Broué's abelian defect group conjecture for the sporadic simple Janko group J_4 revisited

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Dedicated to Geoffrey R. Robinson on the occation of his sixtieth birthday

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Abstract

We show that the 3-block of the sporadic simple Janko group J_4 with defect group $C_3 \times C_3$, and the principal 3-block of the alternating group $\mathfrak{A}_8$ are Puig equivalent, answering a question posed in [15]. To accomplish this, we apply computational techniques, in particular an explicit version of the Brauer construction.

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1 Introduction

In recent years, much impetus in modular representation theory of finite groups has originated from attempts to prove various fascinating deep conjectures. Two of them are Broué's Abelian Defect Group Conjecture [4] and a strengthening, Rickard's Splendidness Conjecture [33], which for the purpose of the present paper may be stated as follows:

(1.1) Conjecture. Let k be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. Let A be a block of kG having an *abelian* defect group P, let $N_G(P)$ be the normaliser of P in G, and let B be the block of $k[N_G(P)]$ which is the Brauer correspondent of A. Then Broué's Conjecture says that A and B are derived equivalent, and Rickard's Conjecture says that there even is a splendid derived equivalence between A and B.

In general, Broué's and Rickard's Conjectures currently are widely open. They have been proven for a number of cases, where for an overview we refer to [5]. In particular, in [12] it is shown that both conjectures hold true whenever A a principal block having a defect group $P \cong C_3 \times C_3$ isomorphic to the elementary abelian group of order 9. This moves non-principal blocks with defect group P into the focus of interest, where, in view of the successful reduction strategy for principal blocks used in [12], and a possible, as yet non-existent, generalisation to non-principal blocks, it seems worthwhile to proceed with blocks of quasi-simple groups. There are a few results already known, see for example [13, 14, 15, 16, 18, 26], which indicate that fairly often a non-principal block with defect

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group P is actually Morita equivalent to a principal block of a different (smaller) group. The present paper is another step in clarifying this relationship:

Letting $p := 3$, here we consider the sporadic simple Janko group J_4. Then $k[J_4]$ has a unique block A of defect 2, hence having a defect group $P \cong C_3 \times C_3$. In order to verify Broué's and Rickard's Conjectures for A, it is shown in [15] that A is actually Morita equivalent to the principal block A' of $k[\mathfrak{A}_8]$, also having defect group P, where $\mathfrak{A}_8$ is the alternating group on 8 letters. But it is left open, see [15, Question 6.14], whether or not A and A' are even Puig equivalent. We are now able to answer this affirmatively:

(1.2) Theorem . The blocks A and A' are Puig equivalent.

This also sheds some light on Rickard's Splendidness Conjecture for A and its Brauer correspondent B in $N_{J_4}(P)$, where in [15, (6.13)] an indirect proof is given, running as follows: Letting B' be the Brauer correspondent of A' in $N_{\mathfrak{A}_8}(P)$, since both A and A', as well as B and B' are mutually Morita equivalent, and Green correspondences between A' and B', as well as A and B are suitably related, the proof of Broué's Conjecture for A' given in [28, Example 4.3], which by [29, Corollary 2] also proves Rickard's Conjecture, works entirely similar for A. Now, our approach provides explicit Puig equivalences between both A and A', as well as B and B', hence these directly transport a splendid tilting complex between A' and B' to such a complex between A and B, thus Rickard's Conjecture for A is completely reduced to A'.

(1.3) Strategy. Our strategy is to rework the approach in [15] explicitly. This means, we fix a concrete realisation of J_4, suitable to work with computationally, and a certain configuration of subgroups, in particular containing a copy of $\mathfrak{A}_8$; this is carried out in Section 3. While doing so, we painstakingly take care that all choices made are unique up to simultaneous J_4-conjugacy.

Having this in place, since the functors provided in [15] inducing stable equivalences between A and B, as well as A' and B' coincide with Green correspondence on simple modules, they can, in principle at least, be evaluated explicitly on simple modules. But the Morita equivalence between B and B' used in [15] is based on the abstract theory of blocks with normal defect groups, see [19, Theorem A] and [32, Proposition 14.6], hence is replaced here by explicit functors relying on our fixed configuration of subgroups of G; this is carried out in Section 4. We would like to mention that we have been led to consider the block A while preparing our earlier paper [18], on the double cover of the sporadic simple Higman-Sims group (which is much smaller than J_4), where a similar, but subtly different problem occurs in the analysis of local subgroups.

Alone, simple A-modules are much too large to be dealt with directly by an explicit approach. (This makes up a decisive difference to our earlier considerations in [18].) Instead, we use the Brauer construction, applied to p-permutation modules, to facilitate the explicit determination of Green correspondents of certain simple A-modules. The underlying theory is presented in Section 2, which is largely based on [3], with a view towards explicit computations.

A few comments on this computational approach seem to be in order: In practice, using this technique we are able to reduce the size of objects to be handled

computationally dramatically, as is seen for example in our main application in (4.4). From a more conceptual point of view, the description in (2.2) shows a formal similarity to so-called *condensation* of permutation modules, a well-known workhorse in computational representation theory, see for example [25, Sections 9, 10]. Condensation is formally described as application of a Schur functor associated with a p'-subgroup K. (The use of the letter 'K' in (2.3) is reminiscent of the German writing of 'Kondensation'.) Here, the role of K is eventually played by the p-subgroup P, so that in a sense we are dealing with a 'p-singular' generalisation of condensation. Moreover, the Brauer construction features prominently in modular representation theory of finite groups; we only mention the comments in [33, Section 4] concerning its application to splendid tilting complexes. Thus, due to its general nature and the gain in computational efficiency, we are sure that this technique will face more applications.

(1.4) In order to facilitate the necessary computations, we make use of the computer algebra system GAP [7], to deal with finite groups, in particular permutation and matrix groups, and with ordinary and Brauer characters of finite groups. In particular, we make use of the character table library [2] of GAP, the GAP-interface [38] to the database [40], and the SmallGroups library [1] of GAP. Moreover, we use the computer algebra system MeatAxe [30, 34], and its extensions [21, 22, 23, 24] to deal with matrix representations over finite fields.

We remark that, although for the theoretical developments we fix an algebraically closed field k of characteristic 3, explicit computations can always be done of a suitably large finite field of characteristic 3, where for the computations to be described here even the prime field $\mathbb{F}_3$ turns out to be large enough. We assume the reader familiar with the relevant concepts of modular representation theory of finite groups and of the various notions of equivalences between categories, as general references see [11, 27, 35]; our group theoretical notation is standard, borrowed from [6].

(1.5) Notation. Throughout this paper a module means a finitely generated *right* module unless stated otherwise. Let G be a finite group. We denote by 1_G the trivial character of G, and by k_G the trivial kG-module. We write $H \leq G$ when H is a subgroup of G, and $H < G$ when H is a proper subgroup of G. Let $H \leq G$, and let V and W be a kG-module and a kH-module, respectively. Then we write $\mathrm{Res}_H^G(V)$ or $\mathrm{Res}_H(V)$ for the restriction of V to H, and $\mathrm{Ind}_H^G(W)$ or $\mathrm{Ind}^G(W)$ for the induced module (induction) $W \otimes_{kH} kG$ of W to G. For a block algebra B of kG, we denote by 1_B the block idempotent of B. For a kG-module X we write $X^\vee$ for the k-dual of X, namely $X^\vee := \mathrm{Hom}_k(X, k)$. For a subset S of G and an element $g \in G$ we write S^g for $g^{-1}Sg$. We write $\mathbb{N}$ and $\mathbb{N}_0$ for the sets of all positive integers and of all non-negative integers, respectively. Let $\mathfrak{S}_n$ and $\mathfrak{A}_n$ denote the symmetric group and the alternating group on n letters, respectively.

