THE 2-MODULAR DECOMPOSITION MATRICES OF THE NON-PRINCIPAL BLOCKS OF MAXIMAL DEFECT OF THE TRIPLE COVER OF THE SPORADIC SIMPLE MCLAUGHLIN GROUP

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Abstract. The 2-modular decomposition numbers of the faithful irreducible ordinary characters of $3.McL$ are determined. The results are obtained by using the computer algebra packages Moc and Meat-Axe, and by applying condensation methods.

Introduction

The 2-modular Brauer characters of the simple group McL have been determined by J. Thackray, see [11]. In the sequel we use the decomposition matrix for the principal block as is given in section 3. The corresponding Brauer characters can be found in [7] and in the library of the program system Gap, see [10], which also contains the Brauer character tables of sporadic simple and related groups as far as they are known. The ordinary character table of $3.McL$ can be found in [1] and also in the library of the program system Gap, where the ordinary character tables of all sporadic simple and related groups can be accessed.

There are four non-trivial 2-blocks of $3.McL$ consisting of faithful characters. We follow the numbering made by the program system Gap. The third and fourth block are of cyclic defect and are already discussed in [2]. The first and the second block are of maximal defect and are complex conjugate to each other. Therefore, it is sufficient to consider only the first block.

In the first section we apply character theoretic methods to find approximations of the decomposition matrix of the first block. The results are written down in terms of bases for the free abelian group of class functions generated by the projective indecomposable characters lying in the first block. The underlying computations have been made using the program system Moc, see [4]. Even though the proofs have been found using a computer, due to the design of the system Moc, we are able to give explicit proofs which can be checked by hand.

In the second section we give the proofs which could not be obtained by purely character theoretic methods. Here we use the program system Meat-Axe, see [8]. Main tools in this section are fixed point condensation and condensation with primitive idempotents. The latter is implicitly used to determine submodule structures. As a reference see [9] and [6].

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Notation: A character is denoted by its degree, a lower case letter and, if it lies in a non-trivial block, by a superscript indicating the block it belongs to according to the numbering mentioned above. The principal block is abbreviated by pb. A projective indecomposable character is denoted by an upper case Φ indexed by a number. Most of the time we will be interested only in the component of a projective character lying in the first block. We denote the projection of a character onto its first block component by ϵ^1.

Conjugacy classes and sums of roots of unity are denoted in the same way as in [1]. Especially, a class of $3.McL$ is denoted by its image under the natural homomorphism onto McL and by an additional superscript ranging from 1 to 3 if necessary. For example, the classes $1A^1$, $1A^2$ and $1A^3$ constitute the center of $3.McL$.

1. The Moc-Part

1.1. The following set ψ^1 of ordinary characters is a basis for the free abelian group of class functions on the 2-regular classes generated by the irreducible Brauer characters in the first block.

$$\begin{aligned}
\psi^1_1 &:= 126a^1, \\
\psi^1_2 &:= 126b^1, \\
\psi^1_3 &:= 1980^1, \\
\psi^1_4 &:= 792^1, \\
\psi^1_5 &:= 5103^1, \\
\psi^1_6 &:= 2376a^1, \\
\psi^1_7 &:= 2376b^1, \\
\psi^1_8 &:= 2520a^1.
\end{aligned}$$

The irreducible ordinary characters in the first block decompose into ψ^1 as follows.

	ψ^1_1	ψ^1_2	ψ^1_3	ψ^1_4	ψ^1_5	ψ^1_6	ψ^1_7	ψ^1_8
$126a^1$	1	.	.	.	.	.	.	.
$126b^1$	.	1	.	.	.	.	.	.
792^1	.	.	.	1	.	.	.	.
1980^1	.	.	1	.	.	.	.	.
$2376a^1$	.	.	.	.	.	1	.	.
$2376b^1$	.	.	.	.	.	.	1	.
$2520a^1$	.	.	.	.	.	.	.	1
$2520b^1$	−2	−2	.	1	.	1	1	−1
2772^1	.	.	1	1	.	.	.	.
4752^1	.	.	2	1	.	.	.	.
5103^1	.	.	.	.	1	.	.	.
7875^1	.	.	1	1	1	.	.	.
$8019a^1$	−1	−1	.	1	1	.	1	.
$8019b^1$	−1	−1	.	1	1	1	.	.
$10395'^1$	−1	−1	.	1	1	1	1	.
$10395a^1$	.	.	1	1	1	.	.	1
$10395b^1$	−2	−2	1	2	1	1	1	−1
12375^1	−1	−1	1	1	1	1	1	.

