On low-degree representations of the symmetric group

Jürgen Müller

Abstract

The aim of the present paper is to obtain a classification of all the irreducible modular representations of the symmetric group on n letters of dimension at most n^3, including dimension formulae. This is achieved by improving an idea, originally due to G. James, to get hands on dimension bounds, by building on the current knowledge about decomposition numbers of symmetric groups and their associated Iwahori-Hecke algebras, and by employing a mixture of theory and computation.

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1 Introduction

It is a major open problem in representation theory of finite groups to understand the irreducible modular representations of the symmetric group $\mathcal{S}_n$ on n letters. Although considerable progress has been made in recent decades, not even their dimensions are in general known today. The situation becomes somewhat better if we restrict ourselves to the representations whose dimensions are bounded above by a polynomial in n: The aim of the present paper is to obtain an explicit classification of all the irreducible modular representations of $\mathcal{S}_n$ of degree at most n^3, together with explicit degree formulae, the final result being described in (6.3), and explicit lists in characteristic $p \leq 7$ being given in the subsequent sections from (6.4) on.

Our original motivation to investigate into this was a particular question asked a while ago by E. O'Brien, in view of the, at that time forthcoming, paper [10]. Accordingly, we are proud to be able to say that the present results are now used significantly in [10, Sect.4]. From a broader perspective, the present paper lies well within the recent philosophy of collecting results on representations of low degree of finite simple groups and their close relatives, and aims at contributing to this programme; see for example [19, 37], where both viewpoints of either fixing a degree bound, or, for the case of groups of Lie type, allowing for a polynomial degree bound in terms of the Lie rank, are pursued.

In order to achieve the goal specified above, it has turned out that quite a few additional pieces of information have to be collected or newly derived, in particular by improving an idea, originally due to G. James, to get hands on dimension bounds, by building on the current knowledge about decomposition numbers of symmetric groups and their associated Iwahori-Hecke algebras, and by applying a mixture of theoretical reasoning and computational techniques.

• The starting point of our considerations is an observation due to G. James [20], describing the growth behavior of irreducible modular representations of the symmetric group $\mathcal{S}_n$ on n letters, when n tends to infinity. More precisely:

Given a rational prime p, let $d^\mu := \dim_{\mathbb{F}_p}(D^\mu)$ be the dimension of the irreducible $\mathcal{S}_n$-module D^μ parameterized by the p-regular partition $\mu = [\mu_1, \mu_2, \ldots]$ of n. Fixing a non-negative integer m, and assuming that the largest part of μ equals $\mu_1 = n - m$, it is shown in [20, Thm.1] that $d^\mu \sim \frac{n^m}{n!} \cdot d^{\overline{\mu}}$, for $n \gg 0$, where $\overline{\mu} = [\mu_2, \mu_3, \ldots]$, a partition of $n - m$. In particular, by [20, Cor.2], d^μ always is bounded above by $d^\mu \leq n^m$, while there only is an asymptotical lower bound $n^{m-1} < d^\mu$ for $n \gg 0$. Since for small n there are cases such that the latter inequality fails, the question arises whether there is an explicit bound n_0 such that $n^{m-1} < d^\mu$ for all $n \geq n_0$.

A general strategy to find effective lower bound functions $f(n)$, fulfilling $f(n) \leq d^\mu$ for all $n \geq n_0$ and some explicitly given n_0, is already introduced and used in the proofs of [20, Thm.5] and its key Lemma [20, La.4]. The major ingredients to these proofs are the branching rule for irreducible ordinary representations of $\mathcal{S}_n$, see for example [22, Thm.9.2], the hook length formula [16] for the degree of irreducible ordinary representations, see also [22, Thm.20.1], and the unitriangularity of the decomposition matrix of $\mathcal{S}_n$. As main theoretical results of the present paper we are going to derive two improvements of [20, La.4]:

• The first main result, being valid for arbitrary rational primes p, is given in (5.2), whose improvement consists of weaker conditions imposed on candidate lower bound functions $f(n)$; it is detailed in (5.3) how this compares to [20, La.4]. In practice, it turns out that the weaker checks necessary save quite a bit of explicit computation, and lends itself to the treatment independent of p carried out in (6.2). The new ingredient to the proof is to replace the ordinary branching rule by the modular branching rule, which is recalled in (2.4); of course, the latter had not been available at the time of writing of [20].

Although the above theorem also holds for $p = 2$, in this case it would not be strong enough for our purposes, the reason being the existence of 'very small' representations escaping the desired growth behavior in low degrees. The representations to be treated separately are the first and second basic spin representations, whose branching behavior is described (5.5), where we also give degree formulae. This leads to our second main result, given in (5.6), which holds only for $p = 2$ but takes care of these exceptional cases. The strategy of proof is reminiscent of [8, Thm.4.3], but again we get away with weaker conditions to be imposed on candidate lower bound functions f, as is detailed in (5.7), which again in practice saves quite a bit of explicit computation.

Actually, we manage to prove both (5.2) and (5.6) in a conceptual and unified manner. In particular, for our major case of interest $m = 3$, in the cases $p = 3$ and $p = 2$ the improvements lead to shorter proofs of [8, Prop.3.1] and

[8, Prop.4.4], together with smaller lower bounds n_0, where regular behavior sets in. Moreover, in the proofs of (5.2) and (5.6) Jantzen-Seitz partitions [28] feature prominently, where an answer to the following problem would yield a small further improvement to (5.2), as is detailed in (5.4):

Open problem: *Classify the Jantzen-Seitz partitions whose largest part occurs at least twice; in particular show that there are none for* $p = 3$.

• The above discussion leads to the question of how to find candidate lower degree functions $f(n)$ in the first place. In our main case of interest $m = 3$, due to G. James's observations [20], it seems reasonable to take as a candidate function $f(n)$ a lower bound function for the degrees d^μ, where μ runs through all p-regular partitions of n with largest part $\mu_1 = n - (m+1) = n - 4$.

Best possible polynomial lower bound functions for the degrees d^μ, where μ runs through all p-regular partitions of n with largest part at least $n-4$, are collected in (6.1). For the cases $p \geq 5$ these follow immediately from the degree formulae given in [7, La.1.21], while the cases $p = 3$ and $p = 2$, which are of most interest for us, are spared there. Actually, lower bound functions, for all $p \geq 2$, can also be derived from a close inspection of the results given without proof in [20, App.], but the degree formulae given there are far from being complete.

Since, apart from the absence of explicit proofs, for our ultimate classification aim we need degree formulae anyway, we are deriving these for the missing cases $p = 3$ and $p = 2$, and all the partitions μ mentioned above, in (3.3) and (4.3), respectively. As it turns out, to collect complete and precise data, but not just estimates, it is necessary to determine most (if not all) of the rows of the decomposition matrix of $\mathcal{S}_n$ being parameterized by partitions λ of n with largest part $\lambda_1 \geq n - 4$. To our knowledge, sufficiently far reaching complete results of this generic kind, where the partitions considered are restricted to interesting classes, but n is treated as a parameter, have not been determined so far. Prototypical computations in this direction are done in [26], from which we in particular use the observation [26, Prop.2.1], being recalled in (2.5), several times, in order to find upper bounds for certain decomposition numbers.

To make our approach to finding the relevant decomposition numbers of $\mathcal{S}_n$ as conceptual as possible, we first determine the corresponding part of the crystallized decomposition matrix of the generic Iwahori-Hecke algebra $\mathcal{H}_n(u)$ associated with $\mathcal{S}_n$, with respect to specializing the indeterminate parameter u to a p-th complex root of unity; the necessary background on the relationship between these decomposition matrices is recalled in (2.6). Crystallized decomposition matrices can be computed recursively by applying the LLT algorithm [36], which is recalled in (2.7). We use a modification of it, as detailed in (2.8):

Let the Fock space $\mathcal{F}$ be the free abelian group having a basis consisting of all partitions of all non-negative integers n, and let $\mathcal{F}^{>m}$ be the subgroup spanned by the partitions whose largest part is smaller than $n - m$, where m is a fixed non-negative integer. Then the truncated Fock space $\mathcal{F}(\leq m) := \mathcal{F}/\mathcal{F}^{>m}$ has a basis consisting of the partitions with largest part at least $n - m$, and thus captures precisely the part of the decomposition matrix we are interested in. As it turns out, instead of running the LLT algorithm in $\mathcal{F}$ and subsequently projecting onto $\mathcal{F}(\leq m)$, a modified version of it can just be applied directly to $\mathcal{F}(\leq m)$. Moreover, $\mathcal{F}(\leq m)$ has components being parameterized by n, which for n large enough are all naturally parameterized by the partitions of the numbers $1, \ldots, m$. Hence it is conceivable that computations can be performed generically, with n as a parameter, in terms of the latter partitions.

Indeed, this is successfully carried out in (3.1) and (4.1) for $m = 4$, and the cases $p = 3$ and $p = 2$, respectively. Then, in (3.2) and (4.2), respectively, the part of the crystallized decomposition matrix found is used, together with *ad hoc* arguments from the toolbox of modular representation theory of symmetric groups, to determine the corresponding part of the decomposition matrix of $\mathcal{S}_n$, Finally, in (3.3) and (4.3), respectively, degree formulae are collected. Moreover, the observations made for the cases treated here lead us to ask the following question, which is discussed in some more detail in (2.9):

Open problem: *Show that the crystallized decomposition matrix corresponding to the n-th component of $\mathcal{F}(\leq m)$ depends only on the congruence class of n modulo p, whenever $n \geq 2m + 1$.*

• Having this theoretical machinery in place, we are finally prepared to tackle our original problem of classifying the p-regular partitions μ of n such that $d^\mu \leq n^3$: As already indicated above, as a test function to employ (5.2) and (5.6), we essentially take the lower bound function for the degrees d^μ where μ has largest part $n - 4$ given in (6.1). Actually, as detailed in (6.2), in order to obtain a description independent of p, and a strong bound for n_0, in low degrees we are using a suitable modification of the natural lower bound function.

This covers the generic behavior, and leaves finitely many n to be considered separately. The computational strategy used to do so, as automatic as possible for fixed p, is described in (6.3). We are employing the computer algebra system GAP [17] as our major computational environment. Here, we also use explicit data on decomposition numbers of symmetric groups accessible in the character table database CTblLib [5] of GAP, and on the web page [49] of the ModularAtlas project [27]. An implementation of the LLT algorithm is provided in the SPECHT package [39], which is available through the computer algebra system CHEVIE [41]. The explicit classification thus obtained for the cases $p = 3$ and $p = 2$ is given in (6.5) and (6.4), respectively.

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2 Background

We assume the reader familiar with ordinary and modular representation theory of symmetric groups, as well as with the representation theory of Iwahori-Hecke algebras; as general references see [22, 25] and [40], respectively. We recall the necessary facts, where in particular we take this opportunity to fix notation.

(2.1) Partitions. For $n \in \mathbb{N}_0$ let $\mathcal{P}_n$ denote the set of all partitions $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_l]$ of n, where we assume $\lambda_1 \geq \cdots \geq \lambda_l > 0$ and $l = l(\lambda) \in \mathbb{N}_0$ is the length of λ; for convenience we let $\lambda_{l+1} := 0$. Moreover, given $\lambda \in \mathcal{P}_n$, whenever $n \geq 1$ let $\overline{\lambda} := [\lambda_2, \lambda_3, \ldots, \lambda_l] \in \mathcal{P}_{n-\lambda_1}$ be the partition obtained from λ by removing its first part; for $n = 0$ we let $\overline{[]} = [] \in \mathcal{P}_0$. Since a partition λ is uniquely determined, as soon as n is understood, by its tail part $\overline{\lambda}$, we may also write it as $(\overline{\lambda})$; here we are using round brackets to distinguish this from our standard square bracket notation for partitions. This convention will prove convenient in view of the following key combinatorial definition of this paper:

For $m \in \mathbb{N}_0$ let

$$\mathcal{P}_n(m) := \{\lambda \in \mathcal{P}_n; \lambda_1 = n - m\} = \{\lambda \in \mathcal{P}_n; \overline{\lambda} \in \mathcal{P}_m\} \subseteq \mathcal{P}_n$$

and $\mathcal{P}_n(\leq m) := \coprod_{j=0}^{m} \mathcal{P}_n(j) \subseteq \mathcal{P}_n$. Hence we get an injective map $\mathcal{P}_n(m) \to \mathcal{P}_m \colon \lambda \mapsto \overline{\lambda}$, where $\overline{\lambda} \in \mathcal{P}_m$ actually is a tail part of some partition in $\mathcal{P}_n$ if and only if $n - m \geq \overline{\lambda}_1$. In particular, we have $\mathcal{P}_n(m) = \emptyset$ whenever $n < m$ or $n = m > 0$, and the map $\mathcal{P}_n(m) \to \mathcal{P}_m$ is surjective if and only if $n \geq 2m$.

In the sequel, a particular role will be played by the sets $\mathcal{P}_n(m)$ for $m \leq 4$, the round bracket notation introduced above yielding a compact description independent of n: we have $\mathcal{P}_n(0) = \{()\}$ for $n \geq 0$, $\mathcal{P}_n(1) = \{(1)\}$ for $n \geq 2$, $\mathcal{P}_n(2) = \{(2), (1^2)\}$ for $n \geq 4$, $\mathcal{P}_n(3) = \{(3), (2,1), (1^3)\}$ for $n \geq 6$, and $\mathcal{P}_n(4) = \{(4), (3,1), (2^2), (2,1^2), (1^4)\}$ for $n \geq 8$.

• Given a rational prime p, let $\mathcal{P}_n^{p\text{-reg}} \subseteq \mathcal{P}_n$ be the set of all p-regular partitions, that is those for which any part occurs less than p-times, and let $\mathcal{P}_n^{p\text{-reg}}(m) := \mathcal{P}_n^{p\text{-reg}} \cap \mathcal{P}_n(m) \subseteq \mathcal{P}_n$ and $\mathcal{P}_n^{p\text{-reg}}(\leq m) := \mathcal{P}_n^{p\text{-reg}} \cap \mathcal{P}_n(\leq m) \subseteq \mathcal{P}_n$. As usual, we assume both $\mathcal{P}_n$ and $\mathcal{P}_n^{p\text{-reg}}$ to be ordered reversed lexicographically. Then the sets $\mathcal{P}_n(\leq m)$ and $\mathcal{P}_n^{p\text{-reg}}(\leq m)$ are order ideals in $\mathcal{P}_n$ and $\mathcal{P}_n^{p\text{-reg}}$, respectively.

Moreover, let $\mathcal{P}_n \to \mathcal{P}_n^{p\text{-reg}} \colon \lambda \mapsto \lambda^R$ be the p-regularisation map, see [25, 6.3.48]; of course, this map restricts to the identity map on $\mathcal{P}_n^{p\text{-reg}}$. Finally, let $\mathcal{P}_n^{p\text{-reg}} \to \mathcal{P}_n^{p\text{-reg}} \colon \mu \mapsto \mu^M$ be the Mullineux map, see [4, 43]; recall that this is the identity map in the case $p = 2$, and by [15] records the effect of tensoring irreducible p-modular representations with the sign representation.

(2.2) Young diagrams. As usual, we identify a partition $\lambda = [\lambda_1, \ldots, \lambda_l] \in \mathcal{P}_n$ with its Young diagram, that is an array of l rows of 'nodes', where row a, counting from top to bottom, contains λ_a nodes, counting from left to right.

The node being in row a and column b of a Young diagram is called its (a,b)-node. Given a rational prime p, the (a,b)-node is called an i-node, or is said to have p-residue $i \in \{0,\ldots,p-1\}$, if $i \equiv b-a \pmod{p}$.

The (a,λ_a)-node x, for some $a \in \{1,\ldots,l\}$, is called removable if $\lambda_a > \lambda_{a+1}$; in this case, we let $\lambda \setminus \{x\} \in \mathcal{P}_{n-1}$ be the partition obtained by removing x. Similarly, the (a,λ_a+1)-node y, for some $a \in \{1,\ldots,l+1\}$, is called addable if either $a=1$ or $\lambda_a < \lambda_{a-1}$; in this case, we let $\lambda \cup \{y\} \in \mathcal{P}_{n+1}$ be the partition obtained by adding y. Let $R_i(\lambda)$ and $A_i(\lambda)$ be the sets of removable and addable i-nodes of λ, respectively. If $x \in R_i(\lambda)$ or $x \in A_i(\lambda)$, then let $r_i(\lambda,x)$ and $a_i(\lambda,x)$ be the number of removable and addable i-nodes strictly to the right of x, respectively.

Let $\mu \in \mathcal{P}_n^{p\text{-reg}}$, and let $i \in \{0,\ldots,p-1\}$. Labelling each removable and addable i-node of μ by '$-$' and '$+$', respectively, and reading off these labels from left to right yields a sequence called the i-signature of μ, and repeatedly cancelling '$-+$' pairs in the sequence we end up with the reduced i-signature of μ. The removable i-nodes surviving in the reduced i-signature are called i-normal, and if there is an i-normal node then the leftmost one is called i-good. Let $N_i(\mu)$ be the set of i-normal nodes of μ; note that $1+r_i(\mu,x)-a_i(\mu,x)$ is the number of i-normal nodes to the right of $x \in N_i(\mu)$, including x. Similarly, the addable i-nodes surviving in the reduced i-signature are called i-conormal, and if there is an i-conormal node then the rightmost one is called i-cogood.

(2.3) Grothendieck groups. For $n \in \mathbb{N}_0$ let $G(\textbf{mod}\text{-}K\mathcal{S}_n)$, where K is a field, be the Grothendieck group of the category of finitely generated $K\mathcal{S}_n$-modules, the equivalence class of a module M being denoted by $[M]$.

$G(\textbf{mod}\text{-}\mathbb{Q}\mathcal{S}_n)$ has the classes of the (absolutely irreducible) Specht modules $\{[S^\lambda];\lambda \in \mathcal{P}_n\}$ as its standard $\mathbb{Z}$-basis, and thus can be naturally identified with the free abelian group $\mathbb{Z}[\mathcal{P}_n]$. Similarly, $G(\textbf{mod}\text{-}\mathbb{F}_p\mathcal{S}_n)$, where p is a rational prime, has the classes of the (absolutely) irreducible modules $\{[D^\lambda];\lambda \in \mathcal{P}_n^{p\text{-reg}}\}$ as its standard $\mathbb{Z}$-basis, and thus can be identified with the free abelian group $\mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}]$. As it turns out, this identification is natural as well:

Associated to this setting there is the decomposition map

$$\Delta_n^p \colon \mathbb{Z}[\mathcal{P}_n] \to \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}] \colon \lambda \mapsto \sum_{\mu \in \mathcal{P}_n^{p\text{-reg}}} [S^\lambda \colon D^\mu] \cdot \mu,$$

where the decomposition number $[S^\lambda \colon D^\mu] \in \mathbb{N}_0$ is the multiplicity of the constituent D^μ in a composition series of a p-modular reduction of S^λ. The map Δ_n^p has the following 'unitriangularity' properties: We have $[S^\lambda \colon D^\mu] > 0$ only if $\lambda \trianglelefteq \mu$, where '$\trianglelefteq$' denotes the dominance partial order of $\mathcal{P}_n$, and $[S^\mu \colon D^\mu] = 1$ for all $\mu \in \mathcal{P}_n^{p\text{-reg}}$. Hence this gives rise to maps $\Delta_n^p(\leq m) \colon \mathbb{Z}[\mathcal{P}_n(\leq m)] \to \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}(\leq m)]$. The matrices $\mathcal{D}_n$ and $\mathcal{D}_n(\leq m)$ of Δ_n^p and $\Delta_n^p(\leq m)$, respectively, with respect to reversed lexicographical ordering of partitions, are called the associated decomposition matrices.