2 Brauer construction

(2.1) Brauer construction. Let k be a field of characteristic $p > 0$, and let G be a finite group. For any finitely generated (right) kG-module V and any subgroup $K \leq G$ let $V^K := \{v \in V; vg = v \text{ for all } g \in K\}$ be the set of K-fixed

points in V. Hence V^K naturally becomes a $k[N_G(K)]$-module, which since K acts trivially induces the structure of a $k[N_G(K)/K]$-module on V^K. Moreover, the k-linear trace map on V is defined as $\mathrm{Tr}_K^G \colon V^K \to V^G \colon v \mapsto \sum_{g \in K \backslash G} vg$, the sum running over a system of representatives g of the right cosets Kg of K in G.

Now, if $P \leq G$ is a p-subgroup, then let

$$V(P) := V^P / \left(\sum_{Q < P} \mathrm{Tr}_Q^P(V^Q) \right),$$

where the sum runs over all proper subgroups Q of P. Since for all $g \in N_G(P)$ and $Q \leq P$ we have $(\mathrm{Tr}_Q^P(v))^g = (\mathrm{Tr}_{Q^g}^P(v^g)) \in V^P$, for all $v \in V^Q$, we conclude that $\sum_{Q < P} \mathrm{Tr}_Q^P(V^Q) \leq V^P$ is a $k[N_G(P)]$-submodule, hence $V(P)$ becomes a $k[N_G(P)]$-module as well. Moreover, the natural map $\mathrm{Br}^P \colon V^P \to V(P)$, being called the associated Brauer map, is a homomorphism of $k[N_G(P)]$-modules; note that in particular Br^P commutes with taking direct sums.

(2.2) p-permutation modules. Recall that V is called a p-permutation kG-module if $\mathrm{Res}_P(V)$ is a permutation kP-module for any p-subgroup $P \leq G$, and that this equivalent to saying that all indecomposable direct summands of V are trivial source kG-modules, see [35, Proposition 27.3].

a) Here, we are content with much less, namely we consider the case where $\mathrm{Res}_P(V)$ is a permutation kP-module for some fixed p-subgroup $P \leq G$. The following facts are well-known, and stated for example in [3, Section 1] and [35, Exercise 11.4]. Still, we include the details which will be needed explicitly later on:

Let Ω be a P-stable k-basis of V; we also write $V = k[\Omega]$. Let $\Omega = \coprod_{i=1}^r \Omega_i$ be the decomposition of Ω into P-orbits, for some $r \in \mathbb{N}_0$, where we assume that

$$\Omega^P := \{\omega \in \Omega; \omega g = \omega \text{ for all } g \in P\} = \coprod_{i=1}^s \Omega_i = \coprod_{i=1}^s \{\omega_i\},$$

for some $s \in \{0, \ldots, r\}$, is the set of P-fixed points in Ω. Then we have $\mathrm{Res}_P(V) = \bigoplus_{i=1}^r k[\Omega_i]$ as kP-modules, and $k[\Omega_i]^P = k[\Omega_i^+]$ where $\Omega_i^+ := \sum_{\omega \in \Omega_i} \omega$ is the associated orbit sum, for all $i \in \{1, \ldots, r\}$. Thus $\{\Omega_1^+, \ldots, \Omega_r^+\}$ is a k-basis of V^P, and as k-vector spaces we get

$$V(P) = \mathrm{Br}^P(V^P) \cong \bigoplus_{i=1}^r \mathrm{Br}^P(k[\Omega_i]^P) = \bigoplus_{i=1}^r \mathrm{Br}^P(k[\Omega_i^+]).$$

If $\omega_0 \in \Omega_i$ for some $i > s$, then for $Q := \mathrm{Stab}_P(\omega_0) < P$ we get $\mathrm{Tr}_Q^P(\omega_0) = \sum_{\omega \in \Omega_i} \omega = \Omega_i^+$, implying that $\mathrm{Br}^P(k[\Omega_i^+]) = 0$; for $i \leq s$ we have $\Omega_i = \{\omega_i\}$ and thus $\mathrm{Tr}_Q^P(\Omega_i^+) = \mathrm{Tr}_Q^P(\omega_i) = [P \colon Q] \cdot \omega_i = 0$, for all $Q < P$, showing that Br^P is the identity map on $k[\Omega_i^+] = k[\omega_i]$, hence so is on $k[\Omega^P]$. Thus we conclude that the Brauer map induces a k-vector space isomorphism $\mathrm{Br}^P \colon k[\Omega^P] \to V(P)$.

b) Now we additionally assume that Ω is $N_G(P)$-stable, that is $\mathrm{Res}_{N_G(P)}(V)$ is a permutation $k[N_G(P)]$-module with respect to Ω; note that this in particular

holds whenever V is a permutation kG-module. Then it follows that $N_G(P)$ permutes the P-orbits in Ω of any fixed length amongst themselves. Hence both Ω^P and $\Omega - \Omega^P$ are $N_G(P)$-stable. Since we have already seen that $k[\Omega - \Omega^P] = \sum_{Q<P} \operatorname{Tr}_Q^P(V^Q)$, as $k[N_G(P)]$-modules we get

$$V^P = k[\Omega^P] \oplus k[\Omega - \Omega^P] = k[\Omega^P] \oplus \left(\sum_{Q<P} \operatorname{Tr}_Q^P(V^Q) \right).$$

Hence we conclude that the the induced map $\operatorname{Br}^P : k[\Omega^P] \to V(P)$ is even an isomorphism of $k[N_G(P)]$-modules; in particular, $V(P)$ is a permutation $k[N_G(P)]$-module. Note that this implies that $V(P)$ is a p-permutation $k[N_G(P)]$-module whenever V is a p-permutation kG-module, see [3, (3.1)].

(2.3) Permutation modules. Let $H \leq G$ be a subgroup, and let $k[\Omega]$ be the permutation kG-module associated with the set $\Omega := H \backslash G$ of right cosets of H in G; we also write $\omega_g \in \Omega$ for the coset $Hg \subseteq G$, where $g \in G$. Moreover, let $K \leq G$ be an arbitrary subgroup. We proceed to derive a description of Ω^K, and of the structure of $k[\Omega^K]$ as a $k[N_G(K)]$-module. The following results are also shown in [3, (1.3), (1.4)], in the case of $K = P$ being a p-group, using Mackey's Formula and Higman's Criterion. Although only the latter case will be relevant in our applications, we present a general straightforward proof, in the spirit of the explicit approach taken here:

Since for any $g \in G$ we have $\operatorname{Stab}_G(\omega_g) = H^g$, we conclude that $\Omega^K \neq \emptyset$ if and only if $K \leq H^g$ for some $g \in G$. Hence, if this is not the case then we have $(k[\Omega])(K) = \{0\}$ anyway, thus, by going over to some G-conjugate of H if necessary, we may assume that $K \leq H$; note that replacing H like this just amounts to going over to an equivalent permutation action of G.