1.2. We obtain projective characters by tensoring defect zero characters and projective indecomposable characters lying in blocks of cyclic defect with ordinary characters and irreducible Brauer characters already known. Let

$$\begin{aligned}
\Omega_1 &:= \epsilon^1(792^1 \otimes 896a),\\
\Omega_2 &:= \epsilon^1(22^{pb} \otimes (6336a^3 + 6336b^3)),\\
\Omega_3 &:= \epsilon^1(22^{pb} \otimes 8064a),\\
\Omega_4 &:= \epsilon^1(126b^1 \otimes 896a).
\end{aligned}$$

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
$126a^1$	.	.	.	.	.
$126b^1$	.	.	.	.	.
792^1	2	2	.	.	2
1980^1	2	.	.	.	1
$2376a^1$	.	.	1	1	1
$2376b^1$	.	.	1	1	1
$2520a^1$	1	1	1	1	2
$2520b^1$	1	1	1	1	2
2772^1	4	2	.	.	3
4752^1	6	2	.	.	4
5103^1	4	1	1	.	3
7875^1	8	3	1	.	6
$8019a^1$	6	3	2	1	6
$8019b^1$	6	3	2	1	6
$10395'^1$	6	3	3	2	7
$10395a^1$	9	4	2	1	8
$10395b^1$	9	4	2	1	8
12375^1	8	3	3	2	8

Since all entries in the sum $\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4$ are even, we obtain the ordinary character Ω_5 by dividing all the entries by two. Since Ω_5 vanishes on 2-singular classes, it is a generalized projective character which can be written as a nonnegative rational linear combination in the projective indecomposable characters of the first block. Since the scalar products with all ordinary and hence with all Brauer characters of this block are integral, it is a projective character.

1.3. Now we are able to give a first basis Ψ^1 for the free abelian group of class functions generated by the projective indecomposable characters lying in the first block.

$$\begin{aligned}
\Psi_1^1 &:= \epsilon^1(2376a^1 \otimes (3520a + 3520b)),\\
\Psi_2^1 &:= \epsilon^1(126b^1 \otimes 896b),\\
\Psi_3^1 &:= \epsilon^1(2520b^1 \otimes 896b),\\
\Psi_4^1 &:= \epsilon^1(126a^1 \otimes 896a),\\
\Psi_5^1 &:= \Omega_5,\\
\Psi_6^1 &:= \Omega_2,\\
\Psi_7^1 &:= \Omega_3,\\
\Psi_8^1 &:= \Omega_4.
\end{aligned}$$

	Ψ_1^1	Ψ_2^1	Ψ_3^1	Ψ_4^1	Ψ_5^1	Ψ_6^1	Ψ_7^1	Ψ_8^1
$126a^1$	1	1	.	.	.	.	.	.
$126b^1$	1	.	1	1	.	.	.	.
792^1	21	.	1	.	2	2	.	.
1980^1	38	.	5	.	1	.	.	.
$2376a^1$	35	1	7	1	1	.	1	1
$2376b^1$	36	1	7	1	1	.	1	1
$2520a^1$	44	.	6	.	2	1	1	1
$2520b^1$	44	.	7	.	2	1	1	1
2772^1	59	.	6	.	3	2	.	.
4752^1	97	.	11	.	4	2	.	.
5103^1	95	1	13	1	3	1	1	.
7875^1	154	1	19	1	6	3	1	.
$8019a^1$	150	1	20	1	6	3	2	1
$8019b^1$	149	1	20	1	6	3	2	1
$10395'^1$	185	2	27	2	7	3	3	2
$10395a^1$	198	1	25	1	8	4	2	1
$10395b^1$	198	1	26	1	8	4	2	1
12375^1	223	2	32	2	8	3	3	2

This is indeed a basis, as the determinant of the following scalar product matrix (ψ^1, Ψ^1) equals 1.

$$\begin{bmatrix} 1 & 1 & . & . & . & . & . & . \\ 1 & . & 1 & 1 & . & . & . & . \\ 38 & . & 5 & . & 1 & . & . & . \\ 21 & . & 1 & . & 2 & 2 & . & . \\ 95 & 1 & 13 & 1 & 3 & 1 & 1 & . \\ 35 & 1 & 7 & 1 & 1 & . & 1 & 1 \\ 36 & 1 & 7 & 1 & 1 & . & 1 & 1 \\ 44 & . & 6 & . & 2 & 1 & 1 & 1 \end{bmatrix}$$

1.4. Now we start to analyze the given situation to obtain a better approximation of the decomposition matrix. Let

$$\Omega_6 := \epsilon^1(2520a^1 \otimes 896a),$$
$$\Omega_7 := \epsilon^1(2376b^1 \otimes (3520a + 3520b)).$$

Then Ω_1, Ω_6 and Ω_7 decompose into Ψ^1 as follows.

	Ψ_1^1	Ψ_2^1	Ψ_3^1	Ψ_4^1	Ψ_5^1	Ψ_6^1	Ψ_7^1	Ψ_8^1
Ω_1	.	.	.	.	2	−1	−1	−1
Ω_6	.	1	−1	1	10	−9	3	−1
Ω_7	−1	2	.	2	76	−55	13	−22

1.5. Assume the ordinary character $126a^1$ were reducible. Using the scalar product matrix given above, we see that there is only one possibility to decompose $126a^1$. But one of the summands would have a negative scalar product with Ω_7. So $126a^1$ is irreducible. Since the composition of the outer automorphism of $3.McL$ and complex conjugation transforms the character $126a^1$ into $126b^1$, the latter is also irreducible. If 1980^1 were reducible, one of the summands would have a negative scalar product with Ω_6 or Ω_7. Hence 1980^1 is irreducible, too. Now we use the decomposition of the ordinary character $2520b^1$ in the basis ψ^1 given above to obtain the following new Brauer characters, since $126a^1$ and $126b^1$ cannot be constituents of $\psi_4^1 = 792^1$.

$$\begin{aligned} 2124a^1 &:= 2376a^1 - 126a^1 - 126b^1, \\ 2124b^1 &:= 2376b^1 - 126a^1 - 126b^1. \end{aligned}$$

We get the following scalar products $(\{2124a^1, 2124b^1\}, \Psi^1)$.

$$\begin{bmatrix} 33 & . & 6 & . & 1 & . & 1 & 1 \\ 34 & . & 6 & . & 1 & . & 1 & 1 \end{bmatrix}$$

If $2124a^1$ or $2124b^1$ were reducible, one of the summands would have a negative scalar product with Ω_1, Ω_6 or Ω_7. Altogether, we have now determined five of the eight irreducible Brauer characters in the first block.