Moreover, for $n \geq 1$, the ordinary branching rule for Specht modules, see for example [22, Thm.9.2], gives rise to i-restriction maps, where $i \in \{0, \ldots, p-1\}$,

$$\downarrow_i \colon \mathbb{Z}[\mathcal{P}_n] \to \mathbb{Z}[\mathcal{P}_{n-1}] \colon \lambda \mapsto \sum_{x \in R_i(\lambda)} \lambda \setminus \{x\}.$$

(Similarly, there are i-induction maps, but these we will not need.) By way of p-modular reduction this induces maps

$$\downarrow_i \colon \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}] \to \mathbb{Z}[\mathcal{P}_{n-1}^{p\text{-reg}}] \colon \mu \mapsto \sum_{\tau \in \mathcal{P}_{n-1}^{p\text{-reg}}} [D^\mu \downarrow_i \colon D^\tau] \cdot \tau,$$

where $[D^\mu \downarrow_i \colon D^\tau] \in \mathbb{N}_0$ is the multiplicity of the constituent D^τ in a composition series of $D^\mu \downarrow_i$. These maps are not at all well-understood, but are partly described by the following theorem, where we only state the facts needed later:

(2.4) Theorem: Modular branching rule; [31, 33], see also [6, 30].
Let $\mu \in \mathcal{P}_n^{p\text{-reg}}$, where $n \geq 1$, and let $i \in \{0, \ldots, p-1\}$.

- *We have $D^\mu \downarrow_i = \{0\}$ if and only if $N_i(\mu) = \emptyset$. Moreover, $D^\mu \downarrow_i$ is irreducible if and only if $N_i(\mu)$ is a singleton set.*
- *Let $x \in N_i(\mu)$ be an i-normal node such that $\mu \setminus \{x\} \in \mathcal{P}_{n-1}^{p\text{-reg}}$; note that this in particular holds if x is the i-good node. Then we have*

$$[D^\mu \downarrow_i \colon D^{\mu \setminus \{x\}}] = 1 + r_i(\mu, x) - a_i(\mu, x). \qquad \sharp$$

We recall a straightforward observation, relating decomposition numbers of $\mathcal{S}_n$ and $\mathcal{S}_{n-1}$ with constituent multiplicities of restrictions of irreducible $\mathcal{S}_n$-modules to $\mathcal{S}_{n-1}$. This actually proves to be powerful to find upper bounds for certain decomposition numbers by induction:

(2.5) Proposition: [26, Prop.2.1].
Let $\lambda \in \mathcal{P}_n$, where $n \geq 1$, and $\tau \in \mathcal{P}_{n-1}^{p\text{-reg}}$. Then for $i \in \{0, \ldots, p-1\}$ we have

$$\sum_{x \in R_i(\lambda)} [S^{\lambda \setminus \{x\}} \colon D^\tau] = \sum_{\mu \in \mathcal{P}_n^{p\text{-reg}}} [S^\lambda \colon D^\mu] \cdot [D^\mu \downarrow_i \colon D^\tau].$$

In particular, for any $\mu \in \mathcal{P}_n^{p\text{-reg}}$ we have

$$[S^\lambda \colon D^\mu] \cdot [D^\mu \downarrow_i \colon D^\tau] \leq \sum_{x \in R_i(\lambda)} [S^{\lambda \setminus \{x\}} \colon D^\tau],$$

where equality holds if and only if D^μ is the only constituent of S^λ such that $[D^\mu \downarrow_i \colon D^\tau] > 0$. $\sharp$

(2.6) Iwahori-Hecke algebras. In order to find lower bounds for decomposition numbers of $\mathcal{S}_n$ we make use of the generic Iwahori-Hecke algebra $\mathcal{H}_n(u)$ associated with $\mathcal{S}_n$. Here, the parameter u is an indeterminate, and $\mathcal{H}_n(u)$ is an algebra over the field $\mathbb{Q}(u)$. By Tits' Deformation Theorem, the (split semi-simple) module categories **mod**-$\mathbb{Q}(u)\mathcal{S}_n$ and **mod**-$\mathcal{H}_n(u)$ can be identified, hence their Grothendieck groups can be identified as well. More precisely, there again is a natural choice of Specht modules, see [9], which we again denote by S^λ, for $\lambda \in \mathcal{P}_n$. Next, we recall the necessary facts about the decomposition theory of $\mathcal{H}_n(u)$, for more details see [40, Ch.6.2]:

Specializing the parameter u to become a primitive p-th root of unity $\zeta := \zeta_p \in \mathbb{C}$ yields the $\mathbb{Q}(\zeta)$-algebra $\mathcal{H}_n(\zeta)$. Then $G(\mathbf{mod}\text{-}\mathcal{H}_n(\zeta))$ has the classes of the (absolutely) irreducible modules $\{[D_\zeta^\lambda]; \lambda \in \mathcal{P}_n^{p\text{-reg}}\}$ as its standard $\mathbb{Z}$-basis, and can also be identified naturally with the free abelian group $\mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}]$. Again there is an associated decomposition map

$$\Delta_n^\zeta \colon \mathbb{Z}[\mathcal{P}_n] \to \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}] \colon \lambda \mapsto \sum_{\mu \in \mathcal{P}_n^{p\text{-reg}}} [S^\lambda \colon D_\zeta^\mu] \cdot \mu,$$

where the decomposition number $[S^\lambda \colon D_\zeta^\mu] \in \mathbb{N}_0$ is the multiplicity of the constituent D_ζ^μ in a composition series of a ζ-modular reduction of S^λ. The map Δ_n^ζ enjoys the same unitriangularity properties as Δ_n^p, and thus similarly gives rise to decomposition matrices $\mathcal{D}_n^\zeta$ and $\mathcal{D}_n^\zeta(\leq m)$.

• Considering p-modular reduction again, which specializes the p-th root of unity $\zeta \in \mathbb{C}$ to $1 \in \mathbb{F}_p$, it turns out, see [18], that there is a decomposition map

$$A_n^\zeta \colon \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}] \to \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}] \colon \tau \mapsto \sum_{\mu \in \mathcal{P}_n^{p\text{-reg}}} [D_\zeta^\tau \colon D^\mu] \cdot \mu,$$

where the decomposition number $[D_\zeta^\tau \colon D^\mu] \in \mathbb{N}_0$ is the multiplicity of the constituent D^μ in a composition series of a p-modular reduction of D_ζ^τ. Then we have $\Delta_n^p = \Delta_n^\zeta \cdot A_n^\zeta$, which implies that A_n^ζ also has the celebrated unitriangularity properties, giving rise to maps $A_n^\zeta(\leq m) \colon \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}(\leq m)] \to \mathbb{Z}[\mathcal{P}_n^{p\text{-reg}}(\leq m)]$. The matrices $\mathcal{A}_n$ and $\mathcal{A}_n(\leq m)$ associated with A_n^ζ and $A_n^\zeta(\leq m)$, respectively, being square lower unitriangular matrices, are called adjustment matrices.

In particular, from this we indeed get lower bounds for decomposition numbers: we have $[S^\lambda \colon D^\mu] \geq [S^\lambda \colon D_\zeta^\mu]$, for all $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{P}_n^{p\text{-reg}}$; or phrased in terms of dimensions we get $\dim_{\mathbb{F}_p}(D^\mu) \leq \dim_{\mathbb{Q}(\zeta)}(D_\zeta^\mu)$. In the cases of small m we are interested in, it eventually turns out that the adjustment matrices $\mathcal{A}_n(\leq m)$ actually are sparse with very small entries, so that the above inequalities tend to provide good lower bounds. Hence to make this approach effective, we need to get hands on the matrices $\mathcal{D}_n^\zeta$. This is actually achieved as follows:

(2.7) The LLT algorithm. We proceed towards a recursive description of $\mathcal{D}_n^\zeta$, where we only describe the facts necessary; for more details see [40, Ch.6.1]:

We consider the Fock space $\mathcal{F} := \bigoplus_{n\in\mathbb{N}_0} \mathbb{Z}[\mathcal{P}_n]$. Let $\mathcal{F}_q := \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathcal{F}$, where q is an indeterminate. For $i \in \{0, \ldots, p-1\}$ running through the residue classes modulo p and $k \in \mathbb{N}_0$, in view of [40, La.6.16] we define $\mathbb{Z}[q, q^{-1}]$-linear maps by

$$F_i^{(k)} \colon \mathcal{F}_q \to \mathcal{F}_q \colon \lambda \mapsto \sum_{X\subseteq A_i(\lambda);|X|=k} q^{\sum_{x\in X} a_i(\lambda\cup X,x)-r_i(\lambda,x)} \cdot (\lambda \cup X),$$

where we let $F_i := F_i^{(1)}$; note that $F_i^{(0)} = \mathrm{id}_{\mathcal{F}_q}$. Let $\mathcal{U}$ be the $\mathbb{Z}[q, q^{-1}]$-algebra generated by the divided power operators $\{F_i^{(k)}; i \in \{0, \ldots, p-1\}; k \in \mathbb{N}_0\}$, and let $^-$ be the $\mathbb{Z}$-linear ring involution of $\mathcal{U}$ defined by $\overline{q} := q^{-1}$ and $\overline{F_i^{(k)}} := F_i^{(k)}$. Moreover, let $\mathcal{L} \leq \mathcal{F}_q$ be the $\mathcal{U}$-submodule of $\mathcal{F}_q$ generated by the empty partition $[] \in \mathcal{P}_0$, and let $^-$ be the $\mathbb{Z}$-linear involution on $\mathcal{L}$ defined by $\overline{[] \cdot u} := [] \cdot \overline{u}$, for all $u \in \mathcal{U}$. Then, from [29], there exists a unique $\mathbb{Z}[q, q^{-1}]$-basis $\{B^\mu \in \mathcal{F}_q; \mu \in \coprod_{n\in\mathbb{N}_0} \mathcal{P}_n^{p\text{-reg}}\}$ of $\mathcal{L}$, being called its global crystal basis, such that $B^\mu - \mu \in q\mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathcal{F} \subseteq \mathcal{F}_q$ and $\overline{B^\mu} = B^\mu$.

Actually we have $B^\mu \in \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathcal{P}_n]$ whenever $\mu \in \mathcal{P}_n^{p\text{-reg}}$. Writing $B^\mu = \sum_{\lambda\in\mathcal{P}_n} b_{\lambda,\mu}(q) \cdot \lambda$ in terms of the standard basis $\mathcal{P}_n$, where $b_{\lambda,\mu}(q) \in \mathbb{Z}[q, q^{-1}]$ such that $b_{\mu,\mu}(q) = 1$, gives rise to the crystallized decomposition matrices $\mathcal{D}_n^q$. These again, by [36], share the celebrated unitriangularity properties, and by [47] fulfill the positivity condition $b_{\lambda,\mu}(q) \in q\mathbb{N}_0[q]$ whenever $\lambda \lhd \mu$. In particular, restricting to the sets $\mathcal{P}_n(\leq m)$ and $\mathcal{P}_n^{p\text{-reg}}(\leq m)$, we also get crystallized decomposition matrices $\mathcal{D}_n^q(\leq m)$. Moreover, as was conjectured in [36] and proved in [1], the desired connection to the decomposition numbers $[S^\lambda \colon D_\zeta^\mu]$ is simply given by $[S^\lambda \colon D_\zeta^\mu] = b_{\lambda,\mu}(1)$, in other words the matrices $\mathcal{D}_n^\zeta$ are obtained by evaluating the crystallized decomposition matrices $\mathcal{D}_n^q$ at $q = 1$.

• The elements $B^\mu \in \mathcal{F}_q$, for $\mu \in \mathcal{P}_n^{p\text{-reg}}$, are found recursively by the following algorithm [36]: For the empty partition $[] \in \mathcal{P}_0$ we have $B^{[]} = [] \in \mathcal{F}_q$, hence we may assume that $n \geq 1$. We consider the sequence of p-ladders, in the sense of [25, 6.3.45], running from left to right, and starting with the 0-th ladder passing through the $(0, 0)$-node; then the j-th ladder, where $j \in \mathbb{N}_0$, consists of nodes of residue $i \equiv p - j \pmod{p}$. Given $\mu \in \mathcal{P}_n^{p\text{-reg}}$, recording the number of nodes of μ lying on the various p-ladders, yields a sequence $[k_0, \ldots, k_r]$, where $r \in \mathbb{N}$ and $k_i \in \mathbb{N}_0$ such that $k_r \geq 1$ and $\sum_{i=1}^r k_i = n$. Then let

$$F^\mu := F_0^{(k_0)} F_{p-1}^{(k_1)} F_{p-2}^{(k_2)} \cdots F_{p-r}^{(k_r)} \in \mathcal{U},$$

be the associated product of divided power operators, where subscripts are read modulo p. Applying it, from left to right, to the empty partition $[] \in \mathcal{P}_0$ yields $A^\mu := [] \cdot F^\mu \in \mathcal{F}_q$. Using the condition $B^\mu - \mu \in q\mathbb{Z}[q] \otimes_{\mathbb{Z}} \mathcal{F}$, it turns out that $A^\mu = B^\mu + \sum_{\tau\in\mathcal{P}_n^{p\text{-reg}};\tau\lhd\mu} a_{\tau,\mu}(q) \cdot B^\tau \in \mathcal{F}_q$, where $a_{\tau,\mu}(q) \in \mathbb{Z}[q, q^{-1}]$. Thus we have $A^\mu = B^\mu$ whenever μ is smallest in $\mathcal{P}_n^{p\text{-reg}}$ with respect to the dominance partial order, and otherwise the $a_{\tau,\mu}(q)$ are uniquely determined recursively using the elements B^τ for $\tau \in \mathcal{P}_n^{p\text{-reg}}$ such that $\tau \lhd \mu$, and the invariance property $\overline{B^\mu} = B^\mu$.

• In the sequel, we will also need the following alternative description, taken from [40, Sect.6.25], saying that we may also use induction on $n \in \mathbb{N}_0$ as follows: Since $B^{[]} = [] \in \mathcal{F}_q$ anyway, we may assume that $n \geq 1$. Let X be the intersection of the r-th p-ladder with μ, that is the rightmost one intersecting μ non-trivially, hence we have $\mu \backslash X \in \mathcal{P}_{n-k_r}^{p\text{-reg}}$. By induction we may let $A'^{\mu} := B^{\mu \backslash X} \cdot F_i^{(k)} \in \mathcal{F}_q$. Then we have $A'^{\mu} = B^{\mu} + \sum_{\tau \in \mathcal{P}_n^{p\text{-reg}}; \tau \lhd \mu} a'_{\tau,\mu}(q) \cdot B^{\tau} \in \mathcal{F}_q$, where $a'_{\tau,\mu}(q) \in \mathbb{Z}[q, q^{-1}]$, and we may again proceed as above.

(2.8) The truncated Fock space. Let $\mathcal{F}^{>m} := \bigoplus_{n \in \mathbb{N}_0} \mathbb{Z}[\mathcal{P}_n \setminus \mathcal{P}_n(\leq m)]$, where $m \in \mathbb{N}_0$. Hence we have $\mathcal{F}(\leq m) := \mathcal{F}/\mathcal{F}^{>m} \cong \bigoplus_{n \in \mathbb{N}_0} \mathbb{Z}[\mathcal{P}_n(\leq m)]$. Then it follows directly from the definition of the maps $F_i^{(k)} \colon \mathcal{F}_q \to \mathcal{F}_q$, that $\mathcal{F}_q^{>m} := \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} \mathcal{F}^{>m} \leq \mathcal{F}_q$ is a $\mathcal{U}$-submodule, thus $\mathcal{F}_q(\leq m) := \mathcal{F}_q/\mathcal{F}_q^{>m} \cong \mathbb{Z}[q, q^{-1}] \otimes_{\mathbb{Z}} (\mathcal{F}/\mathcal{F}^{>m})$ is a $\mathcal{U}$-module as well. Moreover, $\mathcal{L} \cap \mathcal{F}^{>m} \subseteq \mathcal{L}$ is invariant with respect to $\bar{\ }$, hence we get an induced $\mathbb{Z}$-linear involution on $(\mathcal{L} + \mathcal{F}_q^{>m})/\mathcal{F}_q^{>m} \leq \mathcal{F}_q(\leq m)$. We are going to consider the natural projection $\{B^{\mu} + \mathcal{F}_q^{>m} \in \mathcal{F}_q(\leq m); \mu \in \coprod_{n \in \mathbb{N}_0} \mathcal{P}_n^{p\text{-reg}}\}$ of the global crystal basis to $\mathcal{F}_q(\leq m)$; this is indicated by writing $B^{\mu} \equiv \sum_{\lambda \in \mathcal{P}_n(\leq m)} b_{\lambda,\mu}(q) \cdot \lambda$.

It follows from the description in (2.7), and the fact that for all $n \in \mathbb{N}_0$ the set $\mathcal{P}_n(\leq m)$ is an ideal of $\mathcal{P}_n$ with respect to the reversed dominance partial order, that instead of running the LLT algorithm in $\mathcal{F}_q$ and projecting the global crystal basis to $\mathcal{F}_q(\leq m)$ afterwards, we may just run the LLT algorithm in the quotient space $\mathcal{F}_q(\leq m)$ right from the beginning, and still end up with the elements $B^{\mu} + \mathcal{F}_q^{>m} \in \mathcal{F}_q(\leq m)$. In other words, in terms of the standard basis $\mathcal{P}_n(\leq m)$, this directly computes the matrices $\mathcal{D}_n^q(\leq m)$.

• Let $n \geq 2m+1$ and $\mu \in \mathcal{P}_n^{p\text{-reg}}(m)$; hence we have $\mu_1 = n - m > m \geq \mu_2$. Assume that the rightmost p-ladder intersecting μ has intersection of cardinality at least 2 with μ. Then, due to p-regularity, this ladder intersects μ in its (rightmost) $(1, n-m)$-node x and in the $(p, n-m-1)$-node. Hence we have $m \geq (n-m-1)(p-1)$, implying $n \leq \lfloor \frac{pm}{p-1} \rfloor + 1 \leq 2m+1$. We conclude that, whenever $n \geq 2m+2$, the rightmost p-ladder meeting μ intersects it just in $\{x\}$.

Thus, in this case, since x has residue $r \equiv n-m-1 \pmod{p}$, using the notation of (2.7), we get $A'^{\mu} = B^{\mu \setminus \{x\}} \cdot F_r \in \mathcal{F}_q$. Moreover, since $\mu \setminus \{x\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(m)$ again, we conclude that $F^{\mu} \in \mathcal{U}$ is ultimately periodic of the form $F^{\mu} = \cdots (F_{r-p+1}F_{r-p+2} \cdots F_{r-1}F_r) \cdot (F_{r-p+1}F_{r-p+2} \cdots F_{r-1}F_r)$, where the periodic tail of F^{μ} only depends on the congruence class of n modulo p.

Finally, still assuming $n \geq 2m+1$, the action map $F_i \colon \mathcal{P}_n(m) \to \mathcal{P}_{n+1}(m) \,\dot\cup\, \mathcal{P}_{n+1}(m+1)$, where $i \in \{0, \ldots, p-1\}$, can be described, using the identification $\mathcal{P}_n(m) \to \mathcal{P}_m \colon \lambda \mapsto \overline{\lambda}$ from (2.1), by a map $F_i \colon \mathcal{P}_m \to \mathcal{P}_m \,\dot\cup\, \mathcal{P}_{m+1}$, only depending on the residue class of n modulo p, where actually we have $F_i \colon \mathcal{P}_m \to \mathcal{P}_{m+1}$ whenever $i \not\equiv n-m \pmod{p}$.

(2.9) Remark. Motivated by the above observations on the dependence of the combinatorics just on the residue class of n modulo p, we are wondering whether, for n large enough, the outcome of the LLT algorithm on the truncated Fock space might be described generically in terms of $\coprod_{n\in\mathbb{N}_0}\mathcal{P}_n(\leq m)$, using the identification from (2.1) and treating n as a parameter, possibly depending on the residue class of n modulo p.