Let $K_1, \ldots, K_t \leq H$, for some $t \in \mathbb{N}$, be a set of representatives of the H-conjugacy classes of subgroups of H being G-conjugate to K, and let $g_i \in G$ such that $K_i^{g_i} = K$, for $i \in \{1, \ldots, t\}$, where we may assume that $K_1 = K$ and $g_1 = 1$. Now, for $g \in G$ we have $\omega_g \in \Omega^K$ if and only if $K^{g^{-1}} \leq H$. If this is the case, then there are $i \in \{1, \ldots, t\}$ and $h \in H$ such that $K^{g^{-1}} = K_i^{h^{-1}} = K^{g_i^{-1}h^{-1}}$, implying $g_i^{-1}h^{-1}g \in N_G(K)$, and thus $g \in Hg_iN_G(K) \subseteq G$. Conversely, if $g = hg_in \in Hg_iN_G(K)$, for some $h \in H$ and $n \in N_G(K)$, then we have $K^{g^{-1}} = K^{n^{-1}g_i^{-1}h^{-1}} = K_i^{h^{-1}} \leq H$. Hence in conclusion we get

$$\Omega^K = \{\omega_{g_in} \in \Omega; i \in \{1, \ldots, t\}, n \in N_G(K)\}.$$

Next we note that the double cosets $Hg_1N_G(K), \ldots, Hg_rN_G(K) \subseteq G$ are pairwise distinct: Assume that there are $i \neq j \in \{1, \ldots, t\}$ such that $hg_in = h'g_jn'$, for some $h, h' \in H$ and $n, n' \in N_G(K)$; then

$$K_i^{h^{-1}h'} = K_i^{g_inn'^{-1}g_j^{-1}} = K^{nn'^{-1}g_j^{-1}} = K^{g_j^{-1}} = K_j$$

shows that K_i and K_j are H-conjugate, a contradiction. Thus the above description of Ω^K shows that we have $k[\Omega^K] = \bigoplus_{i=1}^t k[\omega_{g_i}N_G(K)]$ as permutation $k[N_G(K)]$-modules. Moreover, from

$$\operatorname{Stab}_{N_G(K)}(\omega_{g_i}) = N_G(K) \cap H^{g_i} = N_{H^{g_i}}(K) = N_{H^{g_i}}(K_i^{g_i}) = N_H(K_i)^{g_i}$$

and $N_G(K) = N_G(K)^{g_i}$, we get

$$k[\Omega^K] \cong \bigoplus_{i=1}^{t} k[N_{H^{g_i}}(K)\backslash N_G(K)] \cong \bigoplus_{i=1}^{t} k[N_H(K_i)\backslash N_G(K_i)]^{g_i},$$

where the latter summands denote the $k[N_G(K)]$-modules obtained from the $k[N_G(K_i)]$-modules $k[N_H(K_i)\backslash N_G(K_i)]$ by transport via g_i.

This completes our description of $k[\Omega^K]$ as $k[N_G(K)$-module. Note that the i-th summand in the above decomposition is the trivial $k[N_G(K)]$-module if and only if we have $N_G(K_i) = N_H(K_i)$, or equivalently $N_G(K_i) \leq H$. Finally, we remark that going over to cardinalities in particular yields $|\Omega^K| = \sum_{i=1}^{t} \frac{|N_G(K)|}{|N_H(K_i)|}$, showing that the above result is a special case of the 'induction formula for marks' (in the theory of Burnside rings) given in [31, Theorem 2.2].

Underlying our application of the Brauer construction now is the following

(2.4) Theorem : Broué-Puig [3, (3.4)]. Let V be an indecomposable trivial source kG-module having P as a vertex, and let f be the Green correspondence with respect to the triple $(G, P, N_G(P))$, see [27, Theorem 4.4.3]. Then the $k[N_G(P)]$-module $V(P)$ coincides with the Green correspondent $f(V)$ of V.

3 The setting

From now on let k be an algebraically closed field of characteristic 3. Moreover, we fix a 3-modular system $(\mathcal{K}, \mathcal{O}, k)$ which is large enough. That is to say, $\mathcal{O}$ is a complete discrete valuation ring of rank one such that its quotient field $\mathcal{K}$ is of characteristic zero, and its residue field $k = \mathcal{O}/\mathrm{rad}(\mathcal{O})$ is of characteristic 3, and that $\mathcal{K}$ and k are splitting fields for all the (finitely many) groups occurring in the sequel.

(3.1) The group J_4. Let $G := J_4$ be the sporadic simple Janko group J_4. Then using the ordinary character table of G given in [6, p.190], also available electronically in the character table library of GAP, it follows that kG has a unique block A having a defect group P of order 9; hence $P \cong C_3 \times C_3$.

Moreover, G has a unique conjugacy class consisting of elements of order 3, and if $Q < P$ is a subgroup of order 3 then we have $N_G(Q) \cong 6.M_{22}.2$, see [9, Section 3]. Now $6.M_{22}.2$ has precisely two conjugacy classes consisting of elements of order 3, with associated centralisers of shape $6.M_{22}$ and $(2^3 \times 3^2).2$, respectively, see [6, p.41]. Thus we infer that G has a unique conjugacy class of subgroups isomorphic to $C_3 \times C_3$, and we have $C := C_G(P) \cong 2^3 \times P$.

By [36, Theorem 4.5] the decomposition matrix of A is as given in Table 1, where the irreducible ordinary characters belonging to A are numbered as in [6, p.190], and their degrees are recorded as well. The simple A-modules are just named $S_1, \ldots, S_5$; their dimensions are immediately read off the decomposition matrix.

Table 1: The decomposition matrix of A.

degree	A	S_1	S_2	S_3	S_4	S_5
4 290 927	χ_{14}	1	.	.	.	.
95 288 172	χ_{21}	.	1	.	.	.
300 364 890	χ_{25}	.	.	1	.	.
393 877 506	χ_{27}	.	.	.	1	.
394 765 284	χ_{28}	1	.	.	.	1
493 456 605	χ_{30}	1	1	.	1	.
690 839 247	χ_{31}	.	.	1	.	1
789 530 568	χ_{35}	.	1	1	1	.
1 089 007 680	χ_{41}	1	.	1	1	1

Table 2: The decomposition matrices of A' and B'.

A'	1	7	13	28	35
χ_1	1	.	.	.	.
χ_7	.	1	.	.	.
χ_{14}	1	.	1	.	.
χ_{20}	.	1	1	.	.
χ_{28}	.	.	.	1	.
χ_{35}	.	.	.	.	1
χ_{56}	1	1	1	.	1
χ_{64}	1	.	.	1	1
χ_{70}	.	1	.	1	1

B'	$1a$	$1b$	$1c$	$1d$	2
χ_{1a}	1	.	.	.	.
χ_{1b}	.	1	.	.	.
χ_{1c}	.	.	1	.	.
χ_{1d}	.	.	.	1	.
χ_2	.	.	.	.	1
χ_{4a}	1	.	1	.	1
χ_{4b}	.	1	.	1	1
χ_{4c}	1	.	.	1	1
χ_{4d}	.	1	1	.	1

(3.2) The group $\mathfrak{A}_8$. Let $G' := \mathfrak{A}_8$ be the alternating group on 8 letters, and let $P \cong C_3 \times C_3$ be a Sylow 3-subgroup of G'; we will see in (3.4) that we may safely reuse the letter 'P' here. Hence we have $C_{G'}(P) = P$, and G' has precisely two conjugacy classes consisting of elements of order 3, of cycle type $[3^2, 1^2]$ and $[3, 1^5]$, respectively, where P contains four elements from each of these classes. Hence G' has precisely two conjugacy classes of subgroups of order 3.

