_1) \wedge (y_1 \vee \bigvee_{i>1} y_i) = y_1 \vee ((y \vee y_1) \wedge \bigvee_{i>1} y_i).$$

Let $z := (y \vee y_1) \wedge \bigvee_{i>1} y_i$, and let $z_j \in \mathcal{L}$ such that $z = \bigvee_{j=1}^{r} z_j$, for some $r \in \mathbb{N}_0$. Assume that $z_j \notin \mathcal{X}$ for some $j \in \{1, \ldots, r\}$, then we have $z_j \leq y \vee y_1 \leq \bigvee \mathcal{X}$ and $z_j \leq \bigvee_{i>1} y_i$, contradicting the minimality of n_y. Hence we have $z_j \in \mathcal{X}$ for all $j \in \{1, \ldots, r\}$. Assume that $y \vee y_1 \leq z$, then we have $y \leq \bigvee_{i>1} y_i$, again contradicting the minimality of n_y. Hence we have $y \vee y_1 \not\leq z$. Since $y \vee y_1 = y_1 \vee z = y_1 \vee \bigvee_{j=1}^{r} z_j$, we have $y \vee y_1 = y_1 \vee z_j$, for some $j \in \{1, \ldots, r\}$.

In particular we have $y \leq y_1 \vee z_j$, and thus $n_y = 2$ and $y_1 \neq z_j$. Moreover we have $y \vee z_j \leq y \vee y_1$, and by the minimality of $y \vee y_1$ we conclude $y \vee z_j = y \vee y_1$. Hence by (2.3) there is a dotted-line $\mathcal{D}$ such that $\{y, y_1, z_j\} \subseteq \mathcal{D}$, and since $y_1, z_j \in \mathcal{X}$ completeness implies $y \in \mathcal{X}$, the final contradiction. ♯

The refined version we prove next is based on the observation that in general there is more than one dotted-line for a given element of $\mathcal{L}_2$, see (4.8). To decide whether a given ideal of $\mathcal{L}$, which is bounded in $\mathcal{M}$, is complete, the completeness property by definition has to be checked with respect to all dotted-lines. We proceed to show that it actually is sufficient to check it with respect to a single fixed dotted-line for each element of $\mathcal{L}_2$.

(2.6) Definition. For each $z \in \mathcal{L}_2$ choose a dotted-line $\mathcal{D}_z \subseteq \mathcal{L}$ for z. An ideal $\mathcal{X} \subseteq \mathcal{L}$ is called **weakly complete** with respect to $\{\mathcal{D}_z; z \in \mathcal{L}_2\}$, if $\mathcal{X}$ is bounded in $\mathcal{M}$, and if for each $z \in \mathcal{L}_2$ such that $|\mathcal{D}_z \cap \mathcal{X}| \geq 2$ we have $\mathcal{D}_z \subseteq \mathcal{X}$.

(2.7) Theorem. Let $\{\mathcal{D}_z; z \in \mathcal{L}_2\}$ be chosen as in (2.6). Then a subset $\mathcal{X} \subseteq \mathcal{L}$ is complete if and only if it is weakly complete with respect to $\{\mathcal{D}_z; z \in \mathcal{L}_2\}$.

Proof. Let $\mathcal{X}$ be weakly complete, and we proceed by induction on $l_{\mathcal{M}}(\bigvee \mathcal{X})$. If $l_{\mathcal{M}}(\bigvee \mathcal{X}) \leq 1$, then $\mathcal{X} = \{0, \bigvee \mathcal{X}\}$ is complete. Hence let $l_{\mathcal{M}}(\bigvee \mathcal{X}) \geq 2$. Let $x_1, x_2 \in \mathcal{X}$ such that $z := x_1 \vee x_2 \in \mathcal{L}_2$. It is sufficient to show that for all $x \in \mathcal{L}$ such that $x \leq z$ we already have $x \in \mathcal{X}$:

Let $\mathcal{Z}_* := \{y \in \mathcal{X}; y \leq z_*\}$, let $z_i := x_i \vee z_* \lessdot z$ and $\mathcal{Z}_i := \{y \in \mathcal{X}; y \leq z_i\}$, for $i \in \{1, 2\}$. Since $\mathcal{X}$ is an ideal and $z_* = (x_1)_* \vee (x_2)_*$, we have $\bigvee \mathcal{Z}_* = z_*$, and hence we have $\bigvee \mathcal{Z}_i = z_i \lessdot z \leq \bigvee \mathcal{X}$. Moreover, the $\mathcal{Z}_i$ are weakly complete, and since $l_{\mathcal{M}}(\bigvee \mathcal{Z}_i) < l_{\mathcal{M}}(\bigvee \mathcal{X})$ the sets $\mathcal{Z}_i$ are complete by induction.

Let $y_i \in \mathcal{D}_z$ such that $z_i = y_i \vee z_*$, for $i \in \{1, 2\}$. Since the $\mathcal{Z}_i$ are complete such that $\bigvee \mathcal{Z}_i = z_i$, by (2.5) we have $y_i \in \mathcal{Z}_i \subseteq \mathcal{X}$, and as $\mathcal{X}$ is weakly complete we have $\mathcal{D}_z \subseteq \mathcal{X}$. Let $k \in \mathcal{I}_z$ such that $x \leq z_k \lessdot z$, and let $\mathcal{Z}_k := \{y \in \mathcal{X}; y \leq z_k\}$. Since we have $\mathcal{D}_z \subseteq \mathcal{X}$ and $\bigvee \mathcal{Z}_* = z_*$, we obtain $\bigvee \mathcal{Z}_k = z_k \lessdot z \leq \bigvee \mathcal{X}$. Moreover, $\mathcal{Z}_k$ is weakly complete, and since $l_{\mathcal{M}}(\bigvee \mathcal{Z}_k) < l_{\mathcal{M}}(\bigvee \mathcal{X})$ the set $\mathcal{Z}_k$ is complete by induction. As $\bigvee \mathcal{Z}_k = z_k$ by (2.5) we finally have $x \in \mathcal{Z}_k \subseteq \mathcal{X}$. ♯

As an application, (2.5) can be used for the computation of submodule lattices. There are computational techniques to determine the set $\mathcal{L}$, and having found $\mathcal{L}$ it remains to enumerate the complete sets, where by the refined version (2.7) the necessary amount of checking is reduced significantly. Actually, (2.7) is motivated by this application, and has already been proved for the particular case of submodule lattices in [6], where also the algorithmic details of the computation of submodule lattices are given.

We conclude this section by briefly commenting on distributive lattices, whose basic structure theorem is Birkhoff's Representation Theorem, which is a special case of (2.5) and sheds some further light on the significance of the set $\mathcal{L}_2$.

(2.8) Corollary: Birkhoff's Representation Theorem.
The lattice $\mathcal{M}$ is distributive, i. e. for all $x, y, z \in \mathcal{M}$ we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, if and only if $\mathcal{L}_2 = \emptyset$. In this case the following maps are a pair of mutually inverse isomorphisms of lattices:

$$\beta\colon \mathcal{M} \to \{\mathcal{X} \subseteq \mathcal{L}; \mathcal{X} \text{ finite ideal}\}\colon x \mapsto \{y \in \mathcal{L}; y \leq x\},$$

$$\beta^{-1}\colon \{\mathcal{X} \subseteq \mathcal{L}; \mathcal{X} \text{ finite ideal}\} \to \mathcal{M}\colon \mathcal{X} \mapsto \bigvee \mathcal{X}.$$

Proof. If $z \in \mathcal{L}_2 \neq \emptyset$, then $[z_*, z]$ is not distributive, hence neither $\mathcal{M}$ is. Conversely, if $\mathcal{L}_2 = \emptyset$ then $\mathcal{M}(\mathcal{L})$ is the set of ideals of $\mathcal{L}$ which are bounded in $\mathcal{M}$, and whose join and meet operations are given by taking set theoretic unions and intersections, respectively. Hence $\mathcal{M}(\mathcal{L})$ is distributive, and thus by (2.5) $\mathcal{M}$ also is.