1.6. The action of the Frobenius automorphism of a finite field of characteristic 2 on 2-modular Brauer characters is given by taking every complex root of unity to its second power. We denote the composition of the Frobenius map and complex conjugation by $\aleph$. It is easily seen that $\aleph$ fixes both blocks of maximal defect. The pairs of ordinary characters $2376a, b^1$, $2520a, b^1$, $8019a, b^1$ and $10395a, b^1$ are interchanged by $\aleph$, the other ordinary characters in the first block are fixed.

1.7. As we have seen above, $2376a^1$ and $2376b^1$ have only the constituents $126a^1$ and $126b^1$ in common. Hence Ψ_8^1 decomposes as a sum of two projective characters which are conjugate under $\aleph$. Using character values on the 2-singular classes $14A^1$ and $30A^1$, we see that there are exactly two ways to decompose Ψ_8^1 as such a sum, let us say the a-branch and the b-branch.

$$\Psi_8^1 = \Phi_{6a} + \Phi_{7a} = \Phi_{6b} + \Phi_{7b}.$$

The emerging projectives $\Phi_{6a,b}$ and $\Phi_{7a,b}$ are given as follows.

	Ψ_8^1	$\Phi_{6,a}$	$\Phi_{7,a}$	$\Phi_{6,b}$	$\Phi_{7,b}$
$126a^1$	.	.	.	.	.
$126b^1$	.	.	.	.	.
792^1	.	.	.	.	.
1980^1	.	.	.	.	.
$2376a^1$	1	1	.	1	.
$2376b^1$	1	.	1	.	1
$2520a^1$	1	1	.	.	1
$2520b^1$	1	.	1	1	.
2772^1	.	.	.	.	.
4752^1	.	.	.	.	.
5103^1	.	.	.	.	.
7875^1	.	.	.	.	.
$8019a^1$	1	.	1	.	1
$8019b^1$	1	1	.	1	.
$10395'^1$	2	1	1	1	1
$10395a^1$	1	1	.	.	1
$10395b^1$	1	.	1	1	.
12375^1	2	1	1	1	1

Using again the scalar product matrix (ψ^1, Ψ^1), we see that Ψ_8^1 is a sum of at most three projective indecomposable characters, hence the new projective characters are indecomposable in either case.

1.8. If we use the argument on the constituents of $2376a^1$ and $2376b^1$ again, we obtain the following new projective characters. Their decomposition into ordinary characters is given in subsection 1.11.

$$\begin{array}{rcl} \Phi_8 & := & \Psi_7^1 - \Psi_8^1, \\ \Omega_8 & := & \Psi_5^1 - \Psi_8^1. \end{array}$$

Φ_8 is indecomposable, as again a glance at the scalar product matrix (ψ^1, Ψ^1) shows.

1.9. Let us now assume that the a-branch is correct. As $\Phi_{6,a}$ and $\Phi_{7,a}$ imply, $2124a^1$ is a modular constituent of $2520a^1$ and $2124b^1$ is one of $2520b^1$. Hence we get the following new Brauer characters.

$$\begin{array}{rcl} 396a^{1a} & := & 2520a^1 - 2124a^1, \\ 396b^{1a} & := & 2520b^1 - 2124b^1. \end{array}$$

Note that the following relation holds on 2-regular classes.

$$792^1 \quad = \quad 396a^{1a} + 396b^{1a}.$$

Hence 792^1 has at least two different modular constituents. But up to now, we have recognized five irreducible Brauer characters. None of them is a constituent of 792^1, as was remarked earlier. Since Φ_8 is indecomposable, a further Brauer character is contained in the ordinary character 5103^1. So there are exactly two different irreducible Brauer characters ϕ_1 and ϕ_2 which are constituents of 792^1. Hence we have

$$396a^{1a} = x_a\phi_1 + y_a\phi_2 \text{ and } 396b^{1a} = x_b\phi_1 + y_b\phi_2.$$

Since $396a^1$ and $396b^1$ are conjugate under $\aleph$, we have $\phi_1^{\aleph} = \phi_2$ and $x_a = y_b$, $x_b = y_a$, hence

$$792^1 = (x_a + x_b) \cdot (\phi_1 + \phi_2).$$

But the row of the scalar product matrix corresponding to $792^1 = \psi_4^1$ shows that 792^1 contains at least one constituent with multiplicity 1. Therefore, $396a^1$ and $396b^1$ are irreducible.