Let A' be the principal block of kG'. Then the decomposition matrix of A' is as given in Table 2, where the simple A'-modules $1, 7, 13, 28, 35$ are denoted by their dimensions, and similarly the irreducible ordinary characters $\chi_1, \ldots, \chi_{70}$ belonging to A' are indexed by their degrees. The irreducible ordinary and Brauer characters belonging to A' can be found in [6, p.22] and [8, p.49], respectively, and are available electronically via [39] as well as in the character table library of GAP.

(3.3) Local subgroups of G'. Let $H' := N_{G'}(P) \cong P\colon D_8$, see [6, p.22]. Since P is abelian, implying that H' controls G'-fusion in P, we deduce that D_8 permutes transitively the 4-sets of non-trivial elements of P of fixed cycle type, hence H' has precisely two conjugacy classes of subgroups of order 3.

Table 3: The ordinary character table of H'.

centraliser	72	8	12	12	18	18	4	6	6
class	$1A$	$2A$	$2B$	$2C$	$3A$	$3B$	$4A$	$6A$	$6B$
χ_{1a}	1	1	1	1	1	1	1	1	1
χ_{1b}	1	1	-1	-1	1	1	1	-1	-1
χ_{1c}	1	1	-1	1	1	1	-1	-1	1
χ_{1d}	1	1	1	-1	1	1	-1	1	-1
χ_2	2	-2	0	0	2	2	0	0	0
χ_{4a}	4	0	0	2	-2	1	0	0	-1
χ_{4b}	4	0	0	-2	-2	1	0	0	1
χ_{4c}	4	0	2	0	1	-2	0	-1	0
χ_{4d}	4	0	-2	0	1	-2	0	1	0

Using the facilities of GAP to deal with groups and their characters, we determine the ordinary character table of H'. It is given in Table 3, where again the irreducible characters are indexed by their degrees, conjugacy classes are denoted by the orders of the elements they contain, and centraliser orders are recorded as well. We choose notation such that χ_{1a} is the trivial character, and that χ_{1b} is distinguished amongst the three non-trivial linear characters, for example by having an element of order 4 in its kernel. (Of course, the characters of $H' \cong P\colon D_8$ can also be determined easily from those of P and D_8 via Clifford theory, see [27, Chapter 3.3]. But since we need the character table explicitly anyway, we found a direct computation appropriate here.)

Now a computation with GAP shows that $\mathrm{Inn}(H') \cong H'$ is a normal subgroup of index 2 in $\mathrm{Aut}(H')$, and that any non-inner automorphism $\omega \in \mathrm{Aut}(H')$ induces a non-inner automorphism of $H'/P \cong D_8$ and interchanges the two H'-conjugacy classes of subgroups of order 3 of P. In particular, ω induces a table automorphism of the ordinary character table of H', fixing χ_{1a}, χ_{1b}, and χ_2, but interchanging $\chi_{1c} \leftrightarrow \chi_{1d}$, $\chi_{4a} \leftrightarrow \chi_{4c}$, and $\chi_{4b} \leftrightarrow \chi_{4d}$.

Let B' be the principal block of kH', that is the Brauer correspondent of A'. Since A' is the only block of kG' having maximal defect, by Brauer's First Main Theorem , see [27, Theorem 5.2.15], we conclude that B' is the only block of kH', that is $B' = kH'$. Since the irreducible Brauer characters of H' are in bijection with the irreducible ordinary characters of $H'/P \cong D_8$, the decomposition matrix of B' is immediate from the ordinary character table of H'. It is given in Table 2, where again the simple B'-modules are denoted by their dimensions. Note that the non-inner automorphism ω fixes $1a$, $1b$, and 2, but interchanges $1c \leftrightarrow 1d$.

(3.4) Embedding G' into G. Before proceeding to the local subgroups of G we determine an explicit embedding of G', and thus of the whole subgroup chain $P < H' < G'$, into G. Note that by [9, Proof of Corollary 6.5.5] there is a unique conjugacy class of subgroups of G isomorphic to $\mathfrak{A}_8$, and recalling that G has a unique conjugacy class of subgroups isomorphic to $C_3 \times C_3$ we may safely choose $P < G'$ as our favourite defect group of A. In practice, we start with standard generators of G, in the sense of [37], which hence in particular are unique up

to simultaneous G-conjugacy. Then we obtain straight-line programs producing generators of all of the subgroups in question. We describe the computations which have to be done, using GAP and the MeatAxe:

In order to find a subgroup isomorphic to $\mathfrak{A}_8$ of G, we work through the maximal subgroup $M := 2^{11} \colon M_{24}$ of G, since it also follows from [9, Proof of Corollary 6.5.5] that we even have a subgroup chain $G' < M_{24} < M < G$. Hence we start with the absolutely irreducible $\mathbb{F}_2$-representation V of G of dimension 112, standard generators of which are available in [40], together with a straight-line program yielding generators of M. By [6, p.190], the restriction $\mathrm{Res}_M(V)$ is uniserial with radical series $[1a/11b/44b/44a/11a/1a]$. Using the MeatAxe, we compute the action of M on the unique submodule U of dimension 12, which hence has shape $[11a/1a]$. Considering the action of M on the elements of U yields a transitive permutation representation π on 1518 points, from which GAP shows that we indeed have found a faithful representation of M. Further computations in M can now be done using the small representation π, where in particular we have all the facilities of GAP to deal with permutation groups at our disposal.

Now we proceed similar to the approach in [25, (16.2)]: Since the normal 2-subgroup $2^{11} \lhd M$ acts trivially on the constituent $11a$ of $\mathrm{Res}_M(V)$, we thus obtain a representation of $M/2^{11} \cong M_{24}$, although in terms of non-standard generators. Following the recipe given in [40], by a random search we find a straight-line program producing standard generators of the action of M_{24} on $11a$, and running this on the representation π, GAP shows that we have indeed found standard generators of a subgroup M_{24} of M. Having this in place, we apply a straight-line program yielding generators of $2^4 \colon \mathfrak{A}_8$ from standard generators of M_{24}, again available in [40], and finally another random search yields a straight-line program producing generators of a subgroup $G' := \mathfrak{A}_8 < 2^4 \colon \mathfrak{A}_8$.