If $\mathcal{M}$ is distributive, then the join-irreducible elements of $\mathcal{M}(\mathcal{L})$ are the singleton subsets of $\mathcal{L}$. As the elements of $\mathcal{M}(\mathcal{L})$ are bounded in $\mathcal{M}$, we conclude $\mathcal{M}(\mathcal{L}) = \{\mathcal{X} \subseteq \mathcal{L}; \mathcal{X} \text{ finite ideal}\}$. ♯

3 Blocks

As a consequence of (2.5) we derive a description of the centre of a modular lattice, being based on a certain graph, called the block graph, having the join-irreducible elements as its vertices. The name 'block' is reminiscent of the related algebraic notion, see (5.5).

(3.1) Proposition. Let $x \in \mathcal{M}$ and $\mathcal{X} := \beta(x) \in \mathcal{M}(\mathcal{L})$, where β is as in (2.5). Then we have $x \in \mathcal{Z}(\mathcal{M})$ if and only if $\mathcal{X}' := \mathcal{L} \setminus \mathcal{X} \in \mathcal{M}(\mathcal{L})$. In this case, letting $x' := \beta^{-1}(\mathcal{X}') \in \mathcal{M}$, we have $\mathcal{M} \cong [0, x] \times [0, x']$.

Proof. Let $x \in \mathcal{Z}(\mathcal{M})$, let $\mathcal{M} \cong [0, x] \times [0, x']$ for some suitable $x' \in \mathcal{M}$, and let $\mathcal{X}' := \beta(x') \in \mathcal{M}(\mathcal{L})$. Hence for $z \in \mathcal{L}$ we have $z \leq x$ or $z \leq x'$, and thus $\mathcal{L} = \{z \in \mathcal{L}; z \leq x\} \,\dot\cup\, \{z \in \mathcal{L}; z \leq x'\} = \mathcal{X} \,\dot\cup\, \mathcal{X}'$, hence $\mathcal{X}' = \mathcal{L} \setminus \mathcal{X}$.

Let conversely $\mathcal{X}' := \mathcal{L} \setminus \mathcal{X} \in \mathcal{M}(\mathcal{L})$, and let $\mathcal{Y}, \mathcal{Y}' \in \mathcal{M}(\mathcal{L})$ such that $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{X}'$. Then $\mathcal{Y} \,\dot\cup\, \mathcal{Y}' \subseteq \mathcal{L}$ is an ideal which is bounded in $\mathcal{M}$. Let $\mathcal{D} \subseteq \mathcal{L}$ be a dotted-line such that $|\mathcal{D} \cap (\mathcal{Y} \,\dot\cup\, \mathcal{Y}')| \geq 2$. Since $|\mathcal{D}| \geq 3$ we have $|\mathcal{D} \cap \mathcal{X}| \geq 2$ or $|\mathcal{D} \cap \mathcal{X}'| \geq 2$, and hence either $\mathcal{D} \subseteq \mathcal{X}$ or $\mathcal{D} \subseteq \mathcal{X}'$. Thus we have either $|\mathcal{D} \cap \mathcal{Y}| \geq 2$ or $|\mathcal{D} \cap \mathcal{Y}'| \geq 2$, and hence either $\mathcal{D} \subseteq \mathcal{Y}$ or $\mathcal{D} \subseteq \mathcal{Y}'$. Thus $\mathcal{Y} \,\dot\cup\, \mathcal{Y}' \in \mathcal{M}(\mathcal{L})$ and hence $\mathcal{Y} \vee \mathcal{Y}' = \mathcal{Y} \,\dot\cup\, \mathcal{Y}'$.

Let $x' := \beta^{-1}(\mathcal{X}') \in \mathcal{M}$, and $\sigma\colon [0, x] \times [0, x'] \to \mathcal{M}\colon [y, y'] \mapsto y \vee y'$. Hence σ is order-preserving. By the above we have $\beta(y \vee y') = \beta(y) \,\dot\cup\, \beta(y')$, hence $\beta(y) = \beta(y \vee y') \cap \mathcal{X}$ and $\beta(y') = \beta(y \vee y') \cap \mathcal{X}'$, thus σ is injective. For $z \in \mathcal{M}$ and $\mathcal{Z} := \beta(z) \in \mathcal{M}(\mathcal{L})$ we have $\mathcal{Z} \cap \mathcal{X}, \mathcal{Z} \cap \mathcal{X}' \in \mathcal{M}(\mathcal{L})$ and $\mathcal{Z} = (\mathcal{Z} \cap \mathcal{X}) \,\dot\cup\, (\mathcal{Z} \cap \mathcal{X}')$. Hence $z = \beta^{-1}(\mathcal{Z}) = y \vee y'$, where $y := \beta^{-1}(\mathcal{Z} \cap \mathcal{X}) = \beta^{-1}(\mathcal{Z}) \wedge \beta^{-1}(\mathcal{X}) = z \wedge x \in \mathcal{M}$ and $y' := \beta^{-1}(\mathcal{Z} \cap \mathcal{X}') = \beta^{-1}(\mathcal{Z}) \wedge \beta^{-1}(\mathcal{X}') = z \wedge x' \in \mathcal{M}$. Thus σ is surjective as well. ♯

(3.2) Definition. The **block graph** of $\mathcal{M}$ is defined as the undirected simple graph having vertex set $\mathcal{L}$, where vertices $x, y \in \mathcal{L}$ are adjacent if and only if $x < y$ or $y < x$ or $x \vee y \in \mathcal{L}_2$. Vertices being in the same connected component of the block graph are called to be in the same **block**, giving rise to the disjoint union $\mathcal{L} = \coprod_{i=1}^{d} \mathcal{Z}_i$, where $d \in \mathbb{N}_0$ is the number of blocks occurring.

Note that $\mathcal{Z}_i \subseteq \mathcal{L}$ is an ideal, and for each dotted-line $\mathcal{D} \subseteq \mathcal{L}$ we have either $\mathcal{D} \cap \mathcal{Z}_i = \emptyset$ or $\mathcal{D} \subseteq \mathcal{Z}_i$, in general $\mathcal{Z}_i$ is not bounded in $\mathcal{M}$. Hence if $\mathcal{M}$ has a greatest element, then we have $\mathcal{Z}_i \in \mathcal{M}(\mathcal{L})$.

(3.3) Theorem. Let $\mathcal{M}$ have a greatest element, let $\mathcal{L} = \coprod_{i=1}^{d} \mathcal{Z}_i$ be the disjoint union of blocks, and let $z_i := \beta^{-1}(\mathcal{Z}_i) \in \mathcal{M}$, for all $i \in \{1, \ldots, d\}$, where β is as in (2.5). Then, given $x \in \mathcal{M}$, we have $x \in \mathcal{Z}(\mathcal{M})$ if and only if $x = \bigvee_{i; z_i \leq x} z_i$ holds. In particular, $\mathcal{M} \cong \prod_{i=1}^{d} [0, z_i]$ is the unique decomposition of $\mathcal{M}$ into nontrivial indecomposable intervals.