This is indeed true in a strong sense for the following special case: Let $n \geq 2m+1$. Then for $\mu \in \mathcal{P}_n^{p\text{-reg}}(m)$ we have $B^\mu \equiv \sum_{\lambda\in\mathcal{P}_n(m)} b_{\lambda,\mu}(q)\cdot\lambda$, where $b_{\lambda,\lambda}(q)=1$ and $b_{\lambda,\mu}(q)\in q\mathbb{N}_0[q]$ for $\lambda\lhd\mu$, that is modulo $\mathcal{F}_q(>m)$ all partitions occurring in B^μ belong to $\mathcal{P}_n(m)$. Since the $(1,n-m+1)$-node x belongs to $A_i(\lambda)$, for all $\lambda\in\mathcal{P}_n(m)$, where $i\equiv n-m \pmod p$, we get $A'^{(\mu\cup\{x\})} := B^\mu\cdot F_i \equiv \sum_{\lambda\in\mathcal{P}_n(m)} b_{\lambda,\mu}(q)\cdot(\lambda\cup\{x\})$. Thus, from the properties of the $b_{\lambda,\mu}(q)$, we get $B^{\mu\cup\{x\}}\equiv A'^{(\mu\cup\{x\})}$. Hence we infer that $B^\mu \equiv \sum_{\overline{\lambda}\in\mathcal{P}_m} b_{\overline{\lambda},\overline{\mu}}(q)\cdot\overline{\lambda}$, with coefficients only depending on $\overline{\mu}\in\mathcal{P}_m^{p\text{-reg}}$, but being independent of n. ♯

But, in view of the results in (3.1) and (4.1), covering the case $m=4$ for $p=3$ and $p=2$, respectively, it cannot possibly be expected that, in general, for $\mu\in\mathcal{P}_n^{p\text{-reg}}(\leq m)$ the basis element $B^\mu \equiv \sum_{\lambda\in\mathcal{P}_n(\leq m)} b_{\lambda,\mu}(q)\cdot\lambda$ only depends on $\overline{\mu}$. Still, from these results it is tempting to expect that B^μ modulo $\mathcal{F}_q(>m)$ only depends on $\overline{\mu}$ and the congruence class of n modulo p, that is we are wondering whether for $n\geq 2m+1$ and $\mu=[\mu_1,\overline{\mu}]\in\mathcal{P}_n^{p\text{-reg}}(\leq m)$ we have

$$B^{[\mu_1+p,\overline{\mu}]} \equiv \sum_{\lambda=[\lambda_1,\overline{\lambda}]\in\mathcal{P}_n(\leq m)} b_{\lambda,\mu}(q)\cdot[\lambda_1+p,\overline{\lambda}].$$

If this held true, then, viewing it as a statement on the crystallized decomposition matrix $\mathcal{D}_n^q(\leq m)$, this would entail a similar statement on the decomposition matrix $\mathcal{D}_n^\zeta(\leq m)$ of the Iwahori-Hecke algebra $\mathcal{H}_n(u)$, while the analogous statement for the decomposition matrix $\mathcal{D}_n(\leq m)$ of the symmetric group cannot possibly hold, as for example the results in (3.2) and (4.2) show. Anyway, to our knowledge this has not yet been examined in the literature, and we leave it as an open question to the reader.

3 Decomposition numbers in characteristic 3

In order to get an overview over the irreducible representations parameterized by $\mu\in\mathcal{P}_n^{3\text{-reg}}(\leq 4)$, we determine the crystallized decomposition matrices $\mathcal{D}_n^q(\leq 4)$ and the decomposition matrices $\mathcal{D}_n(\leq 4)$ for $p=3$.

(3.1) Crystallized decomposition matrices. We apply the LLT algorithm to the truncated Fock space, according to the description in (2.7) and (2.8).

• The maps $F_i\colon\mathcal{F}_q(\leq 4)\to\mathcal{F}_q(\leq 4)$, where $i\in\{0,1,2\}$ runs through the residue classes modulo 3, are given in Table 1, which should be read as follows: We use the round bracket notation introduced in (2.1). Given $\overline{\lambda}\in\mathcal{P}_m$, for some $m\in\mathbb{N}_0$, then the action of F_i on $(\overline{\lambda})\in\mathcal{P}_n(m)$ only depends on the residue class of n modulo 3 as soon as $n-m\geq\overline{\lambda}_1+1$; in particular, this holds for all $\overline{\lambda}\in\mathcal{P}_m$

Table 1: Action on truncated Fock space.

$()$	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0	$()$		
F_1		$()$	
F_2			$()$
	$q^{-1}(1)$	(1)	$q(1)$

(1)	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0		(1)	
	(2)	$q(2)$	$q^{-1}(2)$
F_1			(1)
	$q^{-1}(1^2)$	(1^2)	$q(1^2)$
F_2	(1)		

(2)	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0			(2)
F_1	(2)		
	$q(3)$	$q^{-1}(3)$	(3)
	$q^2(2,1)$	$(2,1)$	$q(2,1)$
F_2		(2)	

(1^2)	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0			(1^2)
	$q^{-1}(2,1)$	$(2,1)$	$q(2,1)$
	(1^3)	$q(1^3)$	$q^2(1^3)$
F_1	(1^2)		
F_2		(1^2)	

(3)	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0	(3)		
F_1		(3)	
	$q^{-1}(3,1)$	$(3,1)$	$q^{-2}(3,1)$
F_2			(3)
	$q^{-1}(4)$	(4)	$q(4)$

$(2,1)$	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0	$(2,1)$		
	$(2,1^2)$	$q^{-2}(2,1^2)$	$q^{-1}(2,1^2)$
F_1		$(2,1)$	
	$(3,1)$	$q(3,1)$	$q^{-1}(3,1)$
F_2			$(2,1)$
	$q^{-1}(2^2)$	(2^2)	$q(2^2)$

(1^3)	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0	(1^3)		
	$q(2,1^2)$	$q^{-1}(2,1^2)$	$(2,1^2)$
F_1		(1^3)	
F_2			(1^3)
	$q^{-1}(1^4)$	(1^4)	$q(1^4)$

$\overline{\lambda}\in\mathcal{P}_4$	$n\equiv 0$	$n\equiv 1$	$n\equiv 2$
F_0		$\overline{\lambda}$	
F_1			$\overline{\lambda}$
F_2	$\overline{\lambda}$		

---

whenever $n\geq 2m+1$. For example, for $\overline{\lambda}=(2)$, for all $n\geq 5$ such that $n\equiv 2 \pmod 3$, we have $F_1\colon [n-2,2]\mapsto [n-2,3]+q\cdot[n-2,2,1]$.

• Next, given $\mu\in\mathcal{P}_n^{3\text{-reg}}(\leq 4)$, the elements $F^\mu\in\mathcal{U}$ are as shown in Table 2, where we again use the round bracket notation, and silently assume that for $\overline{\mu}\in\mathcal{P}_m^{3\text{-reg}}$ we have $n-m\geq\overline{\mu}_1$, or even $n-m\geq\overline{\mu}_1+1$ to ensure 3-regularity.

We observe that the elements $B^\mu+\mathcal{F}_q^{>4}\in\mathcal{F}_q(\leq 4)$, where $\overline{\mu}\in\coprod_{m=0}^4\mathcal{P}_m^{3\text{-reg}}$ is fixed, are ultimately periodic, and only depend on the residue class of n modulo 3; in the first column of Table 2 we give the bound where periodicity sets in. But since we do not have an *a priori* proof of this fact, we determine $B^\mu+\mathcal{F}_q^{>4}$ explicitly, proving periodicity on the fly, see also (2.9). We proceed through $\overline{\mu}\in\coprod_{m=0}^4\mathcal{P}_m^{3\text{-reg}}$ with decreasing m and in reversed lexicographical ordering:

Table 2: Products of divided power operators

	$\overline{\mu}$	F^μ	$B^\mu \equiv$
$n \geq 6$	$(2,1^2)$	$F_0F_2F_1^{(2)}F_0^{(2)} \cdot F_2F_0F_1 \cdot F_2F_0F_1 \cdots$	$\overline{\mu}$
$n \geq 7$	(2^2)	$F_0F_2F_1^{(2)}F_0F_2^{(2)} \cdot F_0F_1F_2 \cdot F_0F_1F_2 \cdots$	$\overline{\mu} + q(1^4)$
$n \geq 7$	$(3,1)$	$F_0F_2F_1^{(2)}F_0F_2F_1 \cdot F_0F_1F_2 \cdot F_0F_1F_2 \cdots$	$\overline{\mu}$
$n \geq 8$	(4)	$F_0F_2F_1F_0F_2F_1F_0F_2 \cdot F_1F_2F_0 \cdot F_1F_2F_0 \cdots$	$\overline{\mu} + q(2^2)$
$n \geq 6$	$(2,1)$	$F_0F_2F_1^{(2)}F_0F_2 \cdot F_0F_1F_2 \cdot F_0F_1F_2 \cdots$	
$n \geq 7$	(3)	$F_0F_2F_1F_0F_2F_1F_0 \cdot F_1F_2F_0 \cdot F_1F_2F_0 \cdots$	
$n \geq 5$	(1^2)	$F_0F_2F_1^{(2)}F_2 \cdot F_0F_1F_2 \cdot F_0F_1F_2 \cdots$	
$n \geq 6$	(2)	$F_0F_2F_1F_0F_2F_0 \cdot F_1F_2F_0 \cdot F_1F_2F_0 \cdots$	
$n \geq 7$	(1)	$F_0F_2F_1F_2F_0F_1F_2 \cdot F_0F_1F_2 \cdot F_0F_1F_2 \cdots$	
$n \geq 7$	$()$	$F_0F_1F_2F_0F_1F_2F_0 \cdot F_1F_2F_0 \cdot F_1F_2F_0 \cdots$	

• For $\overline{\mu} \in \mathcal{P}_4^{\text{3-reg}}$ we get $B^\mu + \mathcal{F}_q^{>4}$ as indicated in Table 2. This is seen as follows: For example, for $\overline{\mu} := (2,1^2)$, applying $F_0F_2F_1^{(2)}F_0^{(2)}$ to the empty partition yields $B^\mu \equiv \overline{\mu}$ for $n = 6$. Then, successively applying $F_2, F_0, F_1, F_2, F_0, F_1, \ldots$, and using Table 1, by induction we get $B^\mu \equiv \overline{\mu}$ for all $n \geq 6$. For the other elements of $\mathcal{P}_4^{\text{3-reg}}$ we argue similarly.

• For $\overline{\mu} \in \mathcal{P}_n^{\text{3-reg}}(\leq 3)$ we proceed similarly, where the subsequent computations should be read as follows: For example, for $\overline{\mu} := (2,1)$, applying $F_0F_2F_1^{(2)}F_0F_2$ to the empty partition, we get $B^\mu + \mathcal{F}_q^{>4}$ for $n = 6$ as indicated below. Then, proceeding by induction on $n \geq 6$, we successively apply $F_0, F_1, F_2, F_0, F_1, F_2, \ldots$, where after each application of F_0 we additionally have to add a suitable multiple of $B^{(2,1^2)} + \mathcal{F}_q^{>4}$; note that at this stage $B^{(2,1^2)} + \mathcal{F}_q^{>4}$ has already been obtained and proved to behave periodically. Closing the circle, the last line shows that we indeed get $B^\mu + \mathcal{F}_q^{>4}$ in terms of the residue class of n modulo 3.

For $\overline{\mu} := (2,1)$ this for $n \geq 6$ yields:

$$\begin{array}{lcll} B^\mu & \equiv & \overline{\mu} + q(1^3) + q(2^2) + q^2(1^4) & \text{if } n \equiv 0 \pmod 3 \\ \xmapsto{F_0} & & \overline{\mu} + q(1^3) + (1+q^2)(2,1^2) & \\ \xmapsto{B^{(2,1^2)}} B^\mu & \equiv & \overline{\mu} + q(1^3) + q^2(2,1^2) & \text{if } n \equiv 1 \pmod 3 \\ \xmapsto{F_1} B^\mu & \equiv & \overline{\mu} + q(1^3) + q(3,1) & \text{if } n \equiv 2 \pmod 3 \\ \xmapsto{F_2} & & \overline{\mu} + q(1^3) + q(2^2) + q^2(1^4) & \end{array}$$

For $\overline{\mu} := (3)$ this for $n \geq 7$ yields:

$$\begin{array}{rrcll}
 & B^{\mu} & \equiv & \overline{\mu} + q(2,1) + q(2,1^2) & \text{if } n \equiv 1 \pmod 3 \\
\xmapsto{F_1} & & & \overline{\mu} + q(2,1) + (1+q^2)(3,1) & \\
\xmapsto{B^{(3,1)}} & B^{\mu} & \equiv & \overline{\mu} + q(2,1) + q^2(3,1) & \text{if } n \equiv 2 \pmod 3 \\
\xmapsto{F_2} & B^{\mu} & \equiv & \overline{\mu} + q(2,1) + q(4) + q^2(2^2) & \text{if } n \equiv 0 \pmod 3 \\
\xmapsto{F_0} & & & \overline{\mu} + q(2,1) + q(2,1^2) &
\end{array}$$

For $\overline{\mu} := (1^2)$ this for $n \geq 5$ yields:

$$\begin{array}{rrcll}
 & B^{\mu} & \equiv & \overline{\mu} & \text{if } n \equiv 2 \pmod 3 \\
\xmapsto{F_0} & B^{\mu} & \equiv & \overline{\mu} + q(2,1) + q^2(1^3) & \text{if } n \equiv 0 \pmod 3 \\
\xmapsto{F_1} & B^{\mu} & \equiv & \overline{\mu} + q(3,1) & \text{if } n \equiv 1 \pmod 3 \\
\xmapsto{F_2} & & & \overline{\mu} &
\end{array}$$

For $\overline{\mu} := (2)$ this for $n \geq 6$ yields:

$$\begin{array}{rrcll}
 & B^{\mu} & \equiv & \overline{\mu} & \text{if } n \equiv 0 \pmod 3 \\
\xmapsto{F_1} & B^{\mu} & \equiv & \overline{\mu} + q(3) + q^2(2,1) & \text{if } n \equiv 1 \pmod 3 \\
\xmapsto{F_2} & B^{\mu} & \equiv & \overline{\mu} + q(4) + q^2(2^2) & \text{if } n \equiv 2 \pmod 3 \\
\xmapsto{F_0} & & & \overline{\mu} &
\end{array}$$

For $\overline{\mu} := (1)$ this for $n \geq 7$ yields:

$$\begin{array}{rrcll}
 & B^{\mu} & \equiv & \overline{\mu} + q(2^2) & \text{if } n \equiv 1 \pmod 3 \\
\xmapsto{F_0} & B^{\mu} & \equiv & \overline{\mu} + q(2) + q(2^2) & \text{if } n \equiv 2 \pmod 3 \\
\xmapsto{F_1} & B^{\mu} & \equiv & \overline{\mu} + q(1^2) + q(3) + q^2(2,1) + q(2^2) & \text{if } n \equiv 0 \pmod 3 \\
\xmapsto{F_2} & & & \overline{\mu} + (4) + 2q(2^2) & \\
\xmapsto{B^{(4)}} & & & \overline{\mu} + q(2^2) &
\end{array}$$

For $\overline{\mu} := ()$ this for $n \geq 7$ yields:

$$\begin{array}{rrcll}
 & B^{\mu} & \equiv & \overline{\mu} + q(2) + q(2,1) + q(2,1^2) & \text{if } n \equiv 1 \pmod 3 \\
\xmapsto{F_1} & & & \overline{\mu} + (3) + 2q(2,1) + q(3,1) & \\
\xmapsto{B^{(3)}} & B^{\mu} & \equiv & \overline{\mu} + q(2,1) & \text{if } n \equiv 2 \pmod 3 \\
\xmapsto{F_2} & B^{\mu} & \equiv & \overline{\mu} + q(1) + q(2,1) + q(2^2) & \text{if } n \equiv 0 \pmod 3 \\
\xmapsto{F_0} & & & \overline{\mu} + q(2) + q(2,1) + q(2,1^2) &
\end{array}$$

• In conclusion, this yields the crystallized decomposition matrix $\mathcal{D}_n^q(\leq 4)$, for $n \geq 8$, as exhibited in Table 3. Here, for $i \in \{0,1,2\}$ running through the residue classes modulo 3, reminiscent of the Kronecker symbol we let

$$\delta_i = \delta_i(n) := \begin{cases} 1, & \text{if } n \equiv i \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

Table 3: Crystallized decomposition matrix for $\mathcal{P}_n(\leq 4)$.

	$()$	(1)	(2)	(1^2)	(3)	$(2,1)$	(4)	$(3,1)$	(2^2)	$(2,1^2)$
$()$	1									
(1)	$q\delta_0$	1								
(2)	$q\delta_1$	$q\delta_2$	1							
(1^2)	.	$q\delta_0$	.	1						
(3)	.	$q\delta_0$	$q\delta_1$	.	1					
$(2,1)$	q	$q^2\delta_0$	$q^2\delta_1$	$q\delta_0$	q	1				
(1^3)	.	.	.	$q^2\delta_0$	.	q				
(4)	.	.	$q\delta_2$	.	$q\delta_0$	.	1			
$(3,1)$	.	.	.	$q\delta_1$	$q^2\delta_2$	$q\delta_2$	.	1		
(2^2)	$q^2\delta_0$	q	$q^2\delta_2$	.	$q^2\delta_0$	$q\delta_0$	q	.	1	
$(2,1^2)$	$q\delta_1$	.	.	.	$q\delta_1$	$q^2\delta_1$	.	.	.	1
(1^4)	.	.	.	.	.	$q^2\delta_0$	.	.	q	.

We remark that for $n \leq 10$ these results are also contained in the explicit crystallized decomposition matrices given in [36, Sect.10.4].

(3.2) Decomposition matrices. We proceed to determine the decomposition matrix $\mathcal{D}_n(\leq 4)$. The result is given in Table 4, where we assume that $n \geq 8$. Here, we again use the Kronecker notation introduced in (3.1), and let

$$\alpha = \alpha(n) := \begin{cases} 1, & \text{if } n \equiv 2,3,4 \pmod 9, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta = \beta(n) := \begin{cases} 1, & \text{if } n \equiv 4,5,6 \pmod 9, \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition matrices $\mathcal{D}_n$ for $n \leq 7$ are known, and for example given in [22, p.143], or accessible in the databases mentioned in Section 1.

To determine the decomposition matrix $\mathcal{D}_n(\leq 4)$, we make use of the crystallized decomposition matrix $\mathcal{D}_n^q(\leq 4)$. While computing the entries of $\mathcal{D}_n(\leq 4)$ we also determine the adjustment matrix $\mathcal{A}_n(\leq 4)$, the result is given in Table 5. In the subsequently given details we make use of the following notation: Given $\lambda = [\lambda_1, \overline{\lambda}] \in \mathcal{P}_n$ the associated Specht module is also written as $S_n(\overline{\lambda})$, and

Table 4: Decomposition matrix for $\mathcal{P}_n(\leq 4)$.

	$()$	(1)	(2)	(1^2)	(3)	$(2,1)$	(4)	$(3,1)$	(2^2)	$(2,1^2)$
$()$	1									
(1)	δ_0	1								
(2)	δ_1	δ_2	1							
(1^2)	.	δ_0	.	1						
(3)	α	δ_0	δ_1	.	1					
$(2,1)$	$1{+}\alpha$	δ_0	δ_1	δ_0	1	1				
(1^3)	.	.	.	δ_0	.	1				
(4)	$\alpha\delta_0$	β	δ_2	.	δ_0	.	1			
$(3,1)$	$(\alpha{+}\beta)\delta_2$	.	.	δ_1	δ_2	δ_2	.	1		
(2^2)	$(1{+}\alpha)\delta_0$	$1{+}\beta$	δ_2	.	δ_0	δ_0	1	.	1	
$(2,1^2)$	$(1{+}\alpha)\delta_1$	.	.	.	δ_1	δ_1	.	.	.	1
(1^4)	.	.	.	.	.	δ_0	.	.	1	.

similarly for $\mu = [\mu_1, \overline{\mu}] \in \mathcal{P}_n^{3\text{-reg}}$, the associated irreducible module is also written as $D_n(\overline{\mu})$.

• The decomposition numbers of the Specht modules $S^{[n-m,m]} = S_n(m)$, where $m \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$, that is those belonging to partitions having at most two parts, are known by [23], see also [22, Thm.24.15].