(3.5) Local subgroups of G. Let $N := N_G(P)$. It is stated in [9, Section 3 and Proof of Corollary 6.4.4], unfortunately without proof, that $N \cong (2^3 \times P) \colon \mathrm{GL}_2(3)$ and that N can be embedded into the maximal subgroup M. Since we need an explicit realisation of N as a subgroup of G anyway, we are going to verify these facts independently:

We have already fixed $P < G' < M$. We remark that, since, by [9, Proof of Corollary 6.5.5], M has a unique conjugacy class of subgroups isomorphic to $\mathfrak{A}_8$, and $P < G'$ is a Sylow 3-subgroup, this defines a unique conjugacy class of subgroups of M isomorphic to $C_3 \times C_3$. Using GAP, the normaliser $N_M(P)$ can be determined explicitly as a subgroup of M. It turns out that $|N_M(P)| = 3456 = 2^3 \cdot 3^2 \cdot 48$. We already know that $C = C_G(P) \cong 2^3 \times P$, hence from $\mathrm{Out}(P) \cong \mathrm{Aut}(P) \cong \mathrm{GL}_2(3)$ we conclude that $C < M$ and that P is fully automised in M, thus we infer $N = N_M(P)$. Moreover, it is verified using GAP that C has a complement in N.

Let B be the block of kN which is the Brauer correspondent of A. Since A is the only block of kG having P as a defect group, by Brauer's First Main Theorem, see [27, Theorem 5.2.15], we conclude that B is the only block of kN having P as a defect group.

(3.6) Local subgroups of G, continued. Next we proceed to find the blocks of C covered by B: Let $2^3 \cong E < C$ be the elementary abelian Sylow 2-subgroup. Then then blocks of C are in bijection with the irreducible ordinary characters of E. Identifying E with the $\mathbb{F}_2$-vector space $\mathbb{F}_2^3$, and its character group with $(\mathbb{F}_2^3)^* := \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2^3, \mathbb{F}_2)$, the MeatAxe shows that $(\mathbb{F}_2^3)^*$, with respect to the action of $N/C \cong \mathrm{GL}_2(3)$, becomes a uniserial module with radical series $[2/1]$, where $\mathrm{GL}_2(3)$ has three orbits, of lengths $[1, 1, 6]$, on the elements of $(\mathbb{F}_2^3)^*$.
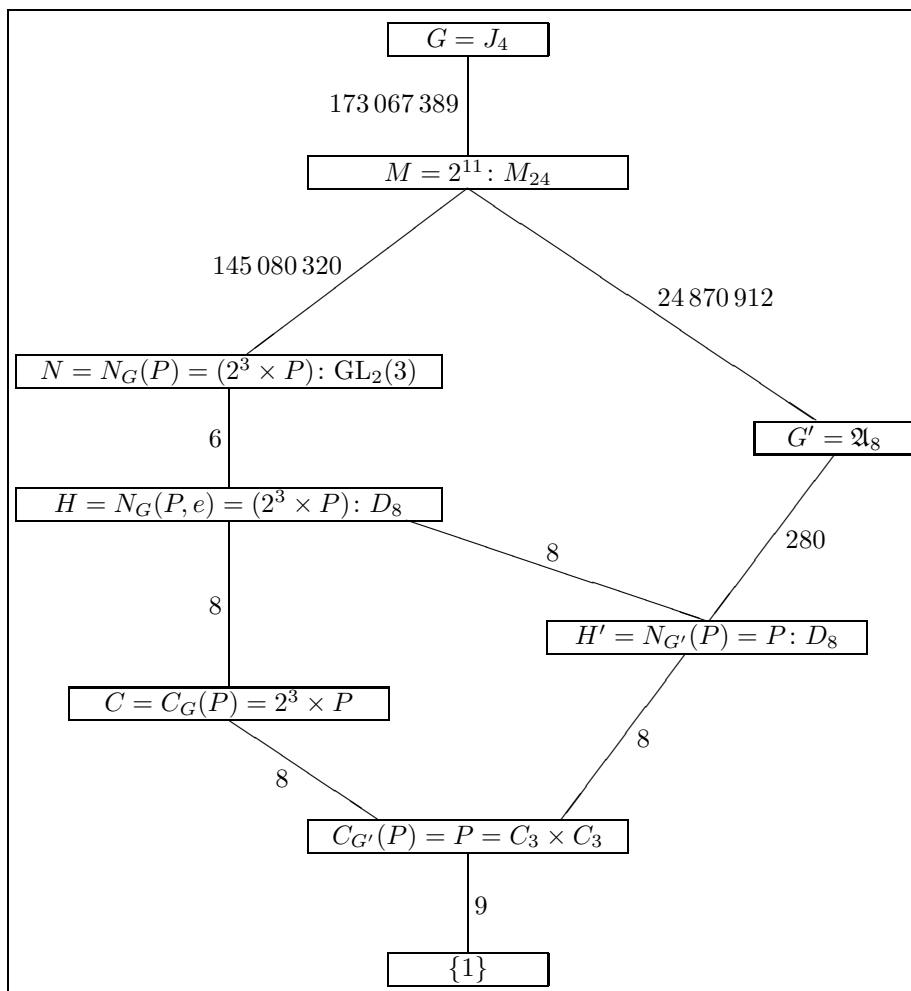
Hence there are two irreducible characters of E which are N-invariant, thus extend to linear characters of N, and hence belong to blocks of (maximal) defect 3 of N. Thus we conclude that B covers an N-orbit of six blocks of C, hence the associated inertia groups have index 6 in N. Now we make a specific choice amongst these blocks:

We observe that $H'C/C \cong H'/(H' \cap C) = H'/C' \cong D_8$, hence $H'C$ has index 6 in N. This is sufficiently suspicious to be tempted to check the following: Indeed, the MeatAxe shows that $(\mathbb{F}_2^3)^*$, with respect to the action of $H'C/C \cong D_8$, has radical series $[1/(1 \oplus 1)]$, where $D_8 < \mathrm{GL}_2(3)$ has five orbits, of lengths $[1, 1, 1, 1, 4]$, on the elements of $(\mathbb{F}_2^3)^*$. This says that there are precisely two blocks of kC amongst those covered by B which are fixed by $H'C$. Moreover, using the identification of the elements of $(\mathbb{F}_2^3)^*$ with the irreducible ordinary characters of E yields the associated primitive idempotents $e_i \in kE$, for $i \in \{1, 2\}$, which are the block idempotents in kC of the blocks in question.

Letting $H_i := N_G(P, e_i) := \{g \in N_G(P); e_i^g = e_i\}$ be the associated inertia groups, this shows that we have $H := H'C = H_1 = H_2 \cong (2^3 \times P) : D_8$. This completes the panorama of subgroups of G we need, the complete picture is shown in Table 4. Finally, let B_i be the block of kH being the Fong-Reynolds correspondent of B with respect to $e_i(kC)e_i$, that is B_i covers $e_i(kC)e_i$ and we have $(B_i)^N = B$, see [27, Theorem 5.5.10]; note that we hence have $1_{B_i} = e_i$.

(3.7) Local structure of G. To determine the decomposition matrices of B_i and B we proceed as follows: Since H is a split extension of $E \cong 2^3$ by $H' \cong P : D_8$, the irreducible ordinary and Brauer characters of H can be determined from those of E and H' via Clifford theory, see [27, Chapter 3.3]. Moreover, since B_i covers $e_i(kE)e_i$, by [27, Lemma 5.5.7] we are only interested in the characters of H covering the irreducible character λ_i of E associated with e_i. Since λ_i is H-invariant, it extends to H by letting $\lambda_i(gh) := \lambda_i(g)$, for all $g \in E$ and $h \in H'$. Then the irreducible ordinary and Brauer characters of B_i are in natural bijection to those of $kH' = B'$. Hence, using the notation in (3.3), we may write the irreducible ordinary characters of B_i as χ_{1x_i}, χ_{2_i}, and χ_{4x_i}, and the simple B_i-modules as $1x_i$ and 2_i, where $x \in \{a, b, c, d\}$, subject to the conditions $\mathrm{Res}_{H'}^H(\chi_{?_i}) = \chi_?$ and $\mathrm{Res}_{H'}^H(?_i) \cong ?$ for characters and modules, respectively. Thus, using these identifications, the decomposition matrices of B_i and B' coincide, see Table 2.