Proof. Let $x \in \mathcal{Z}(\mathcal{M})$ and $\mathcal{X} := \beta(x) \in \mathcal{M}(\mathcal{L})$ as well as $\mathcal{X}' := \mathcal{L} \setminus \mathcal{X} \in \mathcal{M}(\mathcal{L})$ and $x' := \beta^{-1}(\mathcal{X}') \in \mathcal{M}$. Let $\mathcal{X} \cap \mathcal{Z}_i \neq \emptyset$ and $y \in \mathcal{X} \cap \mathcal{Z}_i$, and let $y' \in \mathcal{Z}_i$. If $y' < y$, then by completeness we have $y' \in \mathcal{X}$ as well. If $y' > y$, assume that $y' \notin \mathcal{X}$, hence we have $y' \in \mathcal{X}'$, thus by completeness $y \in \mathcal{X}'$, a contradiction, showing $y' \in \mathcal{X}$. If $y \vee y' \in \mathcal{L}_2$, assume again that $y' \notin \mathcal{X}$, hence we have $y' \in \mathcal{X}'$, thus there is a dotted-line $\mathcal{D} \subseteq \mathcal{L}$ such that $\{y, y'\} \subseteq \mathcal{D}$, hence we have $\mathcal{D} \cap \mathcal{X} \neq \emptyset$ and $\mathcal{D} \cap \mathcal{X}' \neq \emptyset$. Since $|\mathcal{D}| \geq 3$ we have $|\mathcal{D} \cap \mathcal{X}| \geq 2$ or $|\mathcal{D} \cap \mathcal{X}'| \geq 2$, and hence either $\mathcal{D} \subseteq \mathcal{X}$ or $\mathcal{D} \subseteq \mathcal{X}'$, a contradiction, again showing $y' \in \mathcal{X}$.

Hence we have shown that $\mathcal{X} \cap \mathcal{Z}_i \neq \emptyset$ already implies $\mathcal{Z}_i \subseteq \mathcal{X}$. Thus we have $\bigvee_{i; \mathcal{X} \cap \mathcal{Z}_i \neq \emptyset} z_i \leq x$ and $\bigvee_{j; \mathcal{X}' \cap \mathcal{Z}_j \neq \emptyset} z_j \leq x'$. Since $\mathcal{M} \cong [0, x] \times [0, x']$ and $\bigvee_{i=1}^{d} z_i = 1 = x \vee x'$ we conclude $x = \bigvee_{i; z_i \leq x} z_i$.

Let conversely $\{1, \ldots, d\} = \mathcal{I} \,\dot\cup\, \mathcal{J}$ as well as $x := \bigvee_{i \in \mathcal{I}} z_i \in \mathcal{M}$ and $x' := \bigvee_{j \in \mathcal{J}} z_j \in \mathcal{M}$. Since for each dotted-line $\mathcal{D} \subseteq \mathcal{L}$ we have $\mathcal{D} \cap \mathcal{Z}_i = \emptyset$ or $\mathcal{D} \subseteq \mathcal{Z}_i$, we conclude that $\beta(x) = \bigvee_{i \in \mathcal{I}} \mathcal{Z}_i = \coprod_{i \in \mathcal{I}} \mathcal{Z}_i \in \mathcal{M}(\mathcal{L})$ and $\beta(x') = \bigvee_{j \in \mathcal{J}} \mathcal{Z}_j = \coprod_{j \in \mathcal{J}} \mathcal{Z}_j \in \mathcal{M}(\mathcal{L})$. Since $\mathcal{L} = (\coprod_{i \in \mathcal{I}} \mathcal{Z}_i) \,\dot\cup\, (\coprod_{j \in \mathcal{J}} \mathcal{Z}_j)$ we have $x \in \mathcal{Z}(\mathcal{M})$. ♯

These observations lead to a simple proof of Maeda's Theorem, which relates the atoms and the centre of a complemented lattice.

(3.4) Corollary: Maeda's Theorem.
Let $\mathcal{M}$ be complemented, and let $\mathcal{M} \cong \prod_{i=1}^{d} [0, z_i]$ be the decomposition of $\mathcal{M}$ into nontrivial indecomposable intervals. Then for atoms $x \neq y \in \mathcal{M}$ we have $x \vee y \in \mathcal{L}_2$ if and only if there is $i \in \{1, \ldots, d\}$ such that both $x \leq z_i$ and $y \leq z_i$.

Proof. Note that there are unique $i, j \in \{1, \ldots, d\}$ such that $x \leq z_i$ and $y \leq z_j$. If $i \neq j$, then since $[0, z_i \vee z_j] \cong [0, z_i] \times [0, z_j]$ we have $[0, x \vee y] = [0, x] \times [0, y]$, thus $x \vee y \notin \mathcal{L}_2$. If $i = j$, then by (3.3) let $\mathcal{Z}_i := \beta^{-1}(z_i) \subseteq \mathcal{L}$ be the block x and y belong to. As $\mathcal{Z}_i$ is a connected component of the block graph, there is a chain $x = x_0, x_1, \ldots, x_s = y$ in $\mathcal{L}$ such that $x_{i-1} \vee x_i \in \mathcal{L}_2$, for all $i \in \{1, \ldots, s\}$. Hence by [3, La.IV.6.1, La.IV.6.2] we conclude $x \vee y \in \mathcal{L}_2$ as well. ♯

4 Types

We consider a certain subgraph of the block graph, called the $\mathcal{L}_2$-graph, which still has the join-irreducible elements as vertices, but now adjacency is governed by the set $\mathcal{L}_2$ alone. In the light of the Benson-Conway Theorem, this graph should indeed convey interesting information about the underlying modular lattice. This leads to the definition of so-called types for a modular lattice, where again this notion is motivated by an algebraic counterpart: For submodule lattices each simple subquotient of a module has an algebraic isomorphism type attached to it, see (5.1). Lattice theoretic types generalise this and even turn out to be slightly finer than algebraic types. In particular we prove a purely lattice theoretic version of the algebraic Jordan-Hölder Theorem. Having the lattice theoretic notion of types at hand, we finally proceed to obtain a further generalisation of (2.5).

(4.1) Definition. The $\mathcal{L}_2$**-graph** of $\mathcal{M}$ is defined as the undirected simple graph having vertex set $\mathcal{L}$, where vertices $x, y \in \mathcal{L}$ are adjacent if and only if $x \vee y \in \mathcal{L}_2$. Hence the $\mathcal{L}_2$-graph can be considered as a weighted graph, where $\mathcal{L}_2$ is the set of weights, and the edge connecting the vertices x and y has weight $x \vee y$. Vertices being in the same connected component of the $\mathcal{L}_2$-graph are called to be of the same **type**. This gives rise to the **type map** $t_{\mathcal{M}} \colon \mathcal{L} \to \mathcal{T}$, where $\mathcal{T}$ is a suitable index set, and where we assume $t_{\mathcal{M}}$ to be surjective.

Given $z \in \mathcal{L}_2$, we show that the subgraph of the $\mathcal{L}_2$-graph induced by the edges of weight z is connected: Let $x, x', y, y' \in \mathcal{L}$ such that $x \vee y = z = x' \vee y'$, i. e. both x and y as well as x' and y' are connected by an edge of weight z. The elements $x \vee z_* \neq y \vee z_*$ and $x' \vee z_* \neq y' \vee z_*$ are atoms of $[z_*, z]$. Interchanging x' and y' if necessary, we may assume that $x \vee z_* \neq x' \vee z_*$ holds, thus $z = x \vee x' \vee z_* = x \vee x'$. Hence x and x' are also connected by an edge of weight z.