• The decomposition numbers of the Specht modules $S^{[n-m,1^m]} = S_n(1^m)$, where $m \in \{0, \ldots, n-1\}$, that is those belonging to hook partitions, are known by [44], see also [22, Thm.24.1]. More precisely, if $n \not\equiv 0 \pmod 3$, then the Specht module $S_n(1^m)$ is reducible, hence by [25, Thm.6.3.50] is isomorphic to $D^{[n-m,1^m]^R}$. If $n \equiv 0 \pmod 3$, then for $m \in \{1, \ldots, n-2\}$ the Specht module $S_n(1^m)$ is uniserial with two distinct constituents, such that $\mathrm{Hom}_{\mathbb{F}_3\mathcal{S}_n}(S_n(1^m), S_n(1^{m+1})) \neq \{0\}$ for $m \in \{0, \ldots, n-2\}$; hence by induction on $m \in \{0, \ldots, n-1\}$, again using [25, Thm.6.3.50], it follows that the head constituent of $S_n(1^m)$ is isomorphic to $D^{[n-m,1^m]^R}$.

• Whenever $\lambda \in \mathcal{P}_n(m)$ and $\mu \in \mathcal{P}_n^{3\text{-reg}}(m)$ for some $m \in \{0, \ldots, n-1\}$, that is $\lambda_1 = \mu_1 = n - m$, the principle of first row removal, see [21], yields $[S_n(\overline{\lambda})\colon D_n(\overline{\mu})] = [S^{\overline{\lambda}}\colon D^{\overline{\mu}}]$, where the latter are decomposition numbers of $\mathcal{S}_m$, which since $m \leq 4$ are easily determined or can be looked up in [22, p.143].

Table 5: Adjustment matrix for $\mathcal{P}_n(\leq 4)$.

	$()$	(1)	(2)	(1^2)	(3)	$(2,1)$	(4)	$(3,1)$	(2^2)	$(2,1^2)$
$()$	1									
(1)	.	1								
(2)	.	.	1							
(1^2)	.	.	.	1						
(3)	α	.	.	.	1					
$(2,1)$	.	.	.	.	.	1				
(4)	.	β	.	.	.	.	1			
$(3,1)$	$\beta\delta_2$	.	.	.	.	.	.	1		
(2^2)	.	.	.	.	.	.	.	.	1	
$(2,1^2)$	.	.	.	.	.	.	.	.	.	1

• Whenever $\lambda \in \mathcal{P}_n(3) \,\dot{\cup}\, \mathcal{P}_n(4)$ and $\mu \in \mathcal{P}_n^{3\text{-reg}}(2) \,\dot{\cup}\, \mathcal{P}_n^{3\text{-reg}}(3)$ such that $l(\lambda) = l(\mu) = 3$, the principle of first column removal, see [21], yields $[S_n(\overline{\lambda})\colon D_n(\overline{\mu})] = [S_{n-3}(\lambda_2 - 1, \lambda_3 - 1)\colon D_{n-3}(\mu_2 - 1, \mu_3 - 1)]$. This settles the cases

$$\begin{aligned}
[S_n(2,1)\colon D_n(1^2)] &= [S_{n-3}(1)\colon D_{n-3}(())], \\
[S_n(3,1)\colon D_n(2,1)] &= [S_{n-3}(2)\colon D_{n-3}(1)], \\
[S_n(3,1)\colon D_n(1^2)] &= [S_{n-3}(2)\colon D_{n-3}(())], \\
[S_n(2^2)\colon D_n(2,1)] &= [S_{n-3}(1^2)\colon D_{n-3}(1)], \\
[S_n(2^2)\colon D_n(1^2)] &= [S_{n-3}(1^2)\colon D_{n-3}(())],
\end{aligned}$$

where the decomposition numbers of $\mathcal{S}_{n-3}$ involved are two-part or hook partition cases which we have already dealt with above.

• In order to complete $\mathcal{D}_n(\leq 3)$ it remains to determine the row belonging to the partition $(2,1)$. But from $\mathcal{D}_n(\leq 3) = \mathcal{D}_n^{\zeta}(\leq 3) \cdot \mathcal{A}_n(\leq 3)$, using the entries of $\mathcal{D}_n(\leq 3)$ already known, we conclude that $\mathcal{A}_n(\leq 3)$ actually coincides with the upper left-hand (6×6)-sub-matrix of the matrix given in Table 5. This in turn determines $\mathcal{D}_n(\leq 3)$.

• We turn our attention to the rows of $\mathcal{D}_n(\leq 4)$ belonging to partitions in $\mathcal{P}_n(4)$: From $\mathcal{D}_n(\leq 4) = \mathcal{D}_n^{\zeta}(\leq 4) \cdot \mathcal{A}_n(\leq 4)$, a comparison of the known entries of $\mathcal{D}_n(\leq 4)$ with the matrix $\mathcal{D}_n^{\zeta}(\leq 4)$ shows that the rows of $\mathcal{A}_n(\leq 4)$ belonging to the partitions (4) and (2^2) are as given in Table 5. This in turn completes the row of $\mathcal{D}_n(\leq 4)$ belonging to the partition (2^2).

• Considering 3-contents, reflecting the distribution of modules into 3-blocks, we conclude that $[S_n(2,1^2)\colon D_n(\overline{\mu})]=0$ for $\overline{\mu}\in\{(1^2),(1)\}$, and $[S_n(3,1)\colon D_n(\overline{\mu})]=0$ for $\overline{\mu}\in\{(2),(1)\}$. This leaves the six entries $[S_n(2,1^2)\colon D_n(\overline{\mu})]=0$ for $\overline{\mu}\in\{(2,1),(3),(2),()\}$, and $[S_n(3,1)\colon D_n(\overline{\mu})]=0$ for $\overline{\mu}\in\{(3),()\}$ to be determined, which we deal with in turn: First note that for the cases mentioned a consideration of 3-contents shows that $[S_n(2,1^2)\colon D_n(\overline{\mu})]>0$ only if $n\equiv 1 \pmod 3$, and that $[S_n(3,1)\colon D_n(\overline{\mu})]>0$ only if $n\equiv 2 \pmod 3$.

• We consider $[S_n(2,1^2)\colon D_n(2,1)]$ for $n\equiv 1 \pmod 3$: Applying (2.5) with 0-removal leads to the inequality

$$\begin{array}{rl} & [S_n(2,1^2)\colon D_n(2,1)]\cdot[D_n(2,1)\downarrow_0\colon D_{n-1}(1^2)] \\ \leq & [S_{n-1}(2,1)\colon D_{n-1}(1^2)]+[S_{n-1}(1^3)\colon D_{n-1}(1^2)]=2. \end{array}$$

Since by (2.4) we have $[D_n(2,1)\downarrow_0\colon D_{n-1}(1^2)]=2$, using Table 3 implies $1=[S_n(2,1^2)\colon D_n^{\zeta}(2,1)]\leq[S_n(2,1^2)\colon D_n(2,1)]\leq 1$.

• We consider $[S_n(2,1^2)\colon D_n(3)]$ for $n\equiv 1 \pmod 3$: Again applying (2.5) with 0-removal leads to the inequality

$$\begin{array}{rl} & [S_n(2,1^2)\colon D_n(3)]\cdot[D_n(3)\downarrow_0\colon D_{n-1}(3)] \\ \leq & [S_{n-1}(2,1)\colon D_{n-1}(3)]+[S_{n-1}(1^3)\colon D_{n-1}(3)]=1, \end{array}$$

which using Table 3 implies $1=[S_n(2,1^2)\colon D_n^{\zeta}(3)]\leq[S_n(2,1^2)\colon D_n(3)]\leq 1$.

• We consider $[S_n(2,1^2)\colon D_n(2)]$ for $n\equiv 1 \pmod 3$: Applying (2.5) with 0-removal, since by (2.4) we have $[D_n(2)\downarrow_0\colon D_{n-1}(1)]=1$, we get the inequality

$$\begin{array}{rl} & [S_n(2,1^2)\colon D_n(2)]+[S_n(2,1^2)\colon D_n(2,1)]\cdot[D_n(2,1)\downarrow_0\colon D_{n-1}(1)] \\ \leq & [S_{n-1}(2,1)\colon D_{n-1}(1)]+[S_{n-1}(1^3)\colon D_{n-1}(1)]=1. \end{array}$$

We have $[S_n(2,1^2)\colon D_n(2,1)]=1$, and from $[D_n(2,1)\downarrow_0]=[S_n(1^3)\downarrow_0]=[S_{n-1}(1^3)]+[S_{n-1}(1^2)]$ we get

$$[D_n(2,1)\downarrow_0\colon D_{n-1}(1)]=[S_{n-1}(1^3)\colon D_{n-1}(1)]+[S_{n-1}(1^2)\colon D_{n-1}(1)]=1,$$

hence we infer $[S_n(2,1^2)\colon D_n(2)]=0$.

• We consider $[S_n(2,1^2)\colon D_n(())]$ for $n\equiv 1 \pmod 3$: Applying (2.5) with 0-removal, we get the inequality

$$\begin{array}{rcl} [S_n(2,1^2)\colon D_n(())] & \leq & [S_{n-1}(2,1)\colon D_{n-1}(())]+[S_{n-1}(1^3)\colon D_{n-1}(())] \\ & = & \left\{\begin{array}{ll} 2, & \text{if } n\equiv 4 \pmod 9, \\ 1, & \text{if } n\equiv 1,7 \pmod 9. \end{array}\right. \end{array}$$

On the other hand, by what we already know about $\mathcal{A}_n(\leq 4)$, we conclude that

$$[S_n(2,1^2)\colon D_n^{\zeta}(())]+\alpha\cdot[S_n(2,1^2)\colon D_n^{\zeta}(3)]=(1+\alpha)\delta_1\leq[S_n(2,1^2)\colon D_n(())],$$

implying equality. This completes the row of $\mathcal{D}_n(\leq 4)$ belonging to the partition $(2,1^2)$, and shows that the row of $\mathcal{A}_n(\leq 4)$ belonging to $(2,1^2)$ is as in Table 5.

• We consider $[S_n(3,1)\colon D_n(3)]$ for $n \equiv 2 \pmod 3$: Applying (2.5) with 1-removal, we get the inequality

$$\begin{array}{rl} & [S_n(3,1)\colon D_n(3)] \cdot [D_n(3)\downarrow_1\colon D_{n-1}(2)] \\ \leq & [S_{n-1}(3)\colon D_{n-1}(2)] + [S_{n-1}(2,1)\colon D_{n-1}(2)] = 2. \end{array}$$

Since from (2.4) we obtain $[D_n(3)\downarrow_1\colon D_{n-1}(2)] = 2$, using Table 3 thus implies $1 = [S_n(3,1)\colon D_n^\zeta(3)] \leq [S_n(3,1)\colon D_n(3)] \leq 1$.

• We consider $[S_n(3,1)\colon D_n(())]$ for $n \equiv 2 \pmod 3$: Applying (2.5) with 1-removal, since we have $[S_n(3,1)\colon D_n(3)] = 1$, and $[D_n(2,1)\downarrow_1\colon D_{n-1}(())] = 0$ and $[D_n(3,1)\downarrow_1\colon D_{n-1}(())] = 0$ by (2.4), we this time even get the strict equality

$$\begin{array}{rl} & [S_n(3,1)\colon D_n(())] + [D_n(3)\downarrow_1\colon D_{n-1}(())] \\ = & [S_{n-1}(3)\colon D_{n-1}(())] + [S_{n-1}(2,1)\colon D_{n-1}(())] \\ = & \begin{cases} 3, & \text{if } n \equiv 5 \pmod 9, \\ 1, & \text{if } n \equiv 2,8 \pmod 9. \end{cases} \end{array}$$

If $n \equiv 5, 8 \pmod 9$ then $[D_n(3)\downarrow_1] = [S_n(3)\downarrow_1] = [S_{n-1}(2)] + [S_{n-1}(3)]$, hence

$$\begin{array}{rl} & [D_n(3)\downarrow_1\colon D_{n-1}(())] \\ = & [S_{n-1}(2)\colon D_{n-1}(())] + [S_{n-1}(3)\colon D_{n-1}(())] \\ = & \begin{cases} 2, & \text{if } n \equiv 5 \pmod 9, \\ 1, & \text{if } n \equiv 8 \pmod 9. \end{cases} \end{array}$$

Similarly, if $n \equiv 2 \pmod 9$ then $[D_n(3)\downarrow_1] = [S_n(3)\downarrow_1] - [S_n(())\downarrow_1] = [S_{n-1}(2)] + [S_{n-1}(3)] - [S_{n-1}(())]$, hence

$$\begin{array}{rl} & [D_n(3)\downarrow_1\colon D_{n-1}(())] \\ = & [S_{n-1}(2)\colon D_{n-1}(())] + [S_{n-1}(3)\colon D_{n-1}(())] - [S_{n-1}(())\colon D_{n-1}(())] = 0. \end{array}$$

In conclusion we thus have

$$[S_n(3,1)\colon D_n(())] = \begin{cases} 1, & \text{if } n \equiv 2,5 \pmod 9, \\ 0, & \text{if } n \equiv 8 \pmod 9, \end{cases}$$

yielding $[S_n(3,1)\colon D_n(())] = (\alpha+\beta)\delta_2$. This completes $\mathcal{D}_n(\leq 4)$, and shows that the row of $\mathcal{A}_n(\leq 4)$ belonging to the partition $(3,1)$ is as given in Table 5.

(3.3) Degree formulae. We are now prepared to obtain degree formulae for the irreducible modular representations parameterized by $\mathcal{P}_n^{3\text{-reg}}(\leq 4)$. They are given in Tables 6 and 7, where for $\mu \in \mathcal{P}_n^{3\text{-reg}}$ we let $d^\mu := \dim_{\mathbb{F}_3}(D^\mu)$:

In view of the decomposition matrix $\mathcal{D}_n(\leq 4)$ for $n \geq 8$ given in Table 4, and the known decomposition numbers for $n \leq 7$, see the comment at the beginning of (3.2), these follow straightforwardly from the hook length formula for the dimension of Specht modules, see [16] and also [22, Thm.20.1].

Table 6: Degree formulae for $\mathcal{P}_n^{3\text{-reg}}(\leq 3)$.

μ	d^μ	condition	
$[n]$	1		
$[n-1,1]$	$n-2$	$n \equiv 0$	$\pmod 3$
$(n \geq 2)$	$n-1$	$n \equiv 1,2$	$\pmod 3$
$[n-2,2]$	$\frac{1}{2}(n^2-5n+2)$	$n \equiv 2$	$\pmod 3$
$(n \geq 4)$	$\frac{1}{2}(n^2-3n-2)$	$n \equiv 1$	$\pmod 3$
	$\frac{1}{2}(n^2-3n)$	$n \equiv 0$	$\pmod 3$
$[n-2,1^2]$	$\frac{1}{2}(n^2-5n+6)$	$n \equiv 0$	$\pmod 3$
$(n \geq 4)$	$\frac{1}{2}(n^2-3n+2)$	$n \equiv 1,2$	$\pmod 3$
$[n-3,3]$	$\frac{1}{6}(n^3-9n^2+14n)$	$n \equiv 4$	$\pmod 9$
$(n \geq 6)$	$\frac{1}{6}(n^3-9n^2+14n+6)$	$n \equiv 1,7$	$\pmod 9$
	$\frac{1}{6}(n^3-6n^2-n+6)$	$n \equiv 3$	$\pmod 9$
	$\frac{1}{6}(n^3-6n^2-n+12)$	$n \equiv 0,6$	$\pmod 9$
	$\frac{1}{6}(n^3-6n^2+5n-6)$	$n \equiv 2$	$\pmod 9$
	$\frac{1}{6}(n^3-6n^2+5n)$	$n \equiv 5,8$	$\pmod 9$
$[n-3,2,1]$	$\frac{1}{6}(n^3-9n^2+26n-24)$	$n \equiv 0$	$\pmod 3$
$(n \geq 5)$	$\frac{1}{6}(n^3-6n^2+11n-6)$	$n \equiv 1,2$	$\pmod 3$

4 Decomposition numbers in characteristic 2

In order to get an overview over the irreducible representations parameterized by $\mu \in \mathcal{P}_n^{2\text{-reg}}(\leq 4)$, we determine the crystallized decomposition matrices $\mathcal{D}_n^q(\leq 4)$ and the decomposition matrices $\mathcal{D}_n(\leq 4)$ for $p = 2$.

(4.1) Crystallized decomposition matrices. We apply the LLT algorithm to the truncated Fock space, according to the description in (2.7) and (2.8). We proceed entirely similar to (3.1), and thus only record the results:

• The maps $F_i \colon \mathcal{F}_q(\leq 4) \to \mathcal{F}_q(\leq 4)$, where $i \in \{0,1\}$ runs through the residue classes modulo 2, are given in Table 8.

• Given $\mu \in \mathcal{P}_n^{2\text{-reg}}(\leq 4)$, the elements $F^\mu \in \mathcal{U}$ are as shown in Table 9. Again we observe that the elements $B^\mu + \mathcal{F}_q^{>4} \in \mathcal{F}_q(\leq 4)$, where $\overline{\mu} \in \coprod_{m=0}^4 \mathcal{P}_m^{2\text{-reg}}$ is fixed, are ultimately periodic, and only depend on the residue class of n modulo 2; in the first column of Table 9 we give the bound where periodicity sets in.

Table 7: Degree formulae for $\mathcal{P}_n^{3\text{-reg}}(4)$.

μ	d^μ	condition
$[n-4,4]$	$\frac{1}{24}(n^4-14n^3+47n^2-34n)$	$n \equiv 6 \pmod 9$
$(n \geq 8)$	$\frac{1}{24}(n^4-14n^3+47n^2-10n-48)$	$n \equiv 0,3 \pmod 9$
	$\frac{1}{24}(n^4-10n^3+11n^2+22n)$	$n \equiv 5 \pmod 9$
	$\frac{1}{24}(n^4-10n^3+11n^2+46n-24)$	$n \equiv 2,8 \pmod 9$
	$\frac{1}{24}(n^4-10n^3+23n^2-38n+24)$	$n \equiv 5 \pmod 9$
	$\frac{1}{24}(n^4-10n^3+23n^2-14n)$	$n \equiv 1,7 \pmod 9$
$[n-4,3,1]$	$\frac{1}{24}(3n^4-38n^3+129n^2-118n)$	$n \equiv 5 \pmod 9$
$(n \geq 7)$	$\frac{1}{24}(3n^4-38n^3+129n^2-118n+24)$	$n \equiv 2,8 \pmod 9$
	$\frac{1}{24}(3n^4-30n^3+69n^2-18n-24)$	$n \equiv 1 \pmod 3$
	$\frac{1}{24}(3n^4-30n^3+81n^2-54n)$	$n \equiv 0 \pmod 3$
$[n-4,2^2]$	$\frac{1}{24}(n^4-14n^3+71n^2-154n+120)$	$n \equiv 0 \pmod 3$
$(n \geq 7)$	$\frac{1}{24}(n^4-10n^3+35n^2-50n+24)$	$n \equiv 1,2 \pmod 3$
$[n-4,2,1^2]$	$\frac{1}{24}(3n^4-38n^3+153n^2-190n-24)$	$n \equiv 1 \pmod 3$
$(n \geq 6)$	$\frac{1}{24}(3n^4-30n^3+93n^2-90n)$	$n \equiv 0,2 \pmod 3$

• For $\overline{\mu} \in \mathcal{P}_4^{2\text{-reg}}$ we get $B^\mu + \mathcal{F}_q^{>4}$ as follows:

$$\begin{aligned} B^{(3,1)} &\equiv (3,1) + q(2^2) + q^2(2,1^2) \\ B^{(4)} &\equiv (4) + q(3,1) + q(2,1^2) + q^2(1^4) \end{aligned}$$

• For $\overline{\mu} \in \mathcal{P}_n^{2\text{-reg}}(\leq 3)$ we get the following:

For $\overline{\mu} := (2,1)$ this for $n \geq 6$ yields:

$$\begin{array}{lrcll} & B^\mu & \equiv & \overline{\mu} & \text{if } n \equiv 0 \pmod 2 \\ \xmapsto{F_1} & B^\mu & \equiv & \overline{\mu} + q(3,1) + q^2(2^2) + q^3(2,1^2) & \text{if } n \equiv 1 \pmod 2 \\ \xmapsto{F_0} & & & \overline{\mu} & \end{array}$$

For $\overline{\mu} := (3)$ this for $n \geq 7$ yields:

$$\begin{array}{lrcll} & B^\mu & \equiv & \overline{\mu} + q(1^3) & \text{if } n \equiv 1 \pmod 2 \\ \xmapsto{F_0} & B^\mu & \equiv & \overline{\mu} + q(1^3) + q(4) + q^2(3,1) & \\ & & & \quad + q^2(2,1^2) + q^3(1^4) & \text{if } n \equiv 0 \pmod 2 \\ \xmapsto{F_1} & & & \overline{\mu} + q(1^3) & \end{array}$$

Table 8: Action on truncated Fock space.