Moreover, by Fong-Reynolds correspondence, see [27, Theorem 5.5.10], we may write the irreducible ordinary characters of B as χ_{6x}, χ_{12}, and χ_{24x}, and the simple B-modules as $6x$ and 12, where $x \in \{a, b, c, d\}$, subject to the conditions $\mathrm{Ind}_H^N(\chi_{?_i}) = \chi_{6\cdot?}$ and $\mathrm{Ind}_H^N(?_i) \cong 6 \cdot ?$. Since $\mathrm{Res}_{H'}^H(\mathrm{Res}_H^N(\chi_{6\cdot?}) \cdot 1_{B_i}) = \chi_?$ and $\mathrm{Res}_{H'}^H(\mathrm{Res}_H^N(6 \cdot ?) \cdot 1_{B_i}) \cong ?$, the parametrisation of characters and modules of B is indeed independent from the choice of Fong-Reynolds correspondent B_i we

Table 4: Subgroups of G.

work through. Thus, using these identifications, the decomposition matrices of B and B' coincide as well, see Table 2.

4 The final stroke

(4.1) Equivalence between A' and B'. By [15, Lemma 2.4], there is a unique indecomposable direct summand $\mathcal{M}'$ of the kG'-kH'-bimodule $1_{A'}(kG')1_{B'}$ with vertex $\Delta P := \{(g,g) \in G' \times H'; g \in P\} < G' \times H'$. Then, letting **mod**-$A'$ and **mod**-B' denote the categories of finitely generated right A'- and B'-modules, respectively, by [15, Lemma 6.5(iii)] we have a splendid stable equivalence of Morita type given by the tensor functor $? \otimes_{kG'} \mathcal{M}' : \textbf{mod}\text{-}A' \to \textbf{mod}\text{-}B'$; to shorten notation we just write $\mathcal{M}'$ instead. In particular, by [20, Theorem 2.1(ii)], the functor $\mathcal{M}'$ maps any simple A'-module to an indecomposable B'-

module.

Let f' be the Green correspondence with respect to the triple (G', P, H'). Then, by [17, Lemma A.3], for any indecomposable A'-module V having P as a vertex, the unique non-projective indecomposable summand of $\mathcal{M}'(V)$ coincides with the Green correspondent $f'(V)$. Thus, recalling that, by [10, Corollary 3.7], any simple A'-module T indeed has P as a vertex, we conclude that $\mathcal{M}'(T) \cong f'(T)$.

The Green correspondents of the simple A'-modules are known by [28, Example 4.3], and given at the right hand side of Table 5. Note that, using the notation in (3.2) and (3.3), we have $f'(1) \cong 1a$ and $f'(7) \cong 1b$, while $\{1c, 1d\}$ are now distinguished by fixing the embedding $H' < G'$ and specifying f'. Nowadays it is easy to verify this independently, by computing the above Green correspondents explicitly, using GAP and the MeatAxe. Moreover, in the spirit of the present paper, here we already encounter a nice toy application of the Brauer construction, which we cannot resist to include:

We consider the natural permutation action of G' on $\Omega := \{1, \ldots, 8\}$, hence $k[\Omega] \cong 1 \oplus 7$ as $k[G']$-modules. We may assume that $P := \langle (1,2,3), (4,5,6) \rangle <$ $\mathrm{Stab}_{G'}(8) \cong \mathfrak{A}_7$, hence $\Omega^P = \{7, 8\}$. Since $P < \mathfrak{A}_7 < G'$ is a Sylow 3-subgroup, by (2.2) and (2.3) we get $k[\Omega](P) \cong k[\Omega^P] \cong k[N_{\mathfrak{A}_7}(P) \backslash H']$ as $k[H']$-modules. Hence from $P \colon 4 \cong N_{\mathfrak{A}_7}(P) \lhd H'$ we infer

$$k[\Omega](P) \cong k[H'/(P \colon 4)] \cong k[D_8/C_4] \cong 1a \oplus 1b.$$

Thus, it follows from (2.4) that $f'(1) \oplus f'(7) \cong 1a \oplus 1b$ as $k[H']$-modules, and since $f'(1) \cong 1a$ anyway, we get $f'(7) \cong 1b$.

(4.2) Equivalence between A and B. To relate the blocks A and B we work through the Fong-Reynolds correspondents B_i, taking both of them into account: By [15, Lemma 2.4], there is a unique indecomposable direct summand $\mathcal{M}_i$ of the kG-kH-bimodule $1_A(kG)1_{B_i}$ with vertex $\Delta P < G \times H$, which by [15, Lemma 6.3(i)] induces a splendid stable equivalence of Morita type $? \otimes_{kG} \mathcal{M}_i \colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}B_i$, which we abbreviate by writing $\mathcal{M}_i$.

Moreover, arguing as in [14, Theorem 1.5], it follows from Fong-Reynolds correspondence, see [27, Theorem 5.5.10], and [4, Theorem 0.2] that the kN-kH-bimodule $\mathcal{N}_i := 1_B(kN)1_{B_i}$ and its dual, the kH-kN-bimodule $\mathcal{N}_i^\vee :=$ $1_{B_i}(kN)1_B$, induce a pair of mutually inverse Puig equivalences given by $? \otimes_{kN}$ $\mathcal{N}_i \colon \mathbf{mod}\text{-}B \to \mathbf{mod}\text{-}B_i \colon V \mapsto \mathrm{Res}_H^N(V) \cdot 1_{B_i}$ and $? \otimes_{kH} \mathcal{N}_i^\vee \colon \mathbf{mod}\text{-}B_i \to$ $\mathbf{mod}\text{-}B \colon V \mapsto \mathrm{Ind}_H^N(V)$, which we abbreviate by $\mathcal{N}_i$ and $\mathcal{N}_i^\vee$, respectively. Hence by concatenation we get a splendid stable equivalence of Morita type $\mathcal{N}_i^\vee \circ \mathcal{M}_i \colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}B$, again mapping any simple A-module to an indecomposable B-module.

We are now able to relate $\mathcal{N}_i^\vee \circ \mathcal{M}_i$ to the Green correspondence f with respect to the triple (G, P, N): The functor $\mathcal{N}_i^\vee \circ \mathcal{M}_i$ coincides with the tensor functor $? \otimes_{kG} (\mathcal{M}_i \otimes_{kH} \mathcal{N}_i^\vee) \colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}B$, where the kG-kN-bimodule $\mathcal{M}_i \otimes_{kH} \mathcal{N}_i^\vee$ is a direct summand of $1_A(kG \otimes_{kH} kN)1_B \cong 1_A \left(\bigoplus_{i=1}^{[N \colon H]} kG \right) 1_B$. Hence from [17, Lemma A.3] we infer that, for any indecomposable A-module V having P as a vertex, the unique non-projective indecomposable summand of $\mathcal{N}_i^\vee(\mathcal{M}_i(V))$ coincides with the Green correspondent $f(V)$; see [15, Lemma 6.3(iii)].