Since the subgraph induced by the edges of weight z is connected, it belongs to a single connected component of the $\mathcal{L}_2$-graph. Thus the type map can be extended to a map $t_{\mathcal{M}} \colon \mathcal{L} \,\dot{\cup}\, \mathcal{L}_2 \to \mathcal{T}$, such that for $x \in \mathcal{L}$ being incident to an edge of weight $z \in \mathcal{L}_2$ we have $t_{\mathcal{M}}(x) = t_{\mathcal{M}}(z)$. Moreover, the type map can be extended to the set of dotted-lines by letting $t_{\mathcal{M}}(\mathcal{D}) := t_{\mathcal{M}}(z)$, where $\mathcal{D} \subseteq \mathcal{L}$ is a dotted-line for $z \in \mathcal{L}_2$.

A complete subgraph of the $\mathcal{L}_2$-graph all of whose edges have weight $z \in \mathcal{L}_2$ is called a **clique of weight** z. Hence given $z \in \mathcal{L}_2$, by (2.3) the maximal cliques of weight z are in bijection with the dotted-lines for z. Moreover, if $\mathcal{M}$ is distributive, then by (2.8) the $\mathcal{L}_2$-graph has no edges at all, and hence the type map $t_{\mathcal{M}} \colon \mathcal{L} \to \mathcal{T}$ is a bijection.

(4.2) Proposition. Let $y \lessdot z \in \mathcal{M}$, and let $x, x' \in \mathcal{L}$ such that $x \neq x'$ and $x \vee y = z = x' \vee y$. Then we have $t_{\mathcal{M}}(x) = t_{\mathcal{M}}(x')$.

Proof. We have $x \wedge y = x_*$ and $x' \wedge y = (x')_*$. Assume that $x' < x$, then $x' \wedge y = x' \wedge x \wedge y = x' \wedge x_* = x'$, a contradiction. Thus we conclude $x' \not\leq x$, and similarly $x \not\leq x'$. Let $v := x \vee x' \in \mathcal{M}$, hence we have $r_{\mathcal{M}}(v) = 2$. Since $[y, z] = [y, y \vee v] \cong [v \wedge y, v]$, we have $v_* \lessdot v \wedge y \lessdot v$. Moreover, we have $x \vee v_* \neq x' \vee v_*$, as well as $v_* \lessdot x \vee v_* \lessdot v$ and $v_* \lessdot x' \vee v_* \lessdot v$. Since $x, x' \not\leq y$ we have $x \vee v_* \neq v \wedge y$ and $x' \vee v_* \neq v \wedge y$, thus we conclude that $v \in \mathcal{L}_2$. Hence x and x' are adjacent in the $\mathcal{L}_2$-graph. ♯

(4.3) Definition. a) Let $y \lessdot z \in \mathcal{M}$ and let $x \in \mathcal{L}$ such that $x \vee y = z$. As z is the join of the join-irreducible elements contained in z, such an element x indeed exists, and by (4.2) the type $t_{\mathcal{M}}(x)$ is independent of the choice of such an $x \in \mathcal{L}$. Hence the **type** $t_{\mathcal{M}}([y, z]) := t_{\mathcal{M}}(x) \in \mathcal{T}$ is well-defined.

b) Let $x \leq y \in \mathcal{M}$, and let $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l = y$ be a maximal chain in $\mathcal{M}$, where $l = l_{[x,y]}(y) = l_{\mathcal{M}}(y) - l_{\mathcal{M}}(x) \in \mathbb{N}_0$. Letting $\tau_i := t_{\mathcal{M}}([x_{i-1}, x_i]) \in \mathcal{T}$, for all $i \in \{1, \ldots, l\}$, we obtain the **type sequence** $[\tau_1, \ldots, \tau_l] \subseteq \mathcal{T}$ associated to the pair $x \leq y$. By (4.4) below, up to reordering the type sequence only depends on the endpoints x and y, but not on the particular choice of the maximal chain between them. Hence there is a **multiset of types** $t_{\mathcal{M}}([x, y]) \sim [\tau_1, \ldots, \tau_l]$ associated to the pair $x \leq y$, where $\sim$ indicates equality of multisets. In particular, for $x \in \mathcal{L}$ we have $t_{\mathcal{M}}([x_*, x]) \sim t_{\mathcal{M}}(x)$, where we do not distinguish between a singleton multiset and its single element.

Moreover, if $r_\tau \in \mathbb{N}_0$ is the multiplicity with which the type $\tau \in \mathcal{T}$ occurs in the multiset of types $[\tau_1, \ldots, \tau_l]$, then we also use the notation $[\tau_1, \ldots, \tau_l] \sim [\tau^{r_\tau}; \tau \in \mathcal{T}]$, and thus the multiset of types $[\tau_1, \ldots, \tau_l]$ is equivalently described by the **multiplicity vector** $[r_\tau; \tau \in \mathcal{T}] \in \mathbb{N}_0^{|\mathcal{T}|}$.

(4.4) Theorem. Let $x \leq y \in \mathcal{M}$ and let $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l = y$ and $x = x'_0 \lessdot x'_1 \lessdot \cdots \lessdot x'_l = y$ be maximal chains in $\mathcal{M}$, with associated type sequences $[\tau_1, \ldots, \tau_l] \subseteq \mathcal{T}$ and $[\tau'_1, \ldots, \tau'_l] \subseteq \mathcal{T}$, respectively. Then we have $[\tau_1, \ldots, \tau_l] \sim [\tau'_1, \ldots, \tau'_l]$ as multisets, i. e. the type sequence $[\tau'_1, \ldots, \tau'_l]$ is a reordering of the type sequence $[\tau_1, \ldots, \tau_l]$.

Proof. We proceed by induction on $l = l_{\mathcal{M}}(y) - l_{\mathcal{M}}(x) \in \mathbb{N}_0$. The case $l \leq 1$ being trivial, let $l \geq 2$. If $x_1 = x'_1$, then we are done by induction. Hence let $x_1 \neq x'_1$ and $z := x_1 \vee x'_1 \in \mathcal{M}$. Thus $[x_1, z] = [x_1, x_1 \vee x'_1] \cong [x_1 \cap x'_1, x'_1] = [x, x'_1]$, and similarly $[x'_1, z] \cong [x, x_1]$. In particular, we have $l_{\mathcal{M}}(y) - l_{\mathcal{M}}(x_1) = l_{\mathcal{M}}(y) - l_{\mathcal{M}}(x'_1) = l - 1$ and $l_{\mathcal{M}}(y) - l_{\mathcal{M}}(z) = l - 2$. Moreover, if $w \in \mathcal{L}$ such that $w \vee x = x_1$, then $w \vee x'_1 = w \vee x \vee x'_1 = x_1 \vee x'_1 = z$, thus $\tau_1 = t_{\mathcal{M}}([x, x_1]) = t_{\mathcal{M}}([x'_1, z])$, and similarly $\tau'_1 = t_{\mathcal{M}}([x, x'_1]) = t_{\mathcal{M}}([x_1, z])$.

Let $[\tau''_1, \ldots, \tau''_{l-2}] \subseteq \mathcal{T}$ be a type sequence associated to a maximal chain for $z \leq y$ in $\mathcal{M}$. Hence $[\tau'_1, \tau''_1, \ldots, \tau''_{l-2}] \subseteq \mathcal{T}$ and $[\tau_1, \tau''_1, \ldots, \tau''_{l-2}] \subseteq \mathcal{T}$ are type sequences associated to maximal chains for $x_1 \leq y$ and $x_1' \leq y$ in $\mathcal{M}$, respectively. By induction we have $[\tau_2, \ldots, \tau_l] \sim [\tau_1', \tau_1'', \ldots, \tau_{l-2}'']$ as multisets, thus $[\tau_1, \ldots, \tau_l] \sim [\tau_1, \tau_1', \tau_1'', \ldots, \tau_{l-2}'']$ as multisets, and similarly $[\tau_1', \ldots, \tau_l'] \sim [\tau_1', \tau_1, \tau_1'', \ldots, \tau_{l-2}'']$ as multisets. ♯

We derive a few immediate consequences, showing that lattice theoretic types indeed behave as natural generalisations of algebraic types.