$()$	$n \equiv 0$	$n \equiv 1$
F_0	$()$	
F_1		$()$
	$q^{-1}(1)$	$q(1)$

(1)	$n \equiv 0$	$n \equiv 1$
F_0		(1)
	$q^{-1}(2)$	$q(2)$
	(1^2)	$q^2(1^2)$
F_1	(1)	

(2)	$n \equiv 0$	$n \equiv 1$
F_0	(2)	
	$(2,1)$	$q^{-2}(2,1)$
F_1		(2)
	$q^{-1}(3)$	$q(3)$

(1^2)	$n \equiv 0$	$n \equiv 1$
F_0	(1^2)	
	$q(2,1)$	$q^{-1}(2,1)$
F_1		(1^2)
	$q^{-1}(1^3)$	$q(1^3)$

(3)	$n \equiv 0$	$n \equiv 1$
F_0		(3)
	$q^{-1}(4)$	$q(4)$
	$(3,1)$	$q^2(3,1)$
F_1	(3)	

$(2,1)$	$n \equiv 0$	$n \equiv 1$
F_0		$(2,1)$
F_1	$(2,1)$	
	$q(3,1)$	$q^{-1}(3,1)$
	$q^2(2^2)$	(2^2)
	$q^3(2,1^2)$	$q(2,1^2)$

(1^3)	$n \equiv 0$	$n \equiv 1$
F_0		(1^3)
	$q^{-1}(2,1^2)$	$q(2,1^2)$
	(1^4)	$q^2(1^4)$
F_1	(1^3)	

$\overline{\lambda} \in \mathcal{P}_4$	$n \equiv 0$	$n \equiv 1$
F_0	$\overline{\lambda}$	
F_1		$\overline{\lambda}$

For $\overline{\mu} := (2)$ this for $n \geq 7$ yields:

$$\begin{array}{rcll} B^{\mu} & \equiv & \overline{\mu} + q(1^2) + q^2(2,1) + q(2,1^2) + q^2(1^4) & \text{if } n \equiv 1 \pmod 2 \\ \overset{F_1}{\longmapsto} B^{\mu} & \equiv & \overline{\mu} + q(1^2) + q(3) + q^2(1^3) + q(3,1) & \\ & & \quad + q^2(2^2) + (q+q^3)(2,1^2) + q^2(1^4) & \text{if } n \equiv 0 \pmod 2 \\ \overset{F_0}{\longmapsto} & & \overline{\mu} + q(1^2) + (1+q^2)(2,1) + (4) + 2q(3,1) & \\ & & \quad + q^2(2^2) + (2q+q^3)(2,1^2) + 2q^2(1^4) & \\ \overset{B^{(2,1)},B^{(4)}}{\longmapsto} & & \overline{\mu} + q(1^2) + q^2(2,1) + q(2,1^2) + q^2(1^4) & \end{array}$$

For $\overline{\mu} := (1)$ this for $n \geq 5$ yields:

$$\begin{array}{rcll} B^{\mu} & \equiv & \overline{\mu} + q(1^3) & \text{if } n \equiv 1 \pmod 2 \\ \overset{F_0}{\longmapsto} B^{\mu} & \equiv & \overline{\mu} + q(2) + q^2(1^2) + q(1^3) & \\ & & \quad + q^2(2,1^2) + q^3(1^4) & \text{if } n \equiv 0 \pmod 2 \\ \overset{F_1}{\longmapsto} & & \overline{\mu} + (3) + 2q(1^3) & \\ \overset{B^{(3)}}{\longmapsto} & & \overline{\mu} + q(1^3) & \end{array}$$

Table 9: Products of divided power operators

	$\overline{\mu}$	F^{μ}
$n \geq 8$	$(3,1)$	$F_0F_1^{(2)}F_0^{(3)}F_1^{(2)} \cdot F_0F_1 \cdot F_0F_1 \cdots$
$n \geq 9$	(4)	$F_0F_1^{(2)}F_0^{(2)}F_1^{(2)}F_0^{(2)} \cdot F_1F_0 \cdot F_1F_0 \cdots$
$n \geq 6$	$(2,1)$	$F_0F_1^{(2)}F_0^{(3)} \cdot F_1F_0 \cdot F_1F_0 \cdots$
$n \geq 7$	(3)	$F_0F_1^{(2)}F_0^{(2)}F_1^{(2)} \cdot F_0F_1 \cdot F_0F_1 \cdots$
$n \geq 7$	(2)	$F_0F_1^{(2)}F_0^{(2)}F_1F_0 \cdot F_1F_0 \cdot F_1F_0 \cdots$
$n \geq 5$	(1)	$F_0F_1^{(2)}F_0F_1 \cdot F_0F_1 \cdot F_0F_1 \cdots$
$n \geq 5$	$()$	$F_0F_1F_0F_1F_0 \cdot F_1F_0 \cdot F_1F_0 \cdots$

For $\overline{\mu} := ()$ this for $n \geq 5$ yields:

$$\begin{array}{rrcll} & B^{\mu} & \equiv & \overline{\mu} + q(1^2) + q^2(1^4) & \text{if } n \equiv 1 \pmod 2 \\ \xmapsto{F_1} & B^{\mu} & \equiv & \overline{\mu} + q(1) + q(1^2) + q^2(1^3) + q^2(1^4) & \text{if } n \equiv 0 \pmod 2 \\ \xmapsto{F_0} & & & \overline{\mu} + (2) + 2q(1^2) + q^2(2,1) & \\ & & & \quad + q(2,1^2) + 2q^2(1^4) & \\ \xmapsto{B^{(2)}} & & & \overline{\mu} + q(1^2) + q^2(1^4) & \end{array}$$

• In conclusion, this yields the crystallized decomposition matrix $\mathcal{D}_n^q(\leq 4)$, for $n \geq 8$, as exhibited in Table 10. Again, for $i \in \{0,1\}$ running through the residue classes modulo 2, we use the Kronecker type notation

$$\delta = \delta_i(n) := \begin{cases} 1, & \text{if } n \equiv i \pmod 2, \\ 0, & \text{otherwise.} \end{cases}$$

We remark that for $n \leq 13$ these results are also contained in the explicit crystallized decomposition matrices given in [36, Sect.10.3].

(4.2) Decomposition matrices. We proceed to determine the decomposition matrix $\mathcal{D}_n(\leq 4)$, and at the same time the adjustment matrix $\mathcal{A}_n(\leq 4)$. The results are given in Tables 11 and 12, respectively, where we assume that $n \geq 8$. Here, we again use the Kronecker type notation introduced in (4.1), and let

$$\alpha = \alpha(n) := \begin{cases} 1, & \text{if } n \equiv 1,2 \pmod 4, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta = \beta(n) := \begin{cases} 1, & \text{if } n \equiv 3,4,5,6 \pmod 8, \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition matrices $\mathcal{D}_n$ for $n \leq 7$ are known, and for example given in [22, p.137], or accessible in the databases mentioned in Section 1. We again

Table 10: Crystallized decomposition matrix for $\mathcal{P}_n(\leq 4)$.

	$()$	(1)	(2)	(3)	$(2,1)$	(4)	$(3,1)$
$()$	1						
(1)	$q\delta_0$	1					
(2)	.	$q\delta_0$	1				
(1^2)	q	$q^2\delta_0$	q				
(3)	.	.	$q\delta_0$	1			
$(2,1)$	.	.	$q^2\delta_1$	.	1		
(1^3)	$q^2\delta_0$	q	$q^2\delta_0$	q	.		
(4)	.	.	.	$q\delta_0$	.	1	
$(3,1)$	.	.	$q\delta_0$	$q^2\delta_0$	$q\delta_1$	q	1
(2^2)	.	.	$q^2\delta_0$	.	$q^2\delta_1$	.	q
$(2,1^2)$	.	$q^2\delta_0$	$q+q^3\delta_0$	$q^2\delta_0$	$q^3\delta_1$	q	q^2
(1^4)	q^2	$q^3\delta_0$	q^2	$q^3\delta_0$	.	q^2	.

denote the Specht module associated with $\lambda = [\lambda_1, \overline{\lambda}] \in \mathcal{P}_n$ by $S_n(\overline{\lambda})$, and the irreducible module associated with $\mu = [\mu_1, \overline{\mu}] \in \mathcal{P}_n^{2\text{-reg}}$ by $D_n(\overline{\mu})$.

• The decomposition numbers of the Specht modules $S^{[n-m,m]} = S_n(m)$, where $m \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$, are known by [24], see also [22, Thm.24.15].

• Whenever $\lambda \in \mathcal{P}_n(m)$ and $\mu \in \mathcal{P}_n^{2\text{-reg}}(m)$ for some $m \in \{0, \dots, n-1\}$, the principle of first row removal, see [21], yields $[S_n(\overline{\lambda})\colon D_n(\overline{\mu})] = [S^{\overline{\lambda}}\colon D^{\overline{\mu}}]$, where the latter are decomposition numbers of $\mathcal{S}_m$, which since $m \leq 4$ are easily determined or can be looked up in [22, p.137].

• In order to complete $\mathcal{D}_n(\leq 2)$ it remains to determine the row belonging to the partition (1^2). But from $\mathcal{D}_n(\leq 2) = \mathcal{D}_n^{\zeta}(\leq 2) \cdot \mathcal{A}_n(\leq 2)$, using the entries of $\mathcal{D}_n(\leq 2)$ already known, we conclude that $\mathcal{A}_n(\leq 2)$ actually coincides with the upper left-hand (3×3)-sub-matrix of the matrix given in Table 12. This in turn determines $\mathcal{D}_n(\leq 2)$.

• Similarly, from $\mathcal{D}_n(\leq 3) = \mathcal{D}_n^{\zeta}(\leq 3) \cdot \mathcal{A}_n(\leq 3)$, using the entries of $\mathcal{D}_n(\leq 3)$ already known, we conclude that the row of $\mathcal{A}_n(\leq 3)$ belonging to the partition (3) is as given in Table 12. This in turn completes the row of $\mathcal{D}_n(\leq 3)$ belonging to (1^3). In order to complete $\mathcal{D}_n(\leq 3)$ it remains to determine the row belonging to the partition $(2,1)$. Considering 2-contents, reflecting the distribution of modules into 2-blocks, we conclude that $[S_n(2,1)\colon D_n(1)] = 0$, and $[S_n(2,1)\colon D_n(\overline{\mu})] > 0$, where $\overline{\mu} \in \{(2), ()\}$, only if $n \equiv 1 \pmod 2$.

Table 11: Decomposition matrix for $\mathcal{P}_n(\leq 4)$.

	$()$	(1)	(2)	(3)	$(2,1)$	(4)	$(3,1)$
$()$	1						
(1)	δ_0	1					
(2)	α	δ_0	1				
(1^2)	$1+\alpha$	δ_0	1				
(3)	$\alpha\delta_0$	$1-\alpha$	δ_0	1			
$(2,1)$	δ_1	.	δ_1	.	1		
(1^3)	$(1+\alpha)\delta_0$	$2-\alpha$	δ_0	1	.		
(4)	β	$(1-\alpha)\delta_0$	α	δ_0	.	1	
$(3,1)$	$1+(1-\alpha)\delta_0+\alpha\delta_1+\beta$	$(1-\alpha)\delta_0$	$\delta_0+\alpha$	δ_0	δ_1	1	1
(2^2)	$1+(1-\alpha)\delta_0+\alpha\delta_1$	.	δ_0	.	δ_1	.	1
$(2,1^2)$	$1+\delta_0+2\alpha\delta_1+\beta$	$(2-\alpha)\delta_0$	$1+\delta_0+\alpha$	δ_0	δ_1	1	1
(1^4)	$1+\alpha+\beta$	$(2-\alpha)\delta_0$	$1+\alpha$	δ_0	.	1	.

• We consider $[S_n(2,1)\colon D_n(2)]$ for $n \equiv 1 \pmod 2$: Applying (2.5) with 0-removal yields the inequality

$$\begin{aligned} & [S_n(2,1)\colon D_n(2)] \cdot [D_n(2)\downarrow_0\colon D_{n-1}(1)] \\ \leq\ & [S_{n-1}(2)\colon D_{n-1}(1)] + [S_{n-1}(1^2)\colon D_{n-1}(1)] = 2. \end{aligned}$$

Since by (2.4) we have $[D_n(2)\downarrow_0\colon D_{n-1}(1)] = 2$, using Table 10 implies $1 = [S_n(2,1)\colon D_n^{\zeta}(2)] \leq [S_n(2,1)\colon D_n(2)] \leq 1$.

• We consider $[S_n(2,1)\colon D_n(())]$ for $n \equiv 1 \pmod 2$: Applying (2.5) with 0-removal, using $[S_n(2,1)\colon D_n(2)] = 1$ yields the inequality

$$\begin{aligned} & [S_n(2,1)\colon D_n(())] + [D_n(2)\downarrow_0\colon D_{n-1}(())] \\ \leq\ & [S_{n-1}(2)\colon D_{n-1}(())] + [S_{n-1}(1^2)\colon D_{n-1}(())] \\ =\ & 1 + 2\cdot\alpha(n-1) \\ =\ & \begin{cases} 1, & \text{if } n \equiv 1 \pmod 4, \\ 3, & \text{if } n \equiv 3 \pmod 4. \end{cases} \end{aligned}$$

Moreover, if $n \equiv 3 \pmod 4$ then we have $[D_n(2)\downarrow_0] = [S_n(2)\downarrow_0] = [S_{n-1}(2)] + [S_{n-1}(1)]$, thus

$$[D_n(2)\downarrow_0\colon D_{n-1}(())] = [S_{n-1}(2)\colon D_{n-1}(())] + [S_{n-1}(1)\colon D_{n-1}(())] = 2.$$

Table 12: Adjustment matrix for $\mathcal{P}_n(\leq 4)$.

	$()$	(1)	(2)	(3)	$(2,1)$	(4)	$(3,1)$
$()$	1						
(1)	$.$	1					
(2)	α	$.$	1				
(3)	$.$	$1-\alpha$	$.$	1			
$(2,1)$	$(1-\alpha)\delta_1$	$.$	$.$	$.$	1		
(4)	β	$.$	α	$.$	$.$	1	
$(3,1)$	$(2-2\alpha)\delta_0 + 2\alpha\delta_1$	$.$	$.$	$.$	$.$	$.$	1

Hence in any case we have $[S_n(2,1)\colon D_n(())] \leq 1$. On the other hand, the hook length formula [16], see also [22, Thm.20.1], readily yields $\dim_{\mathbb{Q}}(S_n(2,1)) = \frac{1}{3}n(n-2)(n-4)$, which is odd. Since all irreducible 2-modular representations of $\mathcal{S}_n$ are self-contragredient, these all have even degree, apart from the trivial module; see also [22, Thm.11.8]. Thus $[S_n(2,1)\colon D_n(())]$ is odd, hence we infer $[S_n(2,1)\colon D_n(())] = 1$, completing $\mathcal{D}_n(\leq 3)$. We turn our attention to $\mathcal{D}_n(\leq 4)$:

• Whenever $\lambda \in \mathcal{P}_n(4)$ and $\mu \in \mathcal{P}_n^{\text{2-reg}}(3)$ such that $l(\lambda) = l(\mu) = 3$, the principle of first column removal, see [21], this time yields $[S_n(\overline{\lambda})\colon D_n(\overline{\mu})] = [S_{n-3}(\lambda_2 - 1, \lambda_3 - 1)\colon D_{n-3}(\mu_2 - 1, \mu_3 - 1)]$. This settles the cases

$$\begin{aligned} [S_n(3,1)\colon D_n(2,1)] &= [S_{n-3}(2)\colon D_{n-3}(1)], \\ [S_n(2^2)\colon D_n(2,1)] &= [S_{n-3}(1^2)\colon D_{n-3}(1)], \end{aligned}$$

where the latter decomposition numbers belong to $\mathcal{D}_{n-3}(\leq 3)$ which we have already dealt with above. Next we consider the row of $\mathcal{D}_n(\leq 4)$ belonging to the partition (2^2):

• We consider $[S_n(2^2)\colon D_n(3)]$: A consideration of 2-contents shows that we have $[S_n(2^2)\colon D_n(3)] > 0$ only if $n \equiv 0 \pmod 2$. In this case, applying (2.5) with 0-removal, $S_n(2^2)\downarrow_0 = \{0\}$ implies $[S_n(2^2)\colon D_n(3)] \cdot [D_n(3)\downarrow_0\colon D_{n-1}(3)] = 0$. Since by (2.4) we have $[D_n(3)\downarrow_0\colon D_{n-1}(3)] = 1$, we get $[S_n(2^2)\colon D_n(3)] = 0$.

• We consider $[S_n(2^2)\colon D_n(1)]$: Similarly, a consideration of 2-contents shows that $[S_n(2^2)\colon D_n(1)] > 0$ only if $n \equiv 0 \pmod 2$. In this case, applying (2.5) with 0-removal, $S_n(2^2)\downarrow_0 = \{0\}$ implies $[S_n(2^2)\colon D_n(1)] \cdot [D_n(1)\downarrow_0\colon D_{n-1}(1)] = 0$. Since by (2.4) we have $[D_n(1)\downarrow_0\colon D_{n-1}(1)] = 1$, we get $[S_n(2^2)\colon D_n(1)] = 0$.

• We consider $[S_n(2^2)\colon D_n(2)]$: If $n \equiv 1 \pmod 2$, then applying (2.5) with 0-removal yields the inequality

$$[S_n(2^2)\colon D_n(2)] \cdot [D_n(2)\downarrow_0\colon D_{n-1}(1)] \leq [S_{n-1}(2^2)\colon D_{n-1}(1)].$$

Since by (2.4) we have $[D_n(2)\downarrow_0 : D_{n-1}(1)] = 2$, and we have already seen that $[S_{n-1}(2^2) : D_{n-1}(1)] = 0$, we conclude that $[S_n(2^2) : D_n(2)] = 0$ in this case.

Similarly, if $n \equiv 0 \pmod 2$, then applying (2.5) with 1-removal yields

$$\begin{array}{rl} & [S_n(2^2) : D_n(2)] \cdot [D_n(2)\downarrow_1 : D_{n-1}(2)] \\ \leq & [S_{n-1}(2^2) : D_{n-1}(2)] + [S_{n-1}(2,1) : D_{n-1}(2)]. \end{array}$$

Since by (2.4) we have $[D_n(2)\downarrow_1 : D_{n-1}(2)] = 1$, and we have already seen that $[S_{n-1}(2^2) : D_{n-1}(2)] = 0$, using Table 10 we get that $1 \leq [S_n(2^2) : D_n(2)] \leq 1$, thus $[S_n(2^2) : D_n(2)] = 1$ in this case.

• We consider $[S_n(2^2) : D_n(())]$: Letting $x_n := [S_n(2^2) : D_n(())]$, by what we already know we have

$$[S_n(2^2)] = x_n[D_n(())] + \delta_0[D_n(2)] + \delta_1[D_n(2,1)] + [D_n(3,1)].$$

Thus, if $n \equiv 1 \pmod 2$, then by (2.4) we have $[D_n(2,1)\downarrow_0] = [D_{n-1}(2)]$ and $[D_n(3,1)\downarrow_0] = [D_{n-1}(3,1)]$, hence by 0-removal we get

$$[S_{n-1}(2^2)] = [S_n(2^2)\downarrow_0] = x_n[D_{n-1}(())] + [D_{n-1}(2)] + [D_n(3,1)],$$

implying that $x_{n-1} = x_n$. Next, it follows from $\mathcal{D}_n(\leq 4) = \mathcal{D}_n^{\zeta}(\leq 4) \cdot \mathcal{A}_n(\leq 4)$, where $\mathcal{D}_n^{\zeta}(\leq 4)$ is obtained from the crystallized decomposition matrix $\mathcal{D}_n^{q}(\leq 4)$ given in Table 10 by evaluating at $q = 1$, that $[S_n(2^2)] = [S_n(3,1)] - [S_n(4)]$. Hence, if $n \equiv 1 \pmod 2$, then [24, Thm.7.1.(i),(ii)] says that

$$x_n = \begin{cases} 2, & \text{if } n \equiv 1 \pmod 4, \\ 1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

• Thus we have completed the row of $\mathcal{D}_n(\leq 4)$ belonging to the partition (2^2). To conclude we just observe that the results obtained so far show that the adjustment matrix $\mathcal{A}_n(\leq 4)$ is as given in Table 12, which in turn completes $\mathcal{D}_n(\leq 4) = \mathcal{D}_n^{\zeta}(\leq 4) \cdot \mathcal{A}_n(\leq 4)$.