Thus, recalling that, by [10, Corollary 3.7], any simple A-module S indeed has P as a vertex, we conclude that $\mathcal{N}_i^\vee(\mathcal{M}_i(S)) \cong f(S)$, in particular saying that $\mathcal{N}_1^\vee \circ \mathcal{M}_1$ and $\mathcal{N}_2^\vee \circ \mathcal{M}_2$ coincide on simple A-modules. Hence $(\mathcal{N}_2^\vee \circ \mathcal{M}_2)^{-1} \circ (\mathcal{N}_1^\vee \circ \mathcal{M}_1)$ is a stable auto-equivalence of $\mathbf{mod}$-A, which is the identity on simple A-modules, thus by [20, Theorem 2.1(iii)] is equivalent to the identity functor on $\mathbf{mod}$-A. Hence we have $\mathcal{M} := \mathcal{N}_1^\vee \circ \mathcal{M}_1 \cong \mathcal{N}_2^\vee \circ \mathcal{M}_2 \colon \mathbf{mod}$-$A \to \mathbf{mod}$-$B$.

From the equivalence $\mathcal{M}_i \cong \mathcal{N}_i \circ \mathcal{M}$ we get $\mathcal{M}_i(S) \cong \mathrm{Res}_H^N(f(S)) \cdot 1_{B_i}$, for any simple A-module S. Moreover, by [15, Lemmas 6.7–6.10] the modules $\mathcal{M}(S)$ and $\mathcal{M}_i(S)$ are as given in Table 5, where $\{\alpha, \beta, \gamma, \delta\} = \{a, b, c, d\}$. Hence it remains to connect the left and right hand sides of Table 5:

(4.3) Equivalence between B_i and B'. Similar to the equivalence between B and B_i above, we now consider the kH-kH'-bimodule $\mathcal{L}_i := 1_{B_i}(kH)1_{B'}$ and its dual, the kH'-kH-bimodule $\mathcal{L}_i^\vee := 1_{B'}(kH)1_B$. Then it follows from (3.7) and [4, Theorem 0.2] that $? \otimes_{kH} \mathcal{L}_i \colon \mathbf{mod}$-$B_i \to \mathbf{mod}$-$B' \colon V \mapsto \mathrm{Res}_{H'}^H(V)$, and $? \otimes_{kH'} \mathcal{L}_i^\vee \colon \mathbf{mod}$-$B' \to \mathbf{mod}$-$B_i \colon V \mapsto \mathrm{Ind}_{H'}^H(V) \cdot 1_{B_i}$ are a pair of mutually inverse Puig equivalences, which we abbreviate by $\mathcal{L}_i$ and $\mathcal{L}_i^\vee$, respectively.

By [15, Lemma 2.8], tensoring with a linear modules induces a Puig auto-equivalences $? \otimes_k 1z \colon \mathbf{mod}$-$B' \to \mathbf{mod}$-$B' \colon V \mapsto V \otimes 1z$, for $z \in \{a, b, c, d\}$. Here, the trivial module $1a$ induces the identity functor on $\mathbf{mod}$-B', while, the group of linear characters of $H' \cong P \colon D_8$ being isomorphic to $H'/[H', H'] \cong C_2 \times C_2$, for $z \neq a$ we get non-trivial involutory auto-equivalences. This yields Puig equivalences $\mathcal{L}_i^z \colon \mathbf{mod}$-$B_i \to \mathbf{mod}$-$B' \colon V \mapsto \mathrm{Res}_{H'}^H(V) \otimes_k 1z$, where of course $\mathcal{L}_i \cong \mathcal{L}_i^a$; in particular we get Puig equivalences $\mathcal{L}_i^z \circ \mathcal{N}_i \colon \mathbf{mod}$-$B \to \mathbf{mod}$-$B'$.

Moreover, twisting with the non-inner automorphism $\omega \in \mathrm{Aut}(H')$, see (3.3), induces an involutory Morita auto-equivalence $\mathcal{W} \colon \mathbf{mod}$-$B' \to \mathbf{mod}$-$B' \colon V \mapsto V^\omega$. Since applying ω changes the embedding of P into H', this the functor $\mathcal{W}$ is *not* a Puig auto-equivalence; see [15, Lemma 6.12].

Hence, to complete the picture in Table 5, we apply $\mathcal{L}_i^\alpha$, where $1\alpha := \mathrm{Res}_{H'}^H(1\alpha_i)$; note that despite notation $\alpha \in \{a, b, c, d\}$ depends on $i \in \{1, 2\}$. Then we get $\mathcal{L}_i^\alpha(1\alpha_i) = 1a$, and hence it follows from [15, Lemmas 6.8 and 6.10] that $\mathcal{L}_i^\alpha(1\beta_i) = 1b$. This implies $\mathcal{L}_i^\alpha(\{1\gamma_i, 1\delta_i\}) = \{1c, 1d\}$, where $1d \cong (1c)^\omega$. Thus we get a stable equivalence of Morita type

$$\mathcal{F}_i := \mathcal{M}'^{-1} \circ \mathcal{W}^\epsilon \circ \mathcal{L}_i^\alpha \circ \mathcal{N}_i \circ \mathcal{M} \colon \mathbf{mod}\text{-}A \to \mathbf{mod}\text{-}A',$$

which for an appropriate choice of $\epsilon \in \{0, 1\}$, again depending on $i \in \{1, 2\}$, maps simple A-modules to simple A'-modules, hence by [20, Theorem 2.1(iii)] is an equivalence; note that, by [15, Lemma 4.6.(iii)], the uniserial modules appearing in Table 5 are uniquely determined up to isomorphism by their radical series.

Moreover, by construction, $\mathcal{F}_i$ is a splendid equivalence, that is a Puig equivalence, if and only if $\epsilon = 0$. Thus, in our setting, the question left open in [15, Question 6.14] can be reformulated as follows: Is $\epsilon = 0$ or $\epsilon = 1$? Recall that we still have two cases $i \in \{1, 2\}$ at our disposal. Since there does not seem to be a way to answer this by abstract theory alone, we force a decision by explicit computation, which finally is our main application of the Brauer construction:

Table 5: Morita equivalence between A and A'

mod-A	$\xrightarrow{\mathcal{M}}$	**mod-**B	$\xrightarrow{\mathcal{N}_i}$	**mod-**B_i	$\xrightarrow{\mathcal{L}_i^{\alpha}}$	**mod-**B'	$\xleftarrow{\mathcal{M}'}$	**mod-**A'
S_1	$\mapsto$	6α	$\mapsto$	$1\alpha_i$	$\mapsto$	$1a$	$\leftarrow\!\shortmid$	1
S_3	$\mapsto$	6β	$\mapsto$	$1\beta_i$	$\mapsto$	$1b$	$\leftarrow\!\shortmid$	7
S_5	$\mapsto$	$\begin{array}{c} 6\delta \\ 12 \\ 6\delta \end{array}$	$\mapsto$	$\begin{array}{c} 1\delta_i \\ 2 \\ 1\delta_i \end{array}$	$\xmapsto{\mathcal{W}^{\epsilon}}$	$\begin{array}{c} 1d \\ 2 \\ 1d \end{array}$	$\leftarrow\!\shortmid$	13
S_2	$\mapsto$	6γ	$\mapsto$	$1\gamma_i$	$\xmapsto{\mathcal{W}^{\epsilon}}$	$1c$	$\leftarrow\!\shortmid$	28
S_4	$\mapsto$	12	$\mapsto$	2_i	$\mapsto$	2	$\leftarrow\!\shortmid$	35