(4.5) Corollary. a) If $\mathcal{M}$ has a greatest element, then $\mathcal{T}$ is a finite set.
b) For $x, y \in \mathcal{M}$ we have $t_{\mathcal{M}}([x \wedge y, x]) \sim t_{\mathcal{M}}([y, x \vee y])$ as multisets.
c) Let $x = \bigvee_{i=1}^r x_i \in \mathcal{M}$ be irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$. Then we have $t_{\mathcal{M}}([x_*, x]) \sim [t_{\mathcal{M}}(x_1), \ldots, t_{\mathcal{M}}(x_r)]$ as multisets.

Proof. a) Let $\tau \in \mathcal{T}$, and let $x \in \mathcal{L}$ such that $t_{\mathcal{M}}(x) = \tau$. Choosing a maximal chain $0 = x_0' \lessdot \cdots \lessdot x_r' = x_* \lessdot x = x_0 \lessdot \cdots \lessdot x_s = 1$ in $\mathcal{M}$ shows that the type τ occurs in some, and hence any, maximal chain for $0 \leq 1$ in $\mathcal{M}$.
b) Let $x \wedge y = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_l = x$ be a maximal chain in $\mathcal{M}$, and let $z_i \in \mathcal{L}$ such that $z_i \vee x_{i-1} = x_i$, for all $i \in \{1, \ldots, l\}$. Hence we have $t_{\mathcal{M}}([x_{i-1}, x_i]) = t_{\mathcal{M}}(z_i)$. Thus $y = (x_0 \vee y) \lessdot (x_1 \vee y) \lessdot \cdots \lessdot (x_l \vee y) = x \vee y$ is a maximal chain in $\mathcal{M}$. Since $z_i \vee (x_{i-1} \vee y) = x_i \vee y$, for all $i \in \{1, \ldots, l\}$, we conclude $t_{\mathcal{M}}([x_{i-1} \vee y, x_i \vee y]) = t_{\mathcal{M}}(z_i)$.
c) We have $l_{[x_*, x]}(x) = r$, and $x_* \lessdot x_i \vee x_*$ for all $i \in \{1, \ldots, r\}$. Hence $x_* \lessdot x_1 \vee x_* \lessdot (x_1 \vee x_2) \vee x_* \lessdot \cdots \lessdot (\bigvee_{i=1}^r x_i) \vee x_* = x$ is a maximal chain in $\mathcal{M}$. ♯

(4.6) Proposition. Let $r \in \mathbb{N}_0$ and $x \in \mathcal{L}_r$. Then there is $\tau \in \mathcal{T}$ such that $t_{\mathcal{M}}([x_*, x]) \sim [\tau^r]$ as multisets.

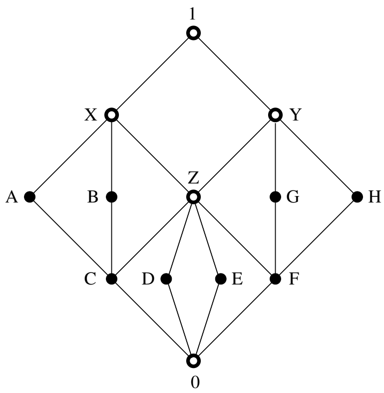
Proof. Let $x = \bigvee_{i=1}^r x_i \in \mathcal{L}_r$ be irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$. Hence we have $x_* \lessdot x_i \vee x_*$, and $x = \bigvee_{i=1}^r (x_i \vee x_*)$ is irredundant in $[x_*, x]$, in particular we have $x_i \vee x_* \neq x_j \vee x_*$ for all $i \neq j$. Hence by (3.4) we have $(x_i \vee x_j) \vee x_* \in \mathcal{L}_2([x_*, x])$, thus $[x_*, (x_i \vee x_j) \vee x_*]$ is indecomposable. Since $[x_*, (x_i \vee x_j) \vee x_*] \cong [(x_i \vee x_j) \wedge x_*, x_i \vee x_j]$ is complemented, we conclude that $(x_i \vee x_j)_* \leq (x_i \vee x_j) \wedge x_* \leq x_i \vee x_j$, hence $(x_i \vee x_j) \wedge x_* = (x_i \vee x_j)_*$, and thus $x_i \vee x_j \in \mathcal{L}_2$. Hence $t_{\mathcal{M}}(x_i) = t_{\mathcal{M}}(x_j) \in \mathcal{T}$ for all $i \neq j$, and thus $t_{\mathcal{M}}([x_*, x]) \sim [t_{\mathcal{M}}(x_1), \ldots, t_{\mathcal{M}}(x_r)] \sim [t_{\mathcal{M}}(x_1)^r]$ as multisets. ♯

(4.7) Remark. If again $x = \bigvee_{i=1}^r x_i \in \mathcal{L}_r$ is irredundant, where $x_i \in \mathcal{L}$ for all $i \in \{1, \ldots, r\}$, then an argument similar to the one in the proof of (4.6) shows that $x_{\mathcal{J}} := \bigvee_{i \in \mathcal{J}} x_i \in \mathcal{L}_{|\mathcal{J}|}$, for all $\mathcal{J} \subseteq \{1, \ldots, r\}$. Moreover, we have $[(x_{\mathcal{J}})_*, x_{\mathcal{J}}] \cong [x_*, x_{\mathcal{J}} \vee x_*]$, and as $[x_*, x]$ is complemented and indecomposable by [3, Thm.IV.7.11] the isomorphism type of $[x_*, x_{\mathcal{J}} \vee x_*]$ only depends on $|\mathcal{J}|$. In particular, for all $i, j \in \{1, \ldots, r\}$ such that $i \neq j$, the cardinality $|\mathcal{I}_{x_i \vee x_j}|$ is independent of the particular choice of $i \neq j$, where $\mathcal{I}_{x_i \vee x_j}$ is as in (2.1).

Hence the $\mathcal{L}_2$-graph can be considered as the 1-dimensional skeleton of a simplicial complex, whose $(r-1)$-dimensional simplices, for $r \in \mathbb{N}$, are the r-subsets $\{x_1, \ldots, x_r\} \subseteq \mathcal{L}$ such that $\bigvee_{i=1}^{r} x_i \in \mathcal{L}_r$. Thus in particular, if $z, z' \in \mathcal{L}_2$ are 1-dimensional faces of an r-dimensional simplex, where $r \geq 2$, then we have $|\mathcal{I}_z| = |\mathcal{I}_{z'}|$. Moreover, (4.6) allows to extend the type map to a map $t\colon \coprod_{r\in\mathbb{N}} \mathcal{L}_r \to \mathcal{T}$ compatible with incidence in this simplicial complex.

But note that the converse of (4.6) does not hold in general, i. e. for $x \in \mathcal{M}_r$ such that $t_\mathcal{M}([x_*, x]) \sim [\tau^r]$, for some $\tau \in \mathcal{T}$, in general we do not have $x \in \mathcal{L}_r$, see (4.8). Moreover, being in the same connected component of the $\mathcal{L}_2$-graph, i. e. we have $t_\mathcal{M}(z) = t_\mathcal{M}(z')$, in general does not suffice to imply equality of $|\mathcal{I}_z|$ and $|\mathcal{I}_{z'}|$, see again (4.8). Actually, in both respects algebraic types for submodule lattices behave more smoothly, see (5.3).