1.10. Let Φ_3 and Φ_4 denote the projective indecomposable characters corresponding to $396a^1$ and $396b^1$. As we will see later on, we obtain the same projective indecomposable characters in case b, so we write $\Phi_{3,4}$ without a superscript. Φ_3 and Φ_4 are conjugate under $\aleph$ and are summands of Ψ_6^1. The projective indecomposable characters corresponding to $126a^1$, $126b^1$ and 1980^1 and $\Phi_{6,a}$, $\Phi_{7,a}$ are not summands of Ψ_6^1. Since the scalar product of Ψ_6^1 and 5103^1 equals 1, Ψ_6^1 decomposes as

$$\Psi_6^1 \quad = \quad \Phi_3 + \Phi_4 + \Phi_8.$$

Hence we get Φ_3 and Φ_4 using character values on the 2-singular classes $14A^1$ and $30A^1$ as they are given in subsection 1.11.

1.11. Since 792^1 decomposes into irreducible Brauer characters as is given above, we obtain the following projective character.

$$\Omega_9 \quad := \quad \Omega_8 - \Phi_3 - \Phi_4.$$

Altogether, the conclusions made above have led us to the following projective characters.

	Ψ_5^1	Ψ_6^1	Ψ_7^1	Ψ_8^1	Φ_8	Φ_3	Φ_4	Ω_8	Ω_9
$126a^1$	.	.	.	.	.	.	.	.	.
$126b^1$	.	.	.	.	.	.	.	.	.
792^1	2	2	.	.	.	1	1	2	.
1980^1	1	.	.	.	.	.	.	1	1
$2376a^1$	1	.	1	1	.	.	.	.	.
$2376b^1$	1	.	1	1	.	.	.	.	.
$2520a^1$	2	1	1	1	.	1	.	1	.
$2520b^1$	2	1	1	1	.	.	1	1	.
2772^1	3	2	.	.	.	1	1	3	1
4752^1	4	2	.	.	.	1	1	4	2
5103^1	3	1	1	.	1	.	.	3	3
7875^1	6	3	1	.	1	1	1	6	4
$8019a^1$	6	3	2	1	1	1	1	5	3
$8019b^1$	6	3	2	1	1	1	1	5	3
$10395'^1$	7	3	3	2	1	1	1	5	3
$10395a^1$	8	4	2	1	1	2	1	7	4
$10395b^1$	8	4	2	1	1	1	2	7	4
12375^1	8	3	3	2	1	1	1	6	4

1.12. Hence a second basis Ψ^2 is given as follows.

$$\begin{aligned}
\Psi_1^2 &:= \Psi_2^1, \\
\Psi_2^2 &:= \Psi_4^1, \\
\Psi_3^2 &:= \Phi_3, \\
\Psi_4^2 &:= \Phi_4, \\
\Psi_5^2 &:= \Omega_9, \\
\Psi_6^2 &:= \Phi_{6,a}, \\
\Psi_7^2 &:= \Phi_{7,a}, \\
\Psi_8^2 &:= \Phi_8.
\end{aligned}$$

	Ψ^2_1	Ψ^2_2	Ψ^2_3	Ψ^2_4	Ψ^2_5	Ψ^2_6	Ψ^2_7	Ψ^2_8
$126a^1$	1	.	.	.	.	.	.	.
$126b^1$	.	1	.	.	.	.	.	.
792^1	.	.	1	1	.	.	.	.
1980^1	.	.	.	.	1	.	.	.
$2376a^1$	1	1	.	.	.	1	.	.
$2376b^1$	1	1	.	.	.	.	1	.
$2520a^1$	.	.	1	.	.	1	.	.
$2520b^1$	.	.	.	1	.	.	1	.
2772^1	.	.	1	1	1	.	.	.
4752^1	.	.	1	1	2	.	.	.
5103^1	1	1	.	.	3	.	.	1
7875^1	1	1	1	1	4	.	.	1
$8019a^1$	1	1	1	1	3	.	1	1
$8019b^1$	1	1	1	1	3	1	.	1
$10395'^1$	2	2	1	1	3	1	1	1
$10395a^1$	1	1	2	1	4	1	.	1
$10395b^1$	1	1	1	2	4	.	1	1
12375^1	2	2	1	1	4	1	1	1

1.13. Let

$$\Omega_{10} \quad := \quad \epsilon^1(2520b^1 \otimes 896b).$$

Ω_{10} decomposes into Ψ^2 as follows.

	Ψ^2_1	Ψ^2_2	Ψ^2_3	Ψ^2_4	Ψ^2_5	Ψ^2_6	Ψ^2_7	Ψ^2_8
Ω_{10}	.	1	.	1	5	6	6	−3

Hence we obtain a new projective character Ω_{11} as

$$\Omega_{11} \quad := \quad \Psi^2_5 - \Phi_8,$$

and a third basis Ψ^3 is given as follows.

$$\begin{aligned}
\Psi^3_1 &:= \Psi^2_1, \\
\Psi^3_2 &:= \Psi^2_2, \\
\Psi^3_3 &:= \Phi_3, \\
\Psi^3_4 &:= \Phi_4, \\
\Psi^3_5 &:= \Omega_{11}, \\
\Psi^3_6 &:= \Phi_{6,a}, \\
\Psi^3_7 &:= \Phi_{7,a}, \\
\Psi^3_8 &:= \Phi_8.
\end{aligned}$$