(4.3) Degree formulae. We are now prepared to obtain degree formulae for the irreducible modular representations parameterized by $\mathcal{P}_n^{2\text{-reg}}(\leq 4)$. They are given in Tables 13 and 14, where for $\mu \in \mathcal{P}_n^{2\text{-reg}}$ we again let $d^\mu := \dim_{\mathbb{F}_2}(D^\mu)$:

In view of the decomposition matrix $\mathcal{D}_n(\leq 4)$ for $n \geq 8$ given in Table 11, and the known decomposition numbers for $n \leq 7$, see the comment at the beginning of (4.2), these follow straightforwardly from the hook length formula for the dimension of Specht modules, see [16] and also [22, Thm.20.1].

5 James's Theorem revisited

Let p be a rational prime. For $n \in \mathbb{N}_0$ and $\mu \in \mathcal{P}_n^{p\text{-reg}}$ we let $d^\mu := \dim_{\mathbb{F}_p}(D^\mu)$. In [20], a description of the growth behavior of d^μ, for $\mu \in \mathcal{P}_n^{p\text{-reg}}(m)$, is given,

Table 13: Degree formulae for $\mathcal{P}_n^{\text{2-reg}}(\leq 3)$.

μ	d^μ	condition
$[n]$	1	
$[n-1,1]$	$n-2$	$n \equiv 0 \pmod 2$
$(n \geq 3)$	$n-1$	$n \equiv 1 \pmod 2$
$[n-2,2]$	$\frac{1}{2}(n^2-5n+2)$	$n \equiv 2 \pmod 4$
$(n \geq 4)$	$\frac{1}{2}(n^2-5n+4)$	$n \equiv 0 \pmod 4$
	$\frac{1}{2}(n^2-3n-2)$	$n \equiv 1 \pmod 4$
	$\frac{1}{2}(n^2-3n)$	$n \equiv 3 \pmod 4$
$[n-3,3]$	$\frac{1}{6}(n^3-9n^2+14n)$	$n \equiv 0 \pmod 4$
$(n \geq 7)$	$\frac{1}{6}(n^3-9n^2+20n-12)$	$n \equiv 2 \pmod 4$
	$\frac{1}{6}(n^3-6n^2-n+6)$	$n \equiv 3 \pmod 4$
	$\frac{1}{6}(n^3-6n^2+5n)$	$n \equiv 1 \pmod 4$
$[n-3,2,1]$	$\frac{1}{6}(2n^3-15n^2+25n-6)$	$n \equiv 3 \pmod 4$
$(n \geq 6)$	$\frac{1}{6}(2n^3-15n^2+25n)$	$n \equiv 1 \pmod 4$
	$\frac{1}{6}(2n^3-12n^2+16n)$	$n \equiv 0 \pmod 2$

Table 14: Degree formulae for $\mathcal{P}_n^{\text{2-reg}}(4)$.

μ	d^μ	condition
$[n-4,4]$	$\frac{1}{24}(n^4-14n^3+47n^2-34n)$	$n \equiv 6 \pmod 8$
$(n \geq 9)$	$\frac{1}{24}(n^4-14n^3+47n^2-34n+24)$	$n \equiv 2 \pmod 8$
	$\frac{1}{24}(n^4-14n^3+59n^2-94n+24)$	$n \equiv 4 \pmod 8$
	$\frac{1}{24}(n^4-14n^3+59n^2-94n+48)$	$n \equiv 0 \pmod 8$
	$\frac{1}{24}(n^4-10n^3+11n^2+22n)$	$n \equiv 5 \pmod 8$
	$\frac{1}{24}(n^4-10n^3+11n^2+22n+24)$	$n \equiv 1 \pmod 8$
	$\frac{1}{24}(n^4-10n^3+23n^2-14n-24)$	$n \equiv 3 \pmod 8$
	$\frac{1}{24}(n^4-10n^3+23n^2-14n)$	$n \equiv 7 \pmod 8$
$[n-4,3,1]$	$\frac{1}{24}(2n^4-28n^3+118n^2-140n-48)$	$n \equiv 1 \pmod 4$
$(n \geq 8)$	$\frac{1}{24}(2n^4-28n^3+118n^2-140n)$	$n \equiv 3 \pmod 4$
	$\frac{1}{24}(2n^4-20n^3+46n^2+20n-96)$	$n \equiv 0 \pmod 4$
	$\frac{1}{24}(2n^4-20n^3+46n^2+20n-48)$	$n \equiv 2 \pmod 4$

where $m \in \mathbb{N}_0$ is fixed and $n \in \mathbb{N}_0$ is allowed to tend to infinity. We are going to present two improvements of [20, La.4], the first one for arbitrary $p \geq 2$, while the second one takes care of particular phenomena occurring in the case $p = 2$.

Later on, we will need the following statement, which is an immediate consequence of [20, La.3] and its proof, and is also contained in [28, Thm.5.1] for $p \geq 3$. We present another proof in the spirit of this work:

(5.1) Proposition: [20, La.3].
Let $n \geq 2$ and $\mu \in \mathcal{P}_n^{p\text{-}reg}$ such that the restriction $D^\mu \downarrow_{\mathcal{S}_{n-2}}$ of D^μ to $\mathcal{S}_{n-2}$ is irreducible. Then we have $\mu = [n]$ or $\mu = [1^n]^R$.

Proof. Let $b := \mu_1$ be the largest part of μ, where we may assume that $b \geq 2$. By (2.4), μ has a single normal node, x say, which is necessarily good. Since the (a, b)-node, where $a \in \{1, \ldots, p-1\}$ is the multiplicity of b in μ, is the rightmost removable one and hence is normal, we infer that x is the (a, b)-node. Let $\mu^* := \mu \setminus \{x\} \in \mathcal{P}_{n-1}^{p\text{-reg}}$ for short. Then, by assumption, μ^* also has a single normal node, x^* say, which also is necessarily good. We distinguish two cases:

• Let $a \geq 2$; note that this only occurs if $p \geq 3$. Then μ^* has largest part b, occurring with multiplicity $a - 1$, and second largest part $b - 1$, occurring with multiplicity $a' \in \{1, \ldots, p-1\}$. By the same reasoning as above we infer that x^* is the $(a-1, b)$-node. Hence the rightmost removable node y of μ^* left of x^*, the $(a + a' - 1, b - 1)$-node, is not normal. As y and the addable node x have distinct p-residues, we conclude that y has the same p-residue as the addable $(1, b + 1)$-node. Thus we get $a + a' \equiv 0 \pmod p$. Since $3 \leq a + a' \leq 2p - 2$ we conclude $a + a' = p$ and hence y is the $(p - 1, b - 1)$-node.

Assume there is a further removable node z of μ^* left of y, and pick the rightmost one amongst those. Since z is not normal, it has a p-residue different from that of the addable $(1, b+1)$-node and the removable node y. Thus it has the same p-residue as the addable node x. But z also is a removable node of μ, contradicting the fact that μ has a unique normal node. Hence μ only has the parts b and $b - 1$, with multiplicities a and $p - a - 1$, respectively, that is $\mu = [1^n]^R$.

• Let $a = 1$. Then μ^* has largest part $b - 1$, occurring with multiplicity $a' \in \{1, \ldots, p - 1\}$, thus x^* is the $(a', b - 1)$-node. We again distinguish two cases:

◦ Let $a' \geq 2$; note again that this only occurs if $p \geq 3$. Then x^* also is a removable node of μ, and since it is not normal, it has the same p-residue as the addable $(1, b + 1)$-node. Thus we get $a' \equiv -1 \pmod p$, thus $a' = p - 1$.

Assume there is a further removable node z of μ^* left of x^*, and pick the rightmost one amongst those. Since z is not normal, it has a p-residue different from that of the addable $(1, b+1)$-node and the removable node x^*. Thus it has the same p-residue as the addable node x. But z also is a removable node of μ, contradicting the fact that μ has a unique normal node. Hence μ only has the parts b and $b-1$, with multiplicities 1 and $p-2$, respectively, that is $\mu = [1^n]^R$.

∘ Let $a' = 1$. Assume there is a further removable node z of μ^* left of x^*, and pick the rightmost one amongst those. Since z is not normal, it has the same p-residue as the addable node x. But z also is a removable node of μ, contradicting the fact that μ has a unique normal node. Hence μ only has the part b, with multiplicity 1, that is $\mu = [n]$. ♯

(5.2) Theorem. *Let $m \in \mathbb{N}_0$, let $n_0 \in \mathbb{N}_0$ such that $n_0 \geq 2m+5$ if $p \geq 3$, and let $f\colon \{n_0, n_0+1, \ldots\} \to \mathbb{R}$ fulfilling $2 \cdot f(n) \geq \max\{f(n+1), f(n+2)\}$ for all $n \geq n_0$. Moreover, for $n \in \mathbb{N}_0$ let*

$$\mathcal{M}_n(\geq m+1) := \{\mu \in \mathcal{P}_n^{p\text{-}reg}; \mu, \mu^M \notin \mathcal{P}_n^{p\text{-}reg}(\leq m)\}.$$

Then we have $d^\mu \geq f(n)$ for all $n \geq n_0$ and $\mu \in \mathcal{M}_n(\geq m+1)$, provided this inequality is known to hold for all $\mu \in \mathcal{M}_n(\geq m+1)$ such that

$$n \in \{n_0, n_0+1\} \quad \text{or} \quad \{\mu, \mu^M\} \cap \mathcal{P}_n^{p\text{-}reg}(m+1) \neq \emptyset.$$

Proof. We proceed by induction on $n \geq n_0$, where by the assumptions made we may assume that $n \geq n_0 + 2$, and $\mu \in \mathcal{P}_n^{p\text{-reg}}(k)$ and $\mu^M \in \mathcal{P}_n^{p\text{-reg}}(k')$, where $k, k' \geq m+2$. We consider the restriction $D^\mu \downarrow_{\mathcal{S}_{n-1}}$ of D^μ to $\mathcal{S}_{n-1}$; recall that $D^{\mu^M} \cong D^\mu \otimes D^{[1^n]^R}$, where $D^{[1^n]^R}$ is the sign representation, and hence $D^{\mu^M} \downarrow_{\mathcal{S}_{n-1}} \cong (D^\mu \downarrow_{\mathcal{S}_{n-1}}) \otimes D^{[1^{n-1}]^R}$. We distinguish two cases:

a) Assume that D^μ restricts reducibly to $\mathcal{S}_{n-1}$. By (2.4) we conclude that μ has at least two normal nodes. We again distinguish two cases:

i) There is an i-good node $x \in N_i(\mu)$, for some $i \in \{0, \ldots, p-1\}$, such that $r_i(\mu, x) > a_i(\mu, x)$. Hence by (2.4) we have $[D^\mu \downarrow_i \colon D^{\mu \setminus \{x\}}] \geq 2$, where, since x cannot possibly belong to the first row of μ, we have

$$\mu \setminus \{x\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(k-1).$$

Moreover, by [32, Alg.4.8], there is an i'-good node $x' \in N_{i'}(\mu^M)$, where $i' \equiv -i \pmod p$ and $r_{i'}(\mu^M, x') - a_{i'}(\mu^M, x') = r_i(\mu, x) - a_i(\mu, x)$, entailing that

$$(\mu \setminus \{x\})^M = \mu^M \setminus \{x'\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(k'-1).$$

Thus by induction we have $d^\mu \geq 2 \cdot d^{\mu \setminus \{x\}} \geq 2 \cdot f(n-1) \geq f(n)$.

ii) There are an i-good node $x \in N_i(\mu)$ and a j-good node $y \in N_j(\mu)$, for some $i \neq j \in \{0, \ldots, p-1\}$, such that $r_i(\mu, x) = a_i(\mu, x)$ and $r_j(\mu, y) = a_j(\mu, y)$. Hence by (2.4) we have $[D^\mu \downarrow_i \colon D^{\mu \setminus \{x\}}] = 1$ and $[D^\mu \downarrow_j \colon D^{\mu \setminus \{y\}}] = 1$, where

$$\mu \setminus \{x\},\ \mu \setminus \{y\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(k-1) \,\dot\cup\, \mathcal{P}_{n-1}^{p\text{-reg}}(k).$$

Moreover, there are an i'-good node $x' \in N_{i'}(\mu^M)$ and a j'-good node $y' \in N_{j'}(\mu^M)$, where $i' \equiv -i \pmod p$ and $j' \equiv -j \pmod p$, and $r_{i'}(\mu^M, x') = a_{i'}(\mu^M, x')$ and $r_{j'}(\mu^M, y') = a_{j'}(\mu^M, y')$, entailing that both

$$(\mu \setminus \{x\})^M = \mu^M \setminus \{x'\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(k'-1) \,\dot\cup\, \mathcal{P}_{n-1}^{p\text{-reg}}(k')$$

and

$$(\mu \setminus \{y\})^M = \mu^M \setminus \{y'\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(k'-1) \,\dot\cup\, \mathcal{P}_{n-1}^{p\text{-reg}}(k').$$

Thus by induction we have $d^\mu \geq d^{\mu\setminus\{x\}} + d^{\mu\setminus\{y\}} \geq 2 \cdot f(n-1) \geq f(n)$.

b) Assume that D^μ restricts irreducibly to $\mathcal{S}_{n-1}$. By (2.4) we conclude that μ has a single normal node, x say, which hence is an i-good node for some $i \in \{0, \ldots, p-1\}$, and we have $r_i(\mu, x) = a_i(\mu, x)$. Letting $\mu^* := \mu \setminus \{x\}$ for short, we have $D^\mu \downarrow_{\mathcal{S}_{n-1}} \cong D^{\mu^*}$, where for some $\epsilon \in \{0, 1\}$ we have

$$\mu^* \in \mathcal{P}_{n-1}^{p\text{-reg}}(k - \epsilon).$$

Moreover, μ^M also has a single normal node, x' say, which, by [32, Alg.4.8], is an i'-good node where $i' \equiv -i \pmod p$ and $r_{i'}(\mu^M, x') = a_{i'}(\mu^M, x')$. Letting $(\mu^M)^* := \mu^M \setminus \{x'\}$, for some $\epsilon' \in \{0, 1\}$ we get

$$(\mu^*)^M = (\mu^M)^* \in \mathcal{P}_{n-1}^{p\text{-reg}}(k' - \epsilon').$$

Assume that D^{μ^*} restricts irreducibly to $\mathcal{S}_{n-2}$. Then from (5.1) we conclude that $\mu = [n]$ or $\mu = [1^n]^R$. In the first case we have $\mu \in \mathcal{P}_n^{p\text{-reg}}(0)$, while in the latter case we have $\mu = [1^n]^R = [n]^M$, hence $\mu^M \in \mathcal{P}_n^{p\text{-reg}}(0)$. Thus in both cases we arrive at a contradiction. Hence we may assume that D^{μ^*} restricts reducibly to $\mathcal{S}_{n-2}$. We again distinguish two cases:

i) We have $k - \epsilon \geq m + 2$ and $k' - \epsilon' \geq m + 2$. Then, by what we have already seen in (a), $D^{\mu^*} \downarrow_{\mathcal{S}_{n-2}}$ has constituents $D^{\mu^*\setminus\{y\}}$ and $D^{\mu^*\setminus\{z\}}$, where y and z are good nodes of μ^*, which are possibly identical, and

$$\mu^* \setminus \{y\},\ \mu^* \setminus \{z\} \in \mathcal{P}_{n-2}^{p\text{-reg}}(k - \epsilon - 1) \,\dot\cup\, \mathcal{P}_{n-2}^{p\text{-reg}}(k - \epsilon).$$

Moreover, there are good nodes y' and z' of $(\mu^*)^M$, which are possibly identical, such that

$$(\mu^* \setminus \{y\})^M = (\mu^*)^M \setminus \{y'\} \in \mathcal{P}_{n-2}^{p\text{-reg}}(k' - \epsilon' - 1) \,\dot\cup\, \mathcal{P}_{n-2}^{p\text{-reg}}(k' - \epsilon')$$

and

$$(\mu^* \setminus \{z\})^M = (\mu^*)^M \setminus \{z'\} \in \mathcal{P}_{n-2}^{p\text{-reg}}(k' - \epsilon' - 1) \,\dot\cup\, \mathcal{P}_{n-2}^{p\text{-reg}}(k' - \epsilon').$$

Thus by induction we have $d^\mu = d^{\mu^*} \geq d^{\mu^*\setminus\{y\}} + d^{\mu^*\setminus\{z\}} \geq 2 \cdot f(n-2) \geq f(n)$.

ii) We have $k - \epsilon = m + 1$ or $k' - \epsilon' = m + 1$. Hence we may assume that

$$\mu \in \mathcal{P}_n^{p\text{-reg}}(m+2) \quad \text{and} \quad \mu^* = \mu \setminus \{x\} \in \mathcal{P}_{n-1}^{p\text{-reg}}(m+1).$$

Thus x, being the rightmost removable node of μ, does not belong to the first row of μ. Hence the rightmost node of the first row is not removable, that is for the first two parts of μ we have $\mu_1 = \mu_2$. This cannot happen if $p = 2$, thus we have $p \geq 3$. Moreover, we have $n - m - 2 = \mu_1 = \mu_2 \leq m + 2$, or equivalently $n \leq 2m + 4$. Hence, since we are assuming that $n \geq 2m + 5$, this case cannot happen either. ♯

(5.3) Remark. The statement of (5.2), and the strategy of proof, are reminiscent of [20, La.4]. But while the proof there employs the ordinary branching rule, see for example [22, Thm.9.2], here we make use of the modular branching rule (2.4), which of course had not been available at the time of writing of [20].

The statement of [20, La.4] is similar to the one given here, but our condition '$\{\mu, \mu^M\} \cap \mathcal{P}_n^{p\text{-reg}}(m+1) \neq \emptyset$' is replaced there by the stronger one

$$\text{`}\{\mu, \mu^M\} \cap \left(\mathcal{P}_n^{p\text{-reg}}(m+1) \,\dot{\cup}\, \mathcal{P}_n^{p\text{-reg}}(m+2)\right) \neq \emptyset\text{'}.$$

As it turns out in practice, only having to check a weaker condition saves quite a bit of explicit computation, and lends itself to a treatment independent of p, see (6.2). In particular, for the case $p = 3$ and $m = 3$ this leads to a shorter proof of [8, Prop.3.1], together with a smaller lower bound n_0.

(5.4) Remark. The assumptions of case (b) of the proof of (5.2) say that the restriction $D^\mu \downarrow_{\mathcal{S}_{n-1}}$ of D^μ to $\mathcal{S}_{n-1}$ is irreducible, hence in theses cases μ is a Jantzen-Seitz partition [28], see [14, 34, 35].

Case (b)(ii) of the proof of (5.2) shows that for $p \geq 3$ we can weaken the conditions slightly more as follows: Note that $\mathcal{M}_n(\geq m+1) = \emptyset$ for $n \leq m+1$ anyway. Then, leaving out the condition '$n_0 \geq 2m+5$', we assume instead that the desired inequality holds for all $n \in \{m+3, \ldots, 2m+4\}$ and all Jantzen-Seitz partitions $\mu \in \mathcal{M}_n(\geq m+2)$ such that $\mu \in \mathcal{P}_n^{p\text{-reg}}(m+2)$ and $\mu_1 = \mu_2$.

For fixed $m \in \mathbb{N}_0$, the latter condition leads to a finite set of cases to be checked additionally. Actually, for small m this set turns out to be very small: For any $m \in \{0, \ldots, 6\}$ there is just the single case $n = 2m+4$ and $\mu = [\frac{n}{2}, \frac{n}{2}] = [m+2, m+2]$ if $p \geq 5$, and no case at all for $p = 3$.