(4.8) Example. Let $\mathcal{M}$ be the modular lattice whose Hasse diagram is as follows:



We have

$$\mathcal{L} = \{A, B, C, D, E, F, G, H\} \quad \text{and} \quad \mathcal{L}_2 = \{X, Y, Z\},$$

where $|\mathcal{I}_X| = 3 = |\mathcal{I}_Y|$ and $|\mathcal{I}_Z| = 4$, while $\mathcal{M}_2 = \mathcal{L}_2 \,\dot{\cup}\, \{1\}$ and $\mathcal{M}_3 = \emptyset$. The unique dotted-line for Z is $\{C, D, E, F\}$, while X and Y each have 3 dotted-lines

$$\{\{A, B, D\}, \{A, B, E\}, \{A, B, F\}\} \quad \text{and} \quad \{\{C, G, H\}, \{D, G, H\}, \{E, G, H\}\},$$

respectively. Finally, as $X = A \vee D$ and $Y = D \vee H$ we have $t_\mathcal{M}(A) = t_\mathcal{M}(D) = t_\mathcal{M}(H)$ and hence $|t_\mathcal{M}(\mathcal{L})| = 1$, but we have $A \vee H = 1 \in \mathcal{M}_2 \setminus \mathcal{L}_2$. ♯

We are prepared to derive a further generalisation of (2.5). Actually, again it is motivated by an algebraic counterpart for submodule lattices, see (5.4).

(4.9) Definition. a) For $\tau \in \mathcal{T}$ let

$$\mathcal{M}_\tau := \coprod_{r\in\mathbb{N}_0} \{x \in \mathcal{M}_r; t_\mathcal{M}([x_*, x]) \sim [\tau^r]\} \quad \text{and} \quad \mathcal{L}_\tau := \{x \in \mathcal{L}; t_\mathcal{M}(x) = \tau\}.$$

For $x, x' \in \mathcal{M}_\tau$ let $x = \bigvee_{i=1}^{r} x_i$ and $x' = \bigvee_{j=1}^{s} x'_j$ be irredundant, where $x_i, x'_j \in \mathcal{L}$. for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$. Then by (4.5) we have $t_{\mathcal{M}}(x_i) = \tau = t_{\mathcal{M}}(x'_j)$, and hence $x \vee x' = (\bigvee_{i=1}^{r} x_i) \vee (\bigvee_{j=1}^{s} x'_j) \in \mathcal{M}_\tau$ as well. Thus $\mathcal{M}_\tau$ is closed under taking joins, and becomes a lattice by $x \vee_\tau x' := x \vee x'$ and

$$x \wedge_\tau x' := \bigvee \{y \in \mathcal{M}_\tau; y \leq x \wedge x'\} \in \mathcal{M}_\tau.$$

b) An ideal $\mathcal{X} \subseteq \mathcal{L}_\tau$ is called **τ-complete**, if $\mathcal{X}$ is bounded in $\mathcal{M}$, and if for each dotted-line $\mathcal{D} \subseteq \mathcal{L}_\tau$ such that $|\mathcal{D} \cap \mathcal{X}| \geq 2$ we already have $\mathcal{D} \subseteq \mathcal{X}$. Note that by (4.1) for each dotted-line $\mathcal{D} \subseteq \mathcal{L}$ we have $\mathcal{D} \subseteq \mathcal{L}_{t(\mathcal{D})}$.

Let $\mathcal{M}(\mathcal{L}_\tau)$ be the partially ordered set of τ-complete subsets of $\mathcal{L}_\tau$, where the partial order is given by set theoretic inclusion. Hence $\mathcal{M}(\mathcal{L}_\tau)$ is closed under taking intersections, and becomes a lattice by letting $\mathcal{X} \wedge_\tau \mathcal{X}' := \mathcal{X} \cap \mathcal{X}'$ and

$$\mathcal{X} \vee_\tau \mathcal{X}' := \bigwedge \{\mathcal{Y} \in \mathcal{M}(\mathcal{L}_\tau); \mathcal{X} \cup \mathcal{X}' \subseteq \mathcal{Y}\} \in \mathcal{M}(\mathcal{L}_\tau).$$

(4.10) Theorem. Let $\tau \in \mathcal{T}$. Then $\mathcal{M}_\tau$ is a modular lattice, and the following maps are a pair of mutually inverse isomorphisms of lattices:

$$\beta_\tau \colon \mathcal{M}_\tau \to \mathcal{M}(\mathcal{L}_\tau) \colon x \mapsto \{y \in \mathcal{L}_\tau; y \leq x\} \text{ and } \beta_\tau^{-1} \colon \mathcal{M}(\mathcal{L}_\tau) \to \mathcal{M}_\tau \colon \mathcal{X} \mapsto \bigvee \mathcal{X}.$$

Proof. We show that for all $x, y, z \in \mathcal{M}_\tau$ such that $z \leq x$ the modular law $x \wedge_\tau (y \vee_\tau z) = (x \wedge_\tau y) \vee_\tau z$ holds: We have

$$x \wedge_\tau (y \vee_\tau z) = \bigvee \{v \in \mathcal{M}_\tau; v \leq x \wedge (y \vee z) = (x \wedge y) \vee z\}$$

and $x \wedge_\tau y = \bigvee \{w \in \mathcal{M}_\tau; w \leq x \wedge y\}$, hence

$$(x \wedge_\tau y) \vee_\tau z = \left(\bigvee \{w \in \mathcal{M}_\tau; w \leq x \wedge y\}\right) \vee z \leq x \wedge_\tau (y \vee_\tau z).$$

Conversely, let $v \in \mathcal{M}_\tau$ such that $v \leq (x \wedge y) \vee z$, where we may assume that $v \in \mathcal{L}_\tau$ and $v \not\leq z$. Hence we have $v \vee z \in \mathcal{L}([z, (x \wedge y) \vee z])$, where $z \leq v_* \vee z \lessdot v \vee z$ and $t_{\mathcal{M}}([v_* \vee z, v \vee z]) = \tau$. Since we have $[z, (x \wedge y) \vee z] \cong [y \wedge z, x \wedge y]$, there is $w \in \mathcal{L}_\tau$ such that $w \leq x \wedge y$ and $w \vee z = v \vee z$. Hence we have $v \leq (x \wedge_\tau y) \vee z$, thus $x \wedge_\tau (y \vee_\tau z) \leq (x \wedge_\tau y) \vee_\tau z$, showing that $\mathcal{M}_\tau$ is modular.

For the isomorphisms, replacing $\wedge$ by $\wedge_\tau$ and $\vee$ by $\vee_\tau$, as well as $\mathcal{L}$ by $\mathcal{L}_\tau$ and and completeness by τ-completeness, the proof of (2.5) holds literally. ♯

Note that by (2.5) we have $\operatorname{im}(\beta_\tau) \subseteq \{\mathcal{X} \cap \mathcal{L}_\tau; \mathcal{X} \in \mathcal{M}(\mathcal{L})\} \subseteq \mathcal{M}(\mathcal{L}_\tau)$, and hence by (4.10) we conclude $\mathcal{M}(\mathcal{L}_\tau) = \{\mathcal{X} \cap \mathcal{L}_\tau; \mathcal{X} \in \mathcal{M}(\mathcal{L})\}$.

5 Submodule lattices

We finally consider submodule lattices and the algebraic notions which have motivated the machinery developed in the present work. We assume the reader to be familiar with the basic notions of the theory of rings and their modules. As a general reference see e. g. [5, Ch.I]. The basic observation relating the set $\mathcal{L}_2$ and algebraic types is (5.2), which immediately leads to the statements for algebraic types already promised in (4.7). Subsequently, we comment on the algebraic counterparts of the sets $\mathcal{M}_\tau$ and on the notion of blocks. While in general lattice theoretic blocks are slightly finer than their algebraic counterparts, it turns out that under a natural technical condition these notions coincide.