(4.4) Applying the Brauer construction. We proceed to determine the Green correspondents $f(S_1) \cong \mathcal{M}(S_1) \cong 6\alpha$ and $f(S_2) \cong \mathcal{M}(S_2) \cong 6\gamma$ of S_1 and S_2. Recall that S_1 and S_2 are of dimension $4\,290\,927$ and $95\,288\,172$, respectively, hence we cannot possibly simply proceed as in (4.1). Instead, we consider the G-action on the set $\Omega := M\backslash G$ of cosets of M in G, which has cardinality $173\,067\,389$:

Let $1_M^G := \mathrm{Ind}_M^G(1_M)$ denote the permutation character of the G-action on Ω. Using the ordinary character tables of M and G, contained in the character table library of GAP, it turns out that the component of 1_M^G belonging to the block A is given as $(1_M^G) \cdot 1_A = \chi_{14} + \chi_{21}$; see [15, Lemma 3.12]. Thus, using the decomposition matrix of A, see (3.1), we get $k[\Omega] \cdot 1_A \cong S_1 \oplus S_2$ as kG-modules, hence both S_1 and S_2 are trivial source modules; see [15, Lemma 3.13]. Moreover, by (2.4) we have

$$f(S_1) \oplus f(S_2) \cong S_1(P) \oplus S_2(P) \mid k[\Omega](P).$$

Hence, by (2.2) we proceed to determine $k[\Omega]^P$ and its structure as a permutation kN-module. In order to apply (2.3), we have to find a set of representatives of the M-conjugacy classes of subgroups of M being G-conjugate to P:

Using GAP, a Sylow 3-subgroup of M, being isomorphic to an extraspecial group 3_+^{1+2}, and from that the conjugacy classes of subgroups of M isomorphic to $C_3 \times C_3$, can be determined. It turns out that M has precisely two such conjugacy classes. One of them of course containing P, let $\widetilde{P} < M$ be a representative of the other conjugacy class. Noting that $k[N_M(P)\backslash N_G(P)] = k[N\backslash N] \cong k$ is the trivial kN-module, from (2.3) we thus get, as permutation kN-modules,

$$k[\Omega^P] \cong k \oplus k[N_{M^g}(P)\backslash N] \cong k \oplus k[N_M(\widetilde{P})\backslash N_G(\widetilde{P})]^g,$$

for some $g \in G - M$ such that $\widetilde{P}^g = P$. Note that within our setting, only allowing to compute efficiently in M, we are not able to get hands on a conjugating element easily; hence we circumvent an explicit choice of such an element:

To determine the action of N on the set $N_{M^g}(P)\backslash N$, up to equivalence of permutation actions, it suffices to find a subgroup of N which is N-conjugate

to $N_{M^g}(P)$. To this end, employing GAP again, we first compute $N_M(\widetilde{P})$ as a subgroup of M. It turns out that we have $N_M(\widetilde{P}) \cong 2 \times (P\colon D_{12})$, where $D_{12} \cong 3\colon 2^2$ can be identified with a Borel subgroup of $\mathrm{GL}_2(3) \cong \mathrm{Aut}(P)$, and using the group library of GAP we get $N_M(\widetilde{P}) \cong \mathsf{SmallGroup}(216, 102)$. Now we have $N_M(\widetilde{P}) \cong N_{M^g}(P) < N$, and GAP shows that N has a unique conjugacy class of subgroups isomorphic to $\mathsf{SmallGroup}(216, 102)$. Hence letting $\widetilde{N} < N$ be a representative of this conjugacy class, we have $k[N_{M^g}(P)\backslash N] \cong k[\widetilde{N}\backslash N]$ as permutation kN-modules, and thus $k[\Omega^P] \cong k \oplus k[\widetilde{N}\backslash N]$.

Note that, as far as we see, we are just lucky here: If there were several conjugacy classes of subgroups of N of the above isomorphism type, then it might become necessary to construct a conjugating element in $G - M$ explicitly. Anyway, although we have started with the huge G-permutation domain Ω, we have now managed to reduce the problem to a consideration of the tiny N-permutation domain $\widetilde{N}\backslash N$; note that $[N\colon \widetilde{N}] = 16$:

(4.5) Conclusion. Using GAP, we determine the permutation action of N on the cosets of $\widetilde{N}$. Then it is straightforward, using the MeatAxe, to find the constituents of the permutation module $k[\widetilde{N}\backslash N]$. It turns out that there are precisely two constituents of dimension 6, 6_I and 6_{II} say, each occurring with multiplicity one. Hence by (2.2) and (2.3) we have $\{f(S_1), f(S_2)\} = \{6_I, 6_{II}\}$. Applying $\mathcal{L}_1 \circ \mathcal{N}_1$ and $\mathcal{L}_2 \circ \mathcal{N}_2$ to 6_I and 6_{II}, the MeatAxe shows that

$$\mathrm{Res}_{H'}^{H}(\mathrm{Res}_H^N(6_I) \cdot (1_{B_1} + 1_{B_2})) \quad \cong \quad 1a \oplus 1a,$$
$$\mathrm{Res}_{H'}^{H}(\mathrm{Res}_H^N(6_{II}) \cdot (1_{B_1} + 1_{B_2})) \quad \cong \quad 1c \oplus 1d.$$

Hence we may assume that $\mathrm{Res}_{H'}^{H}(\mathrm{Res}_H^N(6_{II}) \cdot 1_{B_1}) \cong 1c$.

Thus, using the notation from (4.3), the functor $\mathcal{F}_1$ is equipped with the parameter $\alpha = a$ or $\alpha = c$, depending on whether $f(S_1) \cong 6_I$ or $f(S_1) \cong 6_{II}$, where in both cases we have $\epsilon = 0$, thus $\mathcal{F}_1$ is a Puig equivalence. With respect to the same case distinction, $\mathcal{F}_2$ is equipped with the parameter $\alpha = a$ or $\alpha = d$, where in both cases we have $\epsilon = 1$, hence $\mathcal{F}_2$ is *not* a Puig equivalence. Anyway, this proves Theorem (1.2).

(4.6) Conjecture/Question. Finally, we conjecture that, due to our fixed subgroup configuration, we have $f(S_1) \cong 6_I$, that is $\mathrm{Res}_{H'}^{H}(\mathrm{Res}_H^N(f(S_1)) \cdot 1_{B_i}) \cong 1a$ for both $i \in \{1, 2\}$. Again this cannot be answered by abstract theory alone: Indeed, GAP shows that H has four conjugacy classes of subgroups isomorphic to $P\colon D_8$, all of which supplement C, hence each can be used to parametrise the characters of B_i along the lines of (3.7), but the resulting parametrisations are distinct. Unfortunately, our explicit computations, as far as we have pursued them, do not tell us either.

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