Unfortunately, this pattern does not continue: For $m = 7$ and $p = 5$, next to $\mu = [9^2] \in \mathcal{P}_{18}^{5\text{-reg}}(9)$, we get $\mu = [8^2, 1] \in \mathcal{P}_{17}^{5\text{-reg}}(9)$ and $\mu = [7^2, 1^2] \in \mathcal{P}_{16}^{5\text{-reg}}(9)$, and there are many more examples for larger m. But we have not been able to find any example for $p = 3$, so that we are wondering whether in this case there are any at all. We are tempted to ask for a classification of the Jantzen-Seitz partitions $\mu \in \mathcal{P}_n^{p\text{-reg}}$ such that $\mu_1 = \mu_2$. To our knowledge this is not available in the literature, and we leave it as an open question to the reader. ♯

Although the above theorem also holds for the case $p = 2$, it would not be strong enough for our purposes, the reason being the existence of 'very small' representations escaping the desired growth behavior in low degrees. These we consider next, in order to proceed to prove an improved theorem.

(5.5) Basic spin representations. Let $p := 2$.

- For $n \in \mathbb{N}$ let $\mu_{\text{bs}}(n) \in \mathcal{P}_n^{2\text{-reg}}$ be defined by

$$\mu_{\text{bs}}(n) := \begin{cases} [\frac{n+2}{2}, \frac{n-2}{2}], & \text{if } n \equiv 0 \pmod 2, \\ [\frac{n+1}{2}, \frac{n-1}{2}], & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The associated irreducible $\mathbb{F}_2\mathcal{S}_n$-module $D_n(\mathrm{bs})$, being called the basic spin module, by [24, Cor.5.7], see also [2, La.5.3] or [48, Tbl.III], has dimension

$$\dim_{\mathbb{F}_2}(D_n(\mathrm{bs})) = 2^{\lfloor \frac{n-1}{2} \rfloor} = \begin{cases} 2^{\frac{n-2}{2}}, & \text{if } n \equiv 0 \pmod 2, \\ 2^{\frac{n-1}{2}}, & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

For later use, in Table 15 we record the results obtained from removing i-good and adding i-cogood nodes to $\mu_{\mathrm{bs}}(n)$, for $n \geq 3$. Here, the rows are indexed by the congruence classes of n modulo 4, and missing entries indicate the non-existence of i-normal and i-conormal nodes, respectively. Using this, the degree formulae and (2.4), for $n \geq 2$ we infer

$$\begin{array}{rcccll} D_n(\mathrm{bs})\downarrow_{\mathcal{S}_{n-1}} & \cong & D_n(\mathrm{bs})\downarrow_0 & \cong & D_{n-1}(\mathrm{bs}), & \text{if } n \equiv 0 \pmod 4, \\ {[D_n(\mathrm{bs})\downarrow_{\mathcal{S}_{n-1}}]} & = & [D_n(\mathrm{bs})\downarrow_0] & = & 2\cdot[D_{n-1}(\mathrm{bs})], & \text{if } n \equiv 1 \pmod 4, \\ D_n(\mathrm{bs})\downarrow_{\mathcal{S}_{n-1}} & \cong & D_n(\mathrm{bs})\downarrow_1 & \cong & D_{n-1}(\mathrm{bs}), & \text{if } n \equiv 2 \pmod 4, \\ {[D_n(\mathrm{bs})\downarrow_{\mathcal{S}_{n-1}}]} & = & [D_n(\mathrm{bs})\downarrow_1] & = & 2\cdot[D_{n-1}(\mathrm{bs})], & \text{if } n \equiv 3 \pmod 4. \end{array}$$

- For $n \geq 6$ let $\mu_{\mathrm{bbs}}(n) \in \mathcal{P}_n^{2\text{-reg}}$ be defined by

$$\mu_{\mathrm{bbs}}(n) := \begin{cases} [\frac{n}{2}, \frac{n-2}{2}, 1], & \text{if } n \equiv 0 \pmod 2, \\ [\frac{n+1}{2}, \frac{n-3}{2}, 1], & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The associated irreducible $\mathbb{F}_2\mathcal{S}_n$-module $D_n(\mathrm{bbs})$, being called the second basic spin module, by [2, Thm.1.2] and [48, Tbl.IV], see also [24, Thm.7.1] or [8, Cor.4.2], has dimension

$$\dim_{\mathbb{F}_2}(D_n(\mathrm{bbs})) = \begin{cases} (n-3)\cdot 2^{\frac{n-2}{2}}, & \text{if } n \equiv 0 \pmod 4, \\ (n-4)\cdot 2^{\frac{n-3}{2}}, & \text{if } n \equiv 1 \pmod 4, \\ (n-2)\cdot 2^{\frac{n-2}{2}}, & \text{if } n \equiv 2 \pmod 4, \\ (n-2)\cdot 2^{\frac{n-3}{2}}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Again, in Table 15 we record the results obtained from removing i-good and adding i-cogood nodes to $\mu_{\mathrm{bbs}}(n)$, for $n \geq 6$. Using this, the degree formulae and (2.4), for $n \geq 7$ we infer

$$\begin{array}{rcll} [D_n(\mathrm{bbs})\downarrow_1] & = & 2\cdot[D_{n-1}(\mathrm{bbs})], & \text{if } n \equiv 0 \pmod 4. \\ D_n(\mathrm{bbs})\downarrow_0 & \cong & D_{n-1}(\mathrm{bbs}), & \text{if } n \equiv 1 \pmod 4, \\ {[D_n(\mathrm{bbs})\downarrow_0]} & = & 3\cdot[D_{n-1}(\mathrm{bs})] + 2\cdot[D_{n-1}(\mathrm{bbs})], & \text{if } n \equiv 2 \pmod 4, \\ D_n(\mathrm{bbs})\downarrow_{\mathcal{S}_{n-1}} & \cong & D_{n-1}(\mathrm{bs}) \oplus D_{n-1}(\mathrm{bbs}), & \text{if } n \equiv 3 \pmod 4, \end{array}$$

where we only record non-zero i-restrictions, and for $n \equiv 3 \pmod 4$ the direct summands refer to 0- and 1-restriction, respectively; for $n = 6$ we get $[D_6(\mathrm{bbs})\downarrow_0] = 3\cdot[D_5(\mathrm{bs})] + 4\cdot[D([5])]$.

- We conclude from Table 15 that adding an i-cogood node to $\mu_{\mathrm{bs}}(n)$, we end up with partitions in $\{\mu_{\mathrm{bs}}(n+1), \mu_{\mathrm{bbs}}(n+1)\}$, and repeating this step, possibly with a different i, we end up with partitions in $\{\mu_{\mathrm{bs}}(n+2), \mu_{\mathrm{bbs}}(n+2)\}$.

Table 15: Removal and addition of nodes.

$\mu_{\text{bs}}(n)$				
n	0-good	1-good	0-cogood	1-cogood
0	$\mu_{\text{bs}}(n-1)$		$\mu_{\text{bs}}(n+1)$	
1	$\mu_{\text{bs}}(n-1)$		$\mu_{\text{bbs}}(n+1)$	$\mu_{\text{bs}}(n+1)$
2		$\mu_{\text{bs}}(n-1)$	$\mu_{\text{bbs}}(n+1)$	$\mu_{\text{bs}}(n+1)$
3		$\mu_{\text{bs}}(n-1)$	$\mu_{\text{bs}}(n+1)$	

$\mu_{\text{bbs}}(n)$				
n	0-good	1-good	0-cogood	1-cogood
0		$\mu_{\text{bbs}}(n-1)$	$\mu_{\text{bbs}}(n+1)$	$[\frac{n}{2}, \frac{n-2}{2}, 2]$
1	$\mu_{\text{bbs}}(n-1)$			$[\frac{n+1}{2}, \frac{n-3}{2}, 2]$
2	$\mu_{\text{bs}}(n-1)$			$\mu_{\text{bbs}}(n+1)$
3	$\mu_{\text{bs}}(n-1)$	$\mu_{\text{bbs}}(n-1)$		$\mu_{\text{bbs}}(n+1)$

(5.6) Theorem. *Let $p := 2$, let $m \in \mathbb{N}_0$, let $n_0 \geq 5$, let $f\colon \{n_0, n_0+1, \ldots\} \to \mathbb{R}$ fulfilling $2 \cdot f(n) \geq \max\{f(n+1), f(n+2)\}$ for all $n \geq n_0$, and for $n \in \mathbb{N}$ let*

$$\mathcal{M}'_n(\geq m+1) := \mathcal{P}_n^{p\text{-}reg} \setminus (\mathcal{P}_n^{p\text{-}reg}(\leq m) \cup \{\mu_{bs}(n)\}).$$

Then we have $d^\mu \geq f(n)$ for all $n \geq n_0$ and $\mu \in \mathcal{M}'_n(\geq m+1)$, provided this inequality is known to hold for all $\mu \in \mathcal{M}'_n(\geq m+1)$ such that

$$n \in \{n_0, n_0+1\} \quad or \quad \mu \in \mathcal{P}_n^{p\text{-}reg}(m+1).$$

Proof. We again proceed by induction on $n \geq n_0$, where by the assumptions made we may assume that $n \geq n_0 + 2 \geq 7$, and $\mu \in \mathcal{P}_n^{2\text{-reg}}(k)$ where $k \geq m+2$, and are again going to consider the restriction of D^μ to $\mathcal{S}_{n-1}$. We first assume $\mu \neq \mu_{\text{bbs}}(n)$, and distinguish two cases:

a) Assume that D^μ restricts reducibly to $\mathcal{S}_{n-1}$. By (2.4) we conclude that μ has at least two normal nodes. We again distinguish two cases:

i) There is an i-good node $x \in N_i(\mu)$, for some $i \in \{0,1\}$, such that $r_i(\mu, x) > a_i(\mu, x)$. Hence by (2.4) we have $[D^\mu \downarrow_i : D^{\mu \setminus \{x\}}] \geq 2$, where, since x cannot possibly belong to the first row of μ, we have

$$\mu \setminus \{x\} \in \mathcal{P}_{n-1}^{2\text{-reg}}(k-1).$$

Since from Table 15 we conclude that $\mu \setminus \{x\} \neq \mu_{\text{bs}}(n-1)$, by induction we have $d^\mu \geq 2 \cdot d^{\mu \setminus \{x\}} \geq 2 \cdot f(n-1) \geq f(n)$.

ii) There are an i-good node $x \in N_i(\mu)$ and a j-good node $y \in N_j(\mu)$, for some $i \neq j \in \{0,1\}$, such that $r_i(\mu, x) = a_i(\mu, x)$ and $r_j(\mu, y) = a_j(\mu, y)$. Hence by (2.4) we have $[D^\mu \downarrow_i : D^{\mu \setminus \{x\}}] = 1$ and $[D^\mu \downarrow_j : D^{\mu \setminus \{y\}}] = 1$, where both

$$\mu \setminus \{x\},\, \mu \setminus \{y\} \in \mathcal{P}_{n-1}^{2\text{-reg}}(k-1) \mathbin{\dot\cup} \mathcal{P}_{n-1}^{2\text{-reg}}(k).$$

Since from Table 15 we conclude that $\mu\backslash\{x\} \neq \mu_{\text{bs}}(n-1) \neq \mu\backslash\{y\}$, by induction we have $d^\mu \geq d^{\mu\backslash\{x\}} + d^{\mu\backslash\{y\}} \geq 2 \cdot f(n-1) \geq f(n)$.

b) Assume that D^μ restricts irreducibly to $\mathcal{S}_{n-1}$. By (2.4) we conclude that μ has a single normal node, $x \in N_i(\mu)$ for some $i \in \{0,1\}$, which hence is i-good and we have $r_i(\mu,x) = a_i(\mu,x)$. Since μ is 2-regular, for the first two parts of μ we infer $\mu_1 \neq \mu_2$, implying that the rightmost node of the first row is removable, and hence coincides with the normal node x. Letting $\mu^* := \mu \setminus \{x\}$ for short, we have $D^\mu \downarrow_{\mathcal{S}_{n-1}} \cong D^{\mu^*}$, where $\mu^* \in \mathcal{P}^{\text{2-reg}}_{n-1}(k)$.

Next, in view of Table 15, assume that $n \equiv 1 \pmod 4$ and $\mu = [\frac{n-1}{2}, \frac{n-3}{2}, 2]$. Then μ has three 1-normal nodes, a contradiction. Similarly, assume that $n \equiv 2 \pmod 4$ and $\mu = [\frac{n}{2}, \frac{n-4}{2}, 2]$. Then μ has both a 0-normal and a 1-normal node, a contradiction as well. Thus, since $\mu \notin \{\mu_{\text{bs}}(n), \mu_{\text{bbs}}(n)\}$, from Table 15 we conclude that $\mu^* \notin \{\mu_{\text{bs}}(n-1), \mu_{\text{bbs}}(n-1)\}$.

Assume that D^{μ^*} restricts irreducibly to $\mathcal{S}_{n-2}$. Then from (5.1) we conclude that $\mu = [n] \in \mathcal{P}^{\text{2-reg}}_n(0)$, a contradiction. Hence we may assume that D^{μ^*} restricts reducibly to $\mathcal{S}_{n-2}$. Thus, by what we have already seen in (a), $D^{\mu^*} \downarrow_{\mathcal{S}_{n-2}}$ has constituents $D^{\mu^*\backslash\{y\}}$ and $D^{\mu^*\backslash\{z\}}$, where y and z are good nodes of μ^*, which are possibly identical, and

$$\mu^* \setminus \{y\},\, \mu^* \setminus \{z\} \in \mathcal{P}^{\text{2-reg}}_{n-2}(k-1) \mathbin{\dot\cup} \mathcal{P}^{\text{2-reg}}_{n-2}(k).$$

Thus by induction we have $d^\mu = d^{\mu^*} \geq d^{\mu^*\backslash\{y\}} + d^{\mu^*\backslash\{z\}} \geq 2 \cdot f(n-2) \geq f(n)$.

• Let now $\mu = \mu_{\text{bbs}}(n)$. We again distinguish two cases:

i) Let $n \equiv 0 \pmod 2$, hence $k = \frac{n}{2}$. Then, by (5.5), μ has a normal node x, belonging to the second row of μ, such that

$$\mu \setminus \{x\} = \mu_{\text{bbs}}(n-1) \in \mathcal{P}^{\text{2-reg}}_{n-1}(k-1)$$

and $[D_n(\text{bbs}) \downarrow_{\mathcal{S}_{n-1}} : D_{n-1}(\text{bbs})] = 2$. Hence by induction we from this infer that $d^\mu \geq 2 \cdot d^{\mu\backslash\{x\}} \geq 2 \cdot f(n-1) \geq f(n)$.

ii) Let $n \equiv 1 \pmod 2$, such that $n \geq 9$, hence $k = \frac{n-1}{2}$. Then, by (5.5), μ has a good node x, belonging to the first row of μ, such that

$$\mu \setminus \{x\} = \mu_{\text{bbs}}(n-1) \in \mathcal{P}^{\text{2-reg}}_{n-1}(k)$$

and $[D_n(\text{bbs}) \downarrow_{\mathcal{S}_{n-1}} : D_{n-1}(\text{bbs})] = 1$. By what we have already seen in (i), we have $[D_{n-1}(\text{bbs}) \downarrow_{\mathcal{S}_{n-2}} : D_{n-2}(\text{bbs})] = 2$, where $\mu_{\text{bbs}}(n-2) \in \mathcal{P}^{\text{2-reg}}_{n-1}(k-1)$. Thus by induction we have $d^\mu \geq d^{\mu\backslash\{x\}} \geq 2 \cdot f(n-2) \geq f(n)$.

Finally, for $n = 7$, which occurs only if $n_0 = 5$, we have $\mu = [4,2,1]$, hence $k = 3$, implying $m \leq 1$. We have $d^\mu = 20$, as well as $d^{[6,1]} = 6$ and $d^{[5,2]} = 14$, by [24], see also [22, Thm.24.15], or by looking up the decomposition matrix $\mathcal{D}_7$ in [22, p.137] or in the databases mentioned in Section 1. This implies $d^\mu \geq f(7)$. $\sharp$

(5.7) Remark. Note that using the notation of (5.2) we have $\mathcal{M}'_n(\geq m+1) := \mathcal{M}_n(\geq m+1) \setminus \{\mu_{\text{bs}}(n)\}$, and that $\mathcal{M}'_n(\geq m+1) = \emptyset$ for $n \leq 4$.

The statement of (5.6), and the strategy of proof, are reminiscent of [8, Thm.4.3], but there stronger conditions on the growth behavior of f are needed, and our condition '$\mu \in \mathcal{P}_n^{p\text{-reg}}(m+1)$' is replaced there by the stronger one

$$\text{`}\mu \in \mathcal{P}_n^{p\text{-reg}}(m+1) \,\dot{\cup}\, \mathcal{P}_n^{p\text{-reg}}(m+2)\text{'}.$$

Again, in practice, only having to check weaker conditions saves quite a bit of explicit computation, see (6.2). In particular, for the case $m = 3$ this leads to a shorter proof of [8, Prop.4.4], together with a smaller lower bound n_0.

6 Explicit results

We are now prepared to tackle the problem of classifying all irreducible representations D^μ of $\mathcal{S}_n$, where $\mu \in \mathcal{P}_n^{p\text{-reg}}$, satisfying $d^\mu := \dim_{\mathbb{F}_p}(D^\mu) \leq n^3$. The general strategy to obtain this classification is described in (6.3), and subsequently it is applied to compile explicit lists for $p \leq 7$. To start with, as an immediate corollary of the information collected above, we have the following:

(6.1) Degree formulae. The formulae for d^μ, where $\mu \in \mathcal{P}_n^{p\text{-reg}}(\leq 4)$, given in (3.3) and (4.3) for $p = 3$ and $p = 2$, respectively, together with the results in [7, La.1.21] giving similar formulae for $p \geq 5$, show that for all $\mu \in \mathcal{P}_n^{p\text{-reg}}(m)$, where $m \in \{0, \ldots, 4\}$ and $n \geq 2m$, independent of $p \geq 2$ we get

$$d^\mu = \dim_{\mathbb{F}_p}(D^\mu) \geq \begin{cases} 1 & \text{if } m = 0, \\ n-2 & \text{if } m = 1, \\ \frac{n^2-5n+2}{2} & \text{if } m = 2, \\ \frac{n^3-9n^2+14n}{6} & \text{if } m = 3, \\ \frac{n^4-14n^3+47n^2-34n}{24} & \text{if } m = 4. \end{cases}$$

These lower bounds are best possible in the sense that, for each $m \in \{0, \ldots, 4\}$, equality holds for infinitely many values of $n \geq 2m$. We remark that the cases $m \leq 1$ are already noted in [20, Thm.6], and that the cases $m \in \{2, 3\}$ for $p \geq 5$ also appear in [7, La.1.18, La.1.20]; the remaining lower bounds also follow from a close inspection of the results given without proof in [20, App.].

(6.2) Degree bounds. We are going to apply Theorems (5.2) and (5.6), for $p \geq 3$ and $p = 2$, respectively, in the case $m = 3$. Hence a natural choice for the function $f(n)$ should be closely related to the lower bound function given in (6.1) for the case $m + 1 = 4$; see also [8, Prop.3.1] and [8, Prop.4.4] for the cases $p = 3$ and $p = 2$, respectively. Thus we let

$$f_4 \colon \mathbb{Z} \to \mathbb{Z} \colon n \mapsto \frac{n^4 - 14n^3 + 47n^2 - 34n}{24}.$$

It is easily seen that $f_4(n)$ has integral values, which are positive for $n \geq 10$, that it is strictly increasing for $n \geq 8$, and fulfills $2 \cdot f_4(n) \geq f_4(n+2)$ for $n \geq 16$.