(5.1) Algebraic types. Let A be a right Artinian ring with non-zero identity element. By an A-module we will mean throughout this section a unital right A-module. Let M be a finitely generated A-module, and let $\mathcal{M}(M)$ be the lattice of A-submodules of M. Recall that $\mathcal{M}(M)$ is a modular lattice, whose join and meet operations are given by sum and intersection of submodules, respectively, being denoted by $+$ and $\cap$. Moreover, $\mathcal{M}(M)$ has the least element $\{0\}$ and the greatest element M, and all chains in $\mathcal{M}(M)$ are finite. Hence $\mathcal{M}(M)$ satisfies the assumptions of (1.1). We keep the notation of (1.1), but also indicate the dependence on M, e. g. for the join-irreducible elements we write $\mathcal{L}(M)$.

Let $\mathcal{S}_A$ be the set of simple A-modules up to isomorphism; recall that $\mathcal{S}_A$ is a finite set. For $Y \lessdot Z \leq M$ let the **algebraic type** $s_A(Y,Z) \in \mathcal{S}_A$ be the isomorphism type of the simple A-module Z/Y. This defines the algebraic type map, where $\mathrm{rad}_A(\cdot)$ denotes the Jacobson radical of an A-module,

$$s_A\colon \mathcal{L}(M) \to \mathcal{S}_A\colon X \mapsto s_A(X_*, X) = s_A(\mathrm{rad}_A(X), X).$$

By (5.2) below, the type map $t_M\colon \mathcal{L}(M) \to \mathcal{T}(M)$ is constant on the fibres of s_A. Hence there is a map $\sigma_M\colon \mathcal{T}(M) \to \mathcal{S}_A$ such that $s_A = \sigma_M \circ t_M$. Thus by (4.3) for $Y \lessdot Z \leq M$ we have $s_A(Y,Z) = \sigma_M \circ t_M([Y,Z])$.

The algebraic type map can be extended to a map $s_A\colon \mathcal{L}(M) \dot\cup \mathcal{L}_2(M) \to \mathcal{S}_A$, by letting $s_A(Z) := s_A(X) \in \mathcal{S}_A$, for all $Z = X+Y \in \mathcal{L}_2(M)$, where $X, Y \in \mathcal{L}(M)$. Moreover, the algebraic type map can be extended to the set of dotted-lines by letting $s_A(\mathcal{D}) := s_A(Z)$, where $\mathcal{D} \subseteq \mathcal{L}(M)$ is a dotted-line for $Z \in \mathcal{L}_2(M)$.

Note that in general σ_M is not injective: If e. g. $\mathcal{M}(M)$ is distributive, then $t_M\colon \mathcal{L}(M) \to \mathcal{T}(M)$ is a bijection, while still different elements of $\mathcal{L}(M)$ might have the same algebraic type. The minimal possible example of this situation is as follows: The 2-dimensional algebra $A \subseteq \mathbb{F}_2^{2\times 2}$ over the field $\mathbb{F}_2$ generated by

$$\begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix} \in \mathbb{F}_2^{2\times 2}$$

has a unique simple module, up to isomorphism, hence we have $|\mathcal{S}_A| = 1$. The natural A-module $M = \mathbb{F}_2^2$ is uniserial, i. e. $\mathcal{M}(M)$ has a unique maximal chain. Thus $\mathcal{M}(M)$ is distributive, and we have $|\mathcal{L}(M)| = 2$. Hence the elements of $\mathcal{L}(M)$ have different types, but the same algebraic type.

(5.2) Theorem. Let $X, Y \in \mathcal{L}(M)$ such that $X \not\leq Y \not\leq X$, and let $Z := X + Y \in \mathcal{M}(M)$. Then we have $Z \in \mathcal{L}_2(M)$ if and only if $s_A(X) = s_A(Y)$.

In this case we have $\mathcal{I}_Z = E_S \dot{\cup} \{\infty\}$, where $E_S := \mathrm{End}_A(S)$ denotes the division ring of A-endomorphisms of S, and $\infty \in \mathcal{I}_Z$ just is an additional element disjoint from E_S.

Proof. If $Z \in \mathcal{L}_2(M)$, then let $Z_* \lessdot Z' \lessdot Z$ such that $X + Z_* \neq Z' \neq Y + Z_*$. Hence we have $s_A(X) \cong (X + Z_*)/Z_* \cong Z'/Z_* \cong (Y + Z_*)/Z_* \cong s_A(Y)$ as A-modules. If conversely $S := s_A(X) = s_A(Y) \in \mathcal{S}_A$, then we have $Z/Z_* \cong S \oplus S$ as A-modules, and the submodules $0 \lessdot T \lessdot S \oplus S$ are described as follows:

Let $\pi_1, \pi_2 \colon S \oplus S \to S$ be the natural A-module projections onto the first and second direct summand, respectively. If $T\pi_1 = \{0\}$, then we have $T = T_\infty := \{0\} \oplus S$. Hence let $T\pi_1 \neq \{0\}$, then we have $T\pi_1 = S$. Assume that for some $v \in S$ we have both $[v, w] \in T$ and $[v, w'] \in T$ for some $w, w' \in S$ such that $w \neq w'$. Then $[0, w - w'] \in T$ as well, and thus $\{0\} \oplus S \leq T$, a contradiction. Hence there is a map $\pi \colon S \to S$ such that $T = T_\pi := \{[v, v\pi] \in S \oplus S; v \in S\}$, where it is immediate that $\pi \in E_S$. Conversely, for $\pi \in E_S$ we have $0 \lessdot T_\pi \lessdot S \oplus S$. Hence we have a bijection $E_S \dot{\cup} \{\infty\} \to \{0 \lessdot T \lessdot S \oplus S\} \colon \pi \mapsto T_\pi$. ♯

(5.3) Corollary. a) Let $X \in \mathcal{M}_r(M)$ such that $t_M([X_*, X]) \sim [\tau^r]$ for some $\tau \in \mathcal{T}(M)$. Then we have $X \in \mathcal{L}_r(M)$.
b) Let $Z, Z' \in \mathcal{L}_2(M)$ such that $t_M(Z) = t_M(Z') \in \mathcal{T}(M)$. Then there is a bijection $\mathcal{I}_Z \to \mathcal{I}_{Z'}$.

Proof. a) Let $X = \sum_{i=1}^r X_i$ be irredundant, where $X_i \in \mathcal{L}(M)$ for all $i \in \{1, \ldots, r\}$. Hence we have $s_A(X_*, X_i + X_*) = s_A(X_*, X_j + X_*)$, and thus $[X_*, X_i + X_j + X_*]$ is indecomposable for all $i \neq j$. Hence by (3.4) $[X_*, X]$ is indecomposable as well, and thus we have $X \in \mathcal{L}_r(M)$.
b) We have $S := s_A(Z) = s_A(Z') \in \mathcal{S}_A$, hence $\mathcal{I}_Z = E_S \dot{\cup} \{\infty\} = \mathcal{I}_{Z'}$. ♯

(5.4) The set $\mathcal{M}_S(M)$. Let $S \in \mathcal{S}_A$ and let $e_S \in A$ be a primitive idempotent such that for the projective indecomposable A-module $e_S A \leq A$ we have $e_S A/(e_S A)_* = e_S A/\mathrm{rad}_A(e_S A) \cong S$. The set $e_S A e_S := \{e_S a e_S \in A; a \in A\} \subseteq A$ again is an Artinian ring. Let $Me_S \subseteq M$ denote the image of the action of $e_S \in A$ on M, then Me_S is a finitely generated $e_S A e_S$-module.