	Ψ_1^3	Ψ_2^3	Ψ_3^3	Ψ_4^3	Ψ_5^3	Ψ_6^3	Ψ_7^3	Ψ_8^3
$126a^1$	1	.	.	.	.	.	.	.
$126b^1$	.	1	.	.	.	.	.	.
792^1	.	.	1	1	.	.	.	.
1980^1	.	.	.	.	1	.	.	.
$2376a^1$	1	1	.	.	.	1	.	.
$2376b^1$	1	1	.	.	.	.	1	.
$2520a^1$	.	.	1	.	.	1	.	.
$2520b^1$	.	.	.	1	.	.	1	.
2772^1	.	.	1	1	1	.	.	.
4752^1	.	.	1	1	2	.	.	.
5103^1	1	1	.	.	2	.	.	1
7875^1	1	1	1	1	3	.	.	1
$8019a^1$	1	1	1	1	2	.	1	1
$8019b^1$	1	1	1	1	2	1	.	1
$10395'^1$	2	2	1	1	2	1	1	1
$10395a^1$	1	1	2	1	3	1	.	1
$10395b^1$	1	1	1	2	3	.	1	1
12375^1	2	2	1	1	3	1	1	1

1.14. If we assume the b-branch to be correct, we obtain the following Brauer characters.

$$\begin{aligned} 396a^{1b} &:= 2520a^1 - 2124b^1, \\ 396b^{1b} &:= 2520b^1 - 2124a^1. \end{aligned}$$

All conclusions made for the a-branch can analogously be made in this case, so here the third basis is as follows.

	Ψ_1^3	Ψ_2^3	Ψ_3^3	Ψ_4^3	Ψ_5^3	Ψ_6^3	Ψ_7^3	Ψ_8^3
$126a^1$	1	.	.	.	.	.	.	.
$126b^1$	.	1	.	.	.	.	.	.
792^1	.	.	1	1	.	.	.	.
1980^1	.	.	.	.	1	.	.	.
$2376a^1$	1	1	.	.	.	1	.	.
$2376b^1$	1	1	.	.	.	.	1	.
$2520a^1$	.	.	1	.	.	.	1	.
$2520b^1$	.	.	.	1	.	1	.	.
2772^1	.	.	1	1	1	.	.	.
4752^1	.	.	1	1	2	.	.	.
5103^1	1	1	.	.	2	.	.	1
7875^1	1	1	1	1	3	.	.	1
$8019a^1$	1	1	1	1	2	.	1	1
$8019b^1$	1	1	1	1	2	1	.	1
$10395'^1$	2	2	1	1	2	1	1	1
$10395a^1$	1	1	2	1	3	.	1	1
$10395b^1$	1	1	1	2	3	1	.	1
12375^1	2	2	1	1	3	1	1	1

1.15. The remaining questions are, whether Ψ_8^3 is contained in Ψ_1^3 and Ψ_2^3 and whether and how often it is contained in Ψ_5^3. This is obviously equivalent to determining the constituents of 5103^1, which is the aim of the next section.

2. The Meat-Axe-Part

2.1. If F is an arbitrary field and H is a subgroup of a finite group G with order $|H|$ not divisible by the characteristic of F, let

$$e := |H|^{-1} \cdot \sum_{h \in H} h$$

denote the corresponding idempotent in the group algebra FG. For an FG-module V the $eFGe$-module Ve is called the condensed module with respect to H. If $\{a_1, a_2, \dots\}$ is a set of generators for G, then the subalgebra of $eFGe$ generated by the condensed elements $\{ea_1e, ea_2e, \dots\}$ is called the condensation algebra. It is not necessarily equal to $eFGe$. A constituent of Ve as a module for the condensation algebra is called genuine, if it is also a constituent of Ve as a module for $eFGe$. Furthermore, for $g \in G$ we have the following formula

$$\mathrm{Trace}_{Ve}(ege) = |H|^{-1} \cdot \sum_{h \in H} \mathrm{Trace}_V(gh).$$

Traces on an FG-module V can be computed using the corresponding Brauer character. Recall that the dimension of a condensed module can be calculated using scalar products, provided the Brauer character of the given module is known, since the condensed module is the set of vectors in V which are fixed by H.

2.2. First we construct transitive permutation representations of $3.McL$ on 275 and 66825 points. The representation on 66825 points is faithful and its submodule structure will be used to determine the last missing irreducible Brauer character and to compute traces on certain condensed elements to decide which of the branches a or b is correct. The representation on 275 points is a representation of McL, since the center acts trivially, hence we can identify an element of $3.McL$ with its homomorphic image in McL. We need this representation to make use of the program system Gap. Here we use a Schreier-Sims algorithm to examine certain group elements with respect to conjugay, for instance.

2.3. We obtain the representations mentioned above using the presentation of $3.McL$ and McL given in [1] and a Todd-Coxeter coset enumeration. $\{cef, dcfd\}$ generates a subgroup $U_4(3)$ of index 275 in McL. L. Soicher has given a set of elements which generate a subgroup $2 \cdot A_8$ of index 66825 in $3.McL$, see [3]. Using the Todd-Coxeter process again, we see that

$$A := abc \text{ and } B := def$$

are generators for $3.McL$ and for McL. Now let

$$\begin{aligned} C &:= (AB)^5 B, \\ D &:= (AB)^3 B, \\ E &:= (AB)^4 B, \\ F &:= (AB)^2 B (AB)^3 B, \\ Z &:= (ABA)^{-7}. \end{aligned}$$

C is an element of the $6A$ class of McL. D defines the $15A$ class of McL to be the class it lies in. E is an element of order 7 and F one of order 11. Z is a central element, of order three in $3.McL$ and trivial in McL. Thus the class $1A^2$ of $3.McL$ is defined to be the class Z lies in.