Note that more formally the process of mapping a finitely generated A-module M to the $e_S A e_S$-module Me_S is described by a Schur functor; for more details on Schur functors and how they are used in the computational treatment of modules, see [6], [7].

Let

$$\mathcal{M}_S(M) := \coprod_{\tau \in \sigma_M^{-1}(S)} \mathcal{M}_\tau(M) = \{X \in \mathcal{M}(M); s_M(Y, X) = S \text{ for all } Y \lessdot X\}.$$

Then by [6, Thm.2.3] the following maps are a pair of mutually inverse isomorphisms of lattices:

$$\kappa\colon \mathcal{M}_S(M) \to \mathcal{M}(Me_S)\colon X \mapsto Xe_S \text{ and } \kappa^{-1}\colon \mathcal{M}(Me_S) \to \mathcal{M}_S(M)\colon Y \mapsto Y \cdot A,$$

where for an $e_S A e_S$-submodule $Y \leq Me_S$ we let $Y \cdot A := \{ma \in M; m \in Y, a \in A\} \leq M$. Note that in [6] the ring A is assumed to be a finite-dimensional algebra over a field, but the proof given there holds literally for the general case of Artinian rings considered here.

(5.5) Algebraic blocks. Let $\epsilon_1, \ldots, \epsilon_d \in A$ be the centrally primitive idempotents in A, and let $A_i := \epsilon_i A \epsilon_i \trianglelefteq A$ be the associated block ideals of A. Hence we have $A \cong \bigoplus_{i=1}^d A_i$ as rings. Moreover, letting $M_i := M\epsilon_i \leq M$ be the algebraic block components of M, for all $i \in \{1, \ldots, d\}$, we have $M = \bigoplus_{i=1}^d M_i$ as A-modules, where $M_i \cdot A_j = \{0\}$ for all $i, j \in \{1, \ldots, d\}$ such that $i \neq j$. The computation of algebraic block components is of practical importance, see [6].

Since for all $X \leq M$ we also have $X = \bigoplus_{i=1}^d X\epsilon_i = \bigoplus_{i=1}^d (X \cap M_i)$ as A-modules, we have $M_i \in \mathcal{Z}(\mathcal{M}(M))$, for all $i \in \{1, \ldots, d\}$, and $\mathcal{M}(M) \cong \prod_{i=1}^d \mathcal{M}(M_i)$ as lattices. Thus by (3.3) we have $M_i = \sum_{j; Z_j \leq M_i} Z_j$, where $\{Z_1, \ldots, Z_r\} \subseteq \mathcal{M}(M)$ is the set of minimal elements of $\mathcal{Z}(\mathcal{M}(M)) \setminus \{\{0\}\}$.

Note that in general the algebraic block components $M_i \in \mathcal{M}(M)$ are not indecomposable lattices: Let e. g. $\mathcal{S}_3 := \langle (1,2), (2,3) \rangle$ be the symmetric group on 3 letters, being generated by the adjacent transpositions, and let $A := \mathbb{F}_3 \mathcal{S}_3$ be the associated group algebra over the field $\mathbb{F}_3$. Then we have $\mathcal{S}_A = \{T, S\}$, where T and S are 1-dimensional A-modules given by

$$T\colon (1,2), (2,3) \to \begin{bmatrix} 1 \end{bmatrix} \in \mathbb{F}_3^{1\times 1} \quad \text{and} \quad S\colon (1,2), (2,3) \to \begin{bmatrix} -1 \end{bmatrix} \in \mathbb{F}_3^{1\times 1}.$$

Let $M := T \oplus S$ as A-modules. Then $\mathcal{M}(M) \cong [\{0\}, T] \times [\{0\}, S]$, thus $\mathcal{M}(M)$ is a decomposable lattice. But there is a 2-dimensional A-module R given by

$$R\colon \quad (1,2) \to \begin{bmatrix} 1 & . \\ . & -1 \end{bmatrix} \in \mathbb{F}_3^{2\times 2}, \quad (2,3) \to \begin{bmatrix} 1 & 1 \\ . & -1 \end{bmatrix} \in \mathbb{F}_3^{2\times 2}.$$

Hence R is a uniserial A-module, having constituents T and S. Hence T and S belong to the same block ideal of A, and thus M consists of a single algebraic block component. In view of (5.6) below, note that for the kernel of the associated representation we have $\ker(A \to \mathrm{End}_{\mathbb{F}_3}(M)) = \mathrm{rad}_A(A) \neq \{0\}$.

(5.6) Proposition. Let M be a faithful A-module, i. e. for the kernel of the associated representation we have $\ker(A \to \mathrm{End}_{\mathbb{Z}}(M)) = \{0\} \trianglelefteq A$, and let $M = \bigoplus_{i=1}^d M_i$ be the decomposition of the A-module M into its algebraic block components. Then $\mathcal{M}(M) \cong \prod_{i=1}^d [\{0\}, M_i]$ is the decomposition of $\mathcal{M}(M)$ into nontrivial indecomposable intervals.

Proof. For $S \in \mathcal{S}_A$ let $e_S \in A$ be as in (5.4). Since M is a faithful A-module, we have $\mathrm{Hom}_A(e_S A, M) \cong Me_S \neq \{0\}$, i. e. S is a constituent of M. Hence the algebraic type map $s_A \colon \mathcal{L}(M) \to \mathcal{S}_A$ is surjective.

We may assume that $d = 1$, i. e. A is a block algebra. Hence we have to show that $\mathcal{M}(M)$ is indecomposable: Assume that there are A-submodules $Z, Z' \leq M$ such that $Z \neq \{0\} \neq Z'$ and $\mathcal{M}(M) \cong [\{0\}, Z] \times [\{0\}, Z']$ as lattices. Hence for $X, X' \in \mathcal{L}(M)$ such that $X \leq Z$ and $X' \leq Z'$ we have $X + X' \notin \mathcal{L}_2(M)$, and thus $s_A(X) \neq s_A(X')$. Hence letting $\mathcal{S} := \{S \in \mathcal{S}_A; Ze_S \neq \{0\}\}$ and $\mathcal{S}' := \{S \in \mathcal{S}_A; Z'e_S \neq \{0\}\}$ we have $\mathcal{S} \cap \mathcal{S}' = \emptyset$ and $\mathcal{S} \neq \emptyset \neq \mathcal{S}'$. Thus from the surjectivity of $s_A \colon \mathcal{L}(M) \to \mathcal{S}_A$ we conclude that $\mathcal{S}_A = \mathcal{S} \,\dot{\cup}\, \mathcal{S}'$.

For $S \in \mathcal{S}$ and $S' \in \mathcal{S}'$ we have $M \cdot e_S A e_{S'} = (Z \oplus Z') \cdot e_S A e_{S'} \leq Z e_{S'} = \{0\}$. Since M is a faithful A-module, we from this conclude that $\mathrm{Hom}_A(e_{S'} A, e_S A) \cong e_S A e_{S'} = \{0\}$. Thus for all $S \in \mathcal{S}$ all constituents of the projective indecomposable A-module $e_S A$ are in $\mathcal{S}$, while for all $S \in \mathcal{S}'$ all constituents of $e_S A$ are in $\mathcal{S}'$. Hence by [5, Thm.I.13.11] the ring A is not a block, a contradiction. $\sharp$

Note that in a computational setting M typically is just given by a set of representing matrices. Hence if we let A be the algebra generated by the given matrices, then M indeed is a faithful A-module.

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