2.4. The next task is to construct a suitable condensation subgroup. We define

$$\begin{aligned} A' &:= (A(AB(ABB)^2(AB)^3B)^{-1}BAB(ABB)^2(AB)^3B)^2, \\ B'' &:= (AB(ABB)^2(AB)^3B)^{-4}((AB)^3B)^5(AB(ABB)^2(AB)^3B)^4Z, \\ B''' &:= ((A'B'')^5B'')^{-1}(A'B'')^2B''(A'B'')^5B'', \\ B' &:= B'''^{-1}A'B'''. \end{aligned}$$

Using the representation on 275 points, we see that $\{A', B'\}$ generates an extraspecial group 3^{1+2} in McL, that contains two elements of the $3A$ class and 24 elements of the $3B$ class. Let e denote the idempotent defined by 3^{1+2}.

2.5. Now we examine the condensed module corresponding to the representation of $3.McL$ on 66825 points. We have to do the computations over $GF(4)$, since this is the splitting field of $3.McL$ in characteristic 2. The permutation character is given by

$$1_{2\cdot A_8}\uparrow^{3.McL} \quad := \quad \chi_{pb} + \chi_1 + \chi_2,$$

where

$$\begin{aligned} \chi_{pb} &:= 1 + 252 + 1750 + 5103 + 5544 + 9625, \\ \chi_1 &:= 2772^1 + 5103^1 + 6336a^1 + 8064^1, \\ \chi_2 &:= 2772^2 + 5103^2 + 6336a^2 + 8064^2. \end{aligned}$$

Here χ_{pb} denotes the part of the permutation character that belongs to the principal block, whereas χ_1 and χ_2 denote the parts that belong to blocks for which the central element Z acts by scalar multiplication by the chosen primitive third root of unity $\omega \in GF(4)$ and by ω^2 respectively. Let

$$\epsilon^{pb} := 1 + Z + Z^2.$$

This is the centrally primitive idempotent in the group algebra of $\langle Z \rangle$ over $GF(4)$ which corresponds to the trivial representation of $\langle Z \rangle$. Hence the images of the action of ϵ^{pb} and $1 - \epsilon^{pb}$ on the permution module give rise to summands which correspond to χ_{pb} and $\chi_1 + \chi_2$ respectively. Therefore we obtain two summands U_{855} and U_{1662} of dimensions 855 and 1662 of the condensed module as a module for the condensation algebra generated by $\{eAe, eBe\}$ which correspond to χ_{pb} and $\chi_1 + \chi_2$ respectively.

2.6. We consider U_{1662}. Since eZe does not act trivially on U_{1662}, it follows that the chosen subgroup does not contain the center of $3.McL$, hence $\{A', B'\}$ generates an extraspecial group 3^{1+2} in $3.McL$. We find the following direct sum decomposition

$$U_{1662} \cong U^1_{295} \oplus U^1_{232} \oplus U^1_{304} \oplus U^2_{295} \oplus U^2_{232} \oplus U^2_{304},$$

where the summands correspond to χ_1 and χ_2 as is indicated by the superscripts. Again the summands are indexed by their dimensions. Since $U^{1,2}_{232}$ and $U^{1,2}_{304}$ are irreducible, whereas $U^{1,2}_{295}$ are not, the former ones belong to the ordinary characters of degrees 6336 and 8064 which are in blocks of defect 0 or 1, whereas the latter ones belong to the blocks of maximal defect we are interested in. Now it follows by considering scalar products and the dimensions of the condensed modules corresponding to the character of degrees 6336 and 8064 that 3^{1+2} contains two elements of the $3A^1$ class and 24 elements of the $3B$ class of $3.McL$.

2.7. U^1_{295} has the following constituents as a module for the condensation algebra

$16a$,
$16b$,
$74a$ with multiplicity 2,
115,

and the following socle series

$$74a, 16a \oplus 16b \oplus 115a, 74a.$$

According to a theorem of Zassenhaus, see [5], Theorem 17.3., U^1_{295} has a genuine submodule of dimension 106 that corresponds to the ordinary character 2772^1, so the unique submodule of U^1_{295} of this dimension is genuine. Analogously, using the ordinary character 5103^1, the unique submodule of U^1_{295} of dimension 189 is genuine. Hence their intersection, which equals the socle of U^1_{295}, is also genuine. Therefore the ordinary characters 2772^1 and 5103^1 have a constituent in common, this must be the irreducible Brauer character 1980^1. Furthermore, since 5103^1 is the sum of exactly two irreducible Brauer characters, 5103^1 decomposes as

$$5103^1 = 1980^1 + 3123^1.$$

Hence we obtain projective indecomposable characters as follows.

$$\begin{aligned} \Phi_1 &:= \Psi^3_1 - \Phi_8, \\ \Phi_2 &:= \Psi^3_2 - \Phi_8, \\ \Phi_5 &:= \Psi^3_5 - \Phi_8. \end{aligned}$$

The resulting decomposition matrix for the first block is given at the end of this section.

2.8. Our last aim is to show that the a-branch is correct. If we consider only the ordinary characters of McL, the classes $7A$, $7B$ and $15A$, $15B$ may be exchanged arbitrarily, this amounts to a renumbering of the ordinary characters. But since we assume the irreducible Brauer characters of McL to be as given in section 3, we have already made a choice concerning classes of elements of orders 7 and 15. If we exchange $7A$ and $7B$, we have also to exchange $15A$ and $15B$, if we do not want to alter the set of irreducible Brauer characters. So our task now is to determine the class of McL the element E lies in.

2.9. We consider U_{855}. This module corresponds to a representation of McL, since Z acts trivially on it. It has the following constituents as a module for the condensation algebra.

$1a$ with multiplicity 9,
$4a$ with multiplicity 6,
$14a$ with multiplicity 5,
$28a$ with multiplicity 4,
$28b$ with multiplicity 4,
$72a$,
$72b$,
$128a$ with multiplicity 3.

Since we already know the irreducible Brauer characters of McL, all of these constituents are readily recognized to be genuine. Using scalar products, we see that the constituents of dimension 72 correspond to the irreducible McL-modules of dimension 2124. Furthermore we have

$$\mathrm{Trace}_{72a}(eCe) = \mathrm{Trace}_{72b}(eCe) = 1.$$

2.10. Using the representation of McL on 275 points, we find the coset $C \cdot 3^{1+2}$ to contain the following elements.

4 elements of order 5,
1 element of order 6,
5 elements of order 7,
1 element of order 8,
2 elements of order 9,
3 elements of order 10,
7 elements of order 11,
1 element of order 12,
2 elements of order 14,
1 element of order 15.

The conjugacy class of the element of order 6 is determined by counting its fixed points, it is in the $6A$ class. The element of order 15 is in the $15A$ class, since it is conjugate to D. The elements of order 7 or 14 have the following class distribution

$$7A, 7A, 7A, 7B, 7B, 14B, 14B \text{ or } 7A, 7A, 7B, 7B, 7B, 14A, 14A.$$

The first case is true if and only if E lies in the $7B$ class. Now we compute $\mathrm{Trace}_{72a}(eCe)$ and $\mathrm{Trace}_{72b}(eCe)$ for both class distributions, using the trace formula given in section 2.1 and that the sum of roots of unity

$b7$ reduces to 0,
$b7^{**}$ reduces to 1,
$b15$ reduces to 1,
$b15^{**}$ reduces to 0,
$b11 + b11^{**}$ reduces to 1

modulo 2. In the first case, we find

$$\mathrm{Trace}_{72a}(eCe) = \mathrm{Trace}_{72b}(eCe) = 1,$$

in the second one

$$\mathrm{Trace}_{72a}(eCe) = \mathrm{Trace}_{72b}(eCe) = 0.$$

So the first case is correct, and E is in the $7B$ class.

2.11. Next we consider the constituents $16a$ and $16b$ of U^1_{295}, which correspond to the irreducible Brauer characters of degree 396 lying in the first block as is again seen by taking scalar products. Examining the traces of eFe on these consituents we get ω and ω^2.

2.12. The coset $F \cdot 3^{1+2}$ is found to contain the following elements when examined using the representation on 275 points.

1 element of the $6B$ class,
3 elements of order 7 with class distribution $7A, 7A, 7B$,
2 elements of order 8,
4 elements of order 11,

7 elements of order 12,
4 elements of order 14 with class distribution $14A, 14B, 14B, 14B$,
1 element of the $15A$ class,
5 elements of order 30 with class distribution $30A, 30B, 30B, 30B, 30B$.

The class of the element of order 6 is determined by counting fixed points, the classes of the elements of order 7 or 14 are determined by comparing these elements with the element E which is now known to be in the $7B$ class, and the element D is used to determine the classes of the elements of order 15 or 30. The element distribution of the coset $F \cdot 3^{1+2}$ in the representation on 66825 points is the same one, provided the classes cited above are substituted by their preimages under the natural homomorphism. It is not necessary to know the exact class distribution on the elements of order 6, 8, 11, 12, 24 or 33, since we are only interested in traces on the constituents of dimension 16 and the Brauer characters of degree 396 are constant on the relevant classes. Multiplying by Z gives the following class distribution on the elements of order 7, 14, 21 or 42.

$$7A^2, 7A^3, 7B^1, 14A^1, 14B^2, 14B^3, 14B^3.$$

2.13. Now we have to determine the classes of elements of order 15 and 30 in the coset considered above. Let

$$\begin{aligned}
Y_0 &:= A'^2B'A', \\
Y_1 &:= (FB'A')^{10}, \\
Y_2 &:= (FA'B'^2A'B')^{10}, \\
Y_3 &:= (FB'A'B'^2)^{10}, \\
Y_4 &:= (FB'^2A')^{10}, \\
Y_5 &:= (FA'^2B'^2A'B')^{10}, \\
Y_6 &:= (FB'A'^2)^5,
\end{aligned}$$

where $Y_1, \dots, Y_5$ are the tenth powers of the five elements of order 30 and and Y_6 the fifth power of the element of order 15 in the coset $F \cdot 3^{1+2}$, and Y_0 is one of the elements of the $3A^1$ class in the subgroup 3^{1+2}. Using the uniquely determined tenth and fifth powermaps of $3.McL$, we see that $Y_1, \dots, Y_5$ are elements lying in the $3A^{1,2,3}$ classes. We have the following class multiplication coefficients, which have been computed using a Dixon-Schneider algorithm and the ordinary character table of $3.McL$.

	$4A^1$	$10A^1$	$10A^2$	$10A^3$	$5A^2$	$5A^3$	$5B^2$	$56B^3$
$3A^1, 3A^1$	4	5	.	.	.	.	.	.
$3A^1, 3A^2$	.	.	5	.	.	.	10	.
$3A^1, 3A^3$	.	.	.	5	.	.	.	10
$3A^2, 3A^2$	.	.	.	5	.	.	.	10
$3A^2, 3A^3$	4	5	.	.	.	.	.	.
$3A^2, 3A^3$	.	.	5	.	.	.	10	.

Since Y_0Y_1, Y_1Y_2 and Y_2Y_4 have orders 10, 10 and 4 respectively and Y_0 is in the $3A^1$ class, it follows that the same is true for Y_1, Y_2 and Y_4. Since Y_1Y_5 and $Y_1Y_5Z^2$ have orders 30 and 10 respectively, Y_5 lies in the $3A^2$ class. Since Y_3Y_5 has order 10, Y_3 is in the $3A^3$ class. Finally, since Y_1Y_6 and Y_1Y_6Z have orders 15 and 5 respectively, Y_6 lies in the $3A^3$ class. Using powermaps again, we obtain the class distribution of the elements of order 15 and 30 in the coset $F \cdot 3^{1+2}$ as

$$15A^2, 30A^1, 30B^1, 30B^1, 30B^2, 30B^3.$$

2.14. Now we are able to compute the traces on the condensed modules that correspond to the irreducible Brauer characters of degree 396. The needed reductions of certain sums of roots of unity modulo 2 have been given in section 2.10. For the a-branch we indeed obtain ω and ω^2, but for the b-branch we get 0 and 1. So the a-branch is correct.

2.15. Finally, we can write down the decomposition matrix for the first block.

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
$126a^1$	1	.	.	.	.	.	.	.
$126b^1$	.	1	.	.	.	.	.	.
792^1	.	.	1	1	.	.	.	.
1980^1	.	.	.	.	1	.	.	.
$2376a^1$	1	1	.	.	.	1	.	.
$2376b^1$	1	1	.	.	.	.	1	.
$2520a^1$	.	.	1	.	.	1	.	.
$2520b^1$	.	.	.	1	.	.	1	.
2772^1	.	.	1	1	1	.	.	.
4752^1	.	.	1	1	2	.	.	.
5103^1	.	.	.	.	1	.	.	1
7875^1	.	.	1	1	2	.	.	1
$8019a^1$	.	.	1	1	1	.	1	1
$8019b^1$	.	.	1	1	1	1	.	1
$10395'^1$	1	1	1	1	1	1	1	1
$10395a^1$	.	.	2	1	2	1	.	1
$10395b^1$	.	.	1	2	2	.	1	1
12375^1	1	1	1	1	2	1	1	1

The columns correspond to the following list of irreducible Brauer characters

$$126a^1, 126b^1, 396a^1, 396b^1, 1980^1, 2124a^1, 2124b^1, 3123^1.$$

3. Appendix

The decomposition matrix of the principal block of $3.McL$ as we use it in the previous text is as follows.

1^{pb}	1	.	.	.	.	.	.	.
22^{pb}	.	1	.	.	.	.	.	.
231^{pb}	1	.	1	.	.	.	.	.
252^{pb}	.	1	1	.	.	.	.	.
770^{pb}	.	1	.	1	.	.	.	.
770^{*pb}	.	1	.	.	1	.	.	.
1750^{pb}	2	1	1	1	1	.	.	.
4500^{pb}	.	1	1	.	.	1	1	.
4752^{pb}	.	2	2	.	.	1	1	.
5103^{pb}	1	1	.	1	1	.	.	1
5544^{pb}	4	.	2	1	1	.	.	1
8019^{pb}	1	3	.	1	2	.	1	1
8019^{*pb}	1	3	.	2	1	1	.	1
8250^{pb}	2	3	1	1	2	.	1	1
8250^{*pb}	2	3	1	2	1	1	.	1
9625^{pb}	1	3	1	1	1	1	1	1
10395^{pb}	1	4	1	2	1	1	1	1
10395^{*pb}	1	4	1	1	2	1	1	1

The columns correspond to the following list of irreducible Brauer characters.

$$1, 22, 230, 748, 748^*, 2124, 2124^*, 3584.$$

This decomposition matrix has been determined by J. Thackray, see [11]. The matrix used here is in accordance with the irreducible Brauer characters given in [7]. The decomposition matrix given in [11] is obtained by interchanging the rows corresponding to 8019^{pb}, 8019^{*pb} and 8250^{pb}, 8250^{*pb} and by interchanging the columns corresponding to 2124^{pb} and 2124^{*pb}. This is equivalent to a renumbering of classes of elements of order 7 and 15.

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