• In order to obtain a strong bound n_0 in Theorems (5.2) and (5.6), we use a modification of the above function: Let $g\colon \{11, 12, \ldots\} \to \mathbb{N}$ be defined by $g(11) := 55$ and $g(12) := 89$, and

$$g(2k-1) := 2^{k-6} \cdot g(11) \quad \text{and} \quad g(2k) := 2^{k-6} \cdot g(12),$$

for $k \geq 6$; the choice of $g(11)$ and $g(12)$ will become clear below. Then from $g(11) < g(12) < 2 \cdot g(11)$ it follows that $g(n)$ is strictly increasing, and by construction we have $2 \cdot g(n) = g(n+2)$, for all $n \geq 11$. Having this, we let

$$f\colon \{11, 12, \ldots\} \to \mathbb{N}\colon n \mapsto \min\{f_4(n), g(n)\},$$

where is it is easily seen that $g(n) \leq f_4(n)$ if and only if $n \in \{11, \ldots, 25\}$. Hence it follows that $f(n)$ is strictly increasing, and fulfills $2 \cdot f(n) \geq f(n+2)$, for all $n \geq 11$. For convenience, here are a few values:

n	10	11	12	13	14	15	16	17	18	...	25	26
f_4	15	55	121	221	364	560	820	1156	1581	...	8350	10075
g		55	89	110	178	220	356	440	712	...	7040	11392

• By (6.1) we have $d^\mu \geq f_4(n) \geq f(n)$, for all $n \geq 11$ and $\mu \in \mathcal{P}_n^{p\text{-reg}}(4)$. The decomposition matrices of $\mathcal{S}_n$ for $n \in \{11, 12\}$ are known, see the relevant comments in (6.3), and are accessible in the databases mentioned in Section 1. An explicit check, using the computer algebra system GAP [17], yields the following:

If $p \geq 3$, then for $\mu \in \mathcal{M}_{11}(\geq 4)$ we have $d^\mu \geq 55$, where equality is assumed precisely for $p = 5$ and $\{\mu, \mu^M\} = \{[7, 4], [5, 3^2]\}$, and for $\mu \in \mathcal{M}_{12}(\geq 4)$ we have $d^\mu \geq 89$, where equality is assumed precisely for $p = 5$ and $\{\mu, \mu^M\} = \{[6^2], [4^3]\}$. If $p = 2$, then for $\mu \in \mathcal{M}'_{11}(\geq 4)$ we have $d^\mu \geq 144$, and for $\mu \in \mathcal{M}'_{12}(\geq 4)$ we have $d^\mu \geq 164$, while $d^{\mu_{\text{bs}}(11)} = d^{\mu_{\text{bs}}(12)} = 32$.

Hence, letting $m := 3$ and $n_0 := 11$, by Theorem (5.2) for all $p \geq 3$, independent of p, we have $d^\mu \geq f(n)$ for all $n \geq 11$ and $\mu \in \mathcal{M}_n(\geq 4)$. Similarly, by Theorem (5.6) for $p = 2$ we have $d^\mu \geq f(n)$ for all $n \geq 11$ and $\mu \in \mathcal{M}'_n(\geq 4)$.

(6.3) Strategy. We now apply the following strategy:

• Using the function $f\colon \{11, 12, \ldots\} \to \mathbb{N}$ given in (6.2), it is easily seen that for $n \geq 11$ we have $f(n) \leq n^3$ if and only if $n \leq 36$. Hence for $n \geq 37$ we conclude that $d^\mu \leq n^3$ implies $\mu \in \mathcal{P}_n^{p\text{-reg}}(\leq 3)$. Moreover, due to the optimal choice of $f_4(n)$ in the sense of (6.2), it is to be expected that this bound is rather sharp; by the results in (6.4)–(6.7) it is sharp for $p \in \{2, 3, 5\}$, and almost for $p = 7$.

Conversely, for all $n \in \mathbb{N}_0$ and $\mu \in \mathcal{P}_n^{p\text{-reg}}(\leq 3)$, by [20, Cor.2] we indeed have $d^\mu \leq n^3$; note that this also follows from the degree formulae given in (4.3) and (3.3), and in [7, La.1.21] for $p = 2$, $p = 3$ and $p \geq 5$, respectively.

• Hence it remains to consider the cases $n \in \{1, \ldots, 36\}$. Here, we make use of the decomposition matrices of $\mathcal{S}_n$, as far as they are explicitly known. We briefly report on the current state of the art:

Whenever $n \leq p < 2n$, then the Sylow p-subgroups of $\mathcal{S}_n$ are cyclic of order p. Hence the decomposition numbers are described in terms of Brauer trees, see for example [13, Sect.VII]. Moreover, since $\mathcal{S}_n$ only has rational-valued irreducible ordinary characters, see [25, Thm.2.1.3], all the Brauer trees are actually stems. This together with unitriangularity properties determines these decomposition matrices completely. In turn, this yields the degrees d^μ, for all $\mu \in \mathcal{P}_n^{p\text{-reg}}$.

For $p \geq 5$ and $2p \leq n < 5p$, it was proven in a number of steps in [11, 12, 45, 46] that the adjustment matrix $\mathcal{A}_n$, see (2.6), is the identity matrix. (We remark that it follows from [18] that for $p \geq 2$ and $n \leq p < 2n$ the adjustment matrix $\mathcal{A}_n$ also is the identity matrix, but we will not need this fact.) Hence the decomposition matrix of $\mathcal{S}_n$ coincides with the ζ_p-modular decomposition matrix of the generic Iwahori-Hecke algebra $\mathcal{H}_n$, where the latter decomposition matrix can be computed explicitly by the LLT algorithm, see (2.7), and for all $\mu \in \mathcal{P}_n^{p\text{-reg}}$ we have $d^\mu = \dim_{\mathbb{Q}_{\zeta_p}}(D^\mu_{\zeta_p})$.

The 2-modular and 3-modular decomposition matrices of $\mathcal{S}_n$ for $n \leq 13$ are given in [22, p.137–142] and in [22, p.143–152], respectively. Moreover, those for $p = 2$ and $n \in \{14, 15\}$ have been determined in [3], up to an ambiguity for $n = 15$, which was solved, together with the cases $n \in \{16, 17\}$ in [42]. Moreover, by unpublished work of the author, the 2-modular and 3-modular decomposition matrices of $\mathcal{S}_n$ are also known for $n \in \{18, 19\}$ and $n \in \{14, \ldots, 18\}$, respectively, and are accessible in the databases mentioned in Section 1. (In the latter work, a similar technique as in [42] was employed. It consists of a mixture of character theoretic and module theoretic computations, in particular encompassing so-called condensation methods. A recent description of this approach is given in the thesis [38], which has been written under the author's supervision.)

• By the above, the cases $p \geq 11$ are settled, thus we are left with the cases $p \in \{2, 3, 5, 7\}$ and $n \in \{n_p, \ldots, 36\}$, where $n_2 := 20$, $n_3 := 19$, $n_5 := 25$ and $n_7 := 35$. Of course, for $\mu \in \mathcal{P}_n^{p\text{-reg}}(\leq 4)$ we make use of the degree formulae in (4.3), (3.3) and in [7, La.1.21] for $p = 2$, $p = 3$ and $p \geq 5$, respectively, as well as of those for $p = 2$ and $\mu \in \{\mu_{\text{bs}}(n), \mu_{\text{bbs}}(n)\}$ given in (5.5). Moreover, since the decomposition numbers of the Specht modules $S^{[n-m,m]}$, where $m \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$, are known by [23, 24], see also [22, Thm.24.15], this also yields the degree d^μ whenever $\mu = [n - m, m] \in \mathcal{P}_n^{p\text{-reg}}$. Apart from these cases, given $\mu \in \mathcal{P}_n^{p\text{-reg}}$, the task is to find upper and lower bounds for d^μ, which are good enough to ensure that $d^\mu \leq n^3$ or $d^\mu > n^3$, respectively.

To find upper bounds, by (2.6), in general we have $d^\mu \leq \dim_{\mathbb{Q}_{\zeta_p}}(D^\mu_{\zeta_p})$, where the latter dimension can be determined from the ζ_p-modular decomposition matrix of the generic Iwahori-Hecke algebra $\mathcal{H}_n$. As we are dealing with explicit cases, we use the implementation of the LLT algorithm available in the SPECHT package [39], available in the computer algebra system CHEVIE [41], in order to compute the relevant ζ_p-modular decomposition matrices.

To find lower bounds, we proceed recursively, starting from $n = n_p - 1$, and apply the modular branching rules (2.4). As it turns out, these estimates are good enough to exclude all cases where actually $d^\mu > n^3$. Moreover, as soon as the upper and lower bounds found coincide, we have determined d^μ precisely. This is particularly strong, in view of the recursive nature of the procedure, as soon as $D^\mu\downarrow_{\mathcal{S}_{n-1}} \cong \bigoplus_{i=0}^{p-1} D^\mu\downarrow_i$ is semi-simple, that is all the $D^\mu\downarrow_i$ are irreducible. In particular, as it turns out, these techniques are sufficient to determine d^μ precisely, for all the cases where actually $d^\mu \le n^3$.

• We have implemented the strategy described above in the computer algebra system GAP [17], so that it can be carried out explicitly for any fixed prime $p \ge 2$. Below, we are showing the results thus obtained for $p \in \{2, 3, 5, 7\}$. (These results are of course available electronically on request from the author.)

(6.4) Low-degree representations for $p = 2$. We obtain the following:

• Generic cases $\mu \in \mathcal{P}_n^{2\text{-reg}}(\le 3)$, for $n \in \mathbb{N}_0$, with degree formulae in (4.3).

• Basic spin cases $\mu = \mu_{\mathrm{bs}}(n)$ and second basic spin cases $\mu = \mu_{\mathrm{bbs}}(n)$, with degree formulae given in (5.5), as well as 'almost generic' cases $\mu \in \mathcal{P}_n^{2\text{-reg}}(4) \setminus \{\mu_{\mathrm{bs}}(n), \mu_{\mathrm{bbs}}(n)\}$, with degree formulae given in (4.3):

μ	n
$\mu_{\mathrm{bs}}(n)$	$\{9,\dots,30\} \dot\cup \{32\}$
$\mu_{\mathrm{bbs}}(n)$	$\{8,\dots,19\} \dot\cup \{21\}$

$\overline{\mu}$	n
(4)	$\{11,\dots,34\} \dot\cup \{36\}$
$(3,1)$	$\{10,\dots,21\} \dot\cup \{23\}$

Note that $\mu_{\mathrm{bs}}(n)$ for $n \le 8$, and $\mu_{\mathrm{bbs}}(n)$ for $n \in \{6,7\}$ yield generic cases, and that $\mu \in \mathcal{P}_n^{2\text{-reg}}(4)$ for small n yields basic spin or second basic spin cases: $[5,4] = \mu_{\mathrm{bs}}(9)$ and $[6,4] = \mu_{\mathrm{bs}}(10)$, and $[4,3,1] = \mu_{\mathrm{bbs}}(8)$ and $[5,3,1] = \mu_{\mathrm{bbs}}(9)$.

• 'Exceptional' cases $\mu \in \mathcal{P}_n^{2\text{-reg}} \setminus (\mathcal{P}_n^{2\text{-reg}}(\le 4) \cup \{\mu_{\mathrm{bs}}(n), \mu_{\mathrm{bbs}}(n)\})$:

n	μ	d^μ
9	$[4,3,2]$	160
10	$[5,3,2]$	200
	$[4,3,2,1]$	768
11	$[6,3,2]$	848
	$[5,4,2]$	416
	$[5,3,2,1]$	1168
12	$[7,4,1]$	1408
	$[7,3,2]$	1046
	$[6,4,2]$	416

n	μ	d^μ
13	$[8,5]$	560
	$[8,4,1]$	1572
	$[6,5,2]$	1728
14	$[9,5]$	560
	$[8,4,2]$	2510
	$[7,5,2]$	2016
15	$[10,5]$	1288
	$[9,6]$	1912

n	μ	d^μ
16	$[11,5]$	1288
	$[10,6]$	1912
	$[8,6,2]$	4096
17	$[12,5]$	3808
	$[11,6]$	4488
18	$[13,5]$	3808
	$[12,6]$	4488
20	$[15,5]$	6972

(6.5) Low-degree representations for $p = 3$. We obtain the following:

• Generic cases $\mu \in \mathcal{P}_n^{3\text{-reg}}(\le 3)$, for $n \in \mathbb{N}_0$, with degree formulae in (3.3).

• 'Almost generic' cases $\mu \in \mathcal{P}_n^{3\text{-reg}}(4)$, where $\mu^M \notin \mathcal{P}_n^{3\text{-reg}}(\le 3)$, with degree formulae given in (3.3); we also note the cases where $\mu^M \in \mathcal{P}_n^{3\text{-reg}}(\le 4)$, which only occurs for $n \leq 9$:

$\overline{\mu}$	n
(4)	$\{10, \ldots, 33\} \dot\cup \{36\}$
$(3,1)$	$\{9, \ldots, 17\}$
(2^2)	$\{10, \ldots, 33\} \dot\cup \{36\}$
$(2,1^2)$	$\{8, \ldots, 17\}$

$\overline{\mu}$	6	7	8	9
(4)	–	–	$()$	$()$
$(3,1)$	–	(1)	(1)	
(2^2)	–	(1^2)	$(2,1)$	$(2,1)$
$(2,1^2)$	(2)	(2)	$(2,1^2)$	

• 'Exceptional' cases $\mu, \mu^M \in \mathcal{P}_n^{3\text{-reg}} \setminus \mathcal{P}_n^{3\text{-reg}}(\leq 4)$, where we record the lexicographically largest of μ and μ^M:

n	μ	d^μ
10	$[5,3,1^2]$	567
11	$[6,4,1]$	693
	$[6,3,2]$	252
	$[6,3,1^2]$	791
	$[6,2^2,1]$	714
12	$[7,5]$	297
	$[7,4,1]$	1013
	$[7,3,2]$	252
	$[7,3,1^2]$	1431
	$[7,2^2,1]$	1728
	$[6,3,2,1]$	1428

n	μ	d^μ
13	$[8,5]$	428
	$[8,4,1]$	1275
	$[8,3,2]$	792
	$[8,2^2,1]$	1938
	$[7,3^2]$	924
	$[7,3,2,1]$	1428
14	$[9,5]$	428
	$[9,3,2]$	1287
	$[8,6]$	1000
	$[8,3^2]$	1716
15	$[10,5]$	1548
	$[10,3,2]$	1287
	$[9,6]$	1428
	$[9,3^2]$	1716

n	μ	d^μ
16	$[11,5]$	2108
	$[11,3,2]$	3003
	$[10,6]$	1428
	$[9,7]$	3417
17	$[12,5]$	2108
	$[12,3,2]$	4368
	$[10,7]$	4845
18	$[13,5]$	5508
	$[13,3,2]$	4368
	$[11,7]$	4845
20	$[15,5]$	7105

(6.6) Low-degree representations for $p = 5$. We obtain the following:

• Generic cases $\mu \in \mathcal{P}_n^{5\text{-reg}}(\leq 3)$, for $n \in \mathbb{N}_0$, with degree formulae in [7, La.1.21].

• 'Almost generic' cases $\mu \in \mathcal{P}_n^{5\text{-reg}}(4)$, where $\mu^M \notin \mathcal{P}_n^{5\text{-reg}}(\leq 3)$, with degree formulae given in [7, La.1.21]; we also note the cases where $\mu^M \in \mathcal{P}_n^{5\text{-reg}}(\leq 4)$, which only occurs for $n \leq 10$:

$\overline{\mu}$	n
(4)	$\{8, \ldots, 33\} \dot\cup \{36\}$
$(3,1)$	$\{8, \ldots, 17\}$
(2^2)	$\{8, \ldots, 20\} \dot\cup \{23\}$
$(2,1^2)$	$\{8, \ldots, 16\}$
(1^4)	$\{9, \ldots, 32\} \dot\cup \{35\}$

$\overline{\mu}$	6	7	8	9	10
(4)	–	–	(2^2)		
$(3,1)$	–	(2)			
(2^2)	(3)	(3)	(4)		
$(2,1^2)$	$()$	$(2,1)$	$(2,1^2)$		
(1^4)	(1)	(1^2)	(1^3)	(1^4)	(1^4)

• 'Exceptional' cases $\mu, \mu^M \in \mathcal{P}_n^{5\text{-reg}} \setminus \mathcal{P}_n^{5\text{-reg}}(\leq 4)$, where we record the lexicographically largest of μ and μ^M:

n	μ	d^μ
9	$[4^2,1]$	83
10	$[5^2]$	34
	$[5,4,1]$	217
	$[5,3,2]$	450
	$[5,3,1^2]$	266
	$[5,2^2,1]$	525
	$[4^2,1^2]$	300
11	$[6,5]$	89
	$[6,4,1]$	372
	$[6,3,2]$	605
	$[6,3,1^2]$	266
	$[6,2^2,1]$	1100
	$[6,2,1^3]$	252
	$[5^2,1]$	285
	$[5,4,2]$	980
	$[5,4,1^2]$	1035
	$[5,3,2,1]$	1330
	$[5,2^3]$	825

n	μ	d^μ
12	$[7,5]$	144
	$[7,4,1]$	372
	$[7,3,1^2]$	1266
	$[7,2^2,1]$	1506
	$[7,2,1^3]$	462
	$[6^2]$	89
	$[6,5,1]$	835
	$[6,3^2]$	1650
	$[6,3,2,1]$	1596
	$[6,2,1^4]$	2100
	$[5^2,2]$	1265
	$[5^2,1^2]$	1320
13	$[8,5]$	144
	$[8,4,1]$	2001
	$[8,2,1^3]$	792
	$[7,6]$	233
	$[7,5,1]$	1495
	$[7,2^2,1^2]$	924
	$[6^2,1]$	924
14	$[9,5]$	1001
	$[9,2,1^3]$	1287
	$[8,6]$	377
	$[8,5,1]$	1639
	$[8,2^2,1^2]$	1716
	$[7^2]$	233
	$[7,6,1]$	2652

n	μ	d^μ
15	$[10,5]$	1625
	$[10,2,1^3]$	1287
	$[9,6]$	377
	$[9,2^2,1^2]$	1716
	$[8,7]$	610
	$[7^2,1]$	2885
16	$[11,5]$	2445
	$[11,2,1^3]$	3003
	$[10,6]$	3640
	$[9,7]$	987
	$[8^2]$	610
17	$[12,5]$	3265
	$[12,2,1^3]$	4368
	$[10,7]$	987
	$[9,8]$	1597
18	$[13,5]$	3265
	$[10,8]$	2584
	$[9,9]$	1597
19	$[11,8]$	2584
	$[10,9]$	4181
20	$[11,9]$	6765
	$[10,10]$	4181
21	$[12,9]$	6765

(6.7) Low-degree representations for $p = 7$. We obtain the following:

- Generic cases $\mu \in \mathcal{P}_n^{7\text{-reg}}(\leq 3)$, for $n \in \mathbb{N}_0$, with degree formulae in [7, La.1.21].
- 'Almost generic' cases $\mu \in \mathcal{P}_n^{7\text{-reg}}(4)$, where $\mu^M \notin \mathcal{P}_n^{7\text{-reg}}(\leq 3)$, with degree formulae given in [7, La.1.21]; we also note the cases where $\mu^M \in \mathcal{P}_n^{7\text{-reg}}(\leq 4)$, which only occurs for $n \leq 9$:

$\overline{\mu}$	n
(4)	$\{8,\ldots,34\}$
$(3,1)$	$\{7,\ldots,16\}$
(2^2)	$\{7,\ldots,20\}$
$(2,1^2)$	$\{9,\ldots,16\}$
(1^4)	$\{9,\ldots,32\} \,\dot\cup\, \{35\}$

$\overline{\mu}$	5	6	7	8	9
(4)	–	–	–		
$(3,1)$	–	–	(2^2)		
(2^2)	–	(3)	$(3,1)$		
$(2,1^2)$	–	(2)	$(2,1)$	$(2,1)$	
(1^4)	$()$	(1)	(1)	(1^3)	(1^4)

- 'Exceptional' cases $\mu, \mu^M \in \mathcal{P}_n^{7\text{-reg}} \setminus \mathcal{P}_n^{7\text{-reg}}(\leq 4)$, where we record the lexicographically largest of μ and μ^M:

n	μ	d^μ
8	$[3^2,2]$	42
9	$[4^2,1]$	84
	$[4,3,2]$	168
	$[3^3]$	42
10	$[5,5]$	42
	$[5,4,1]$	199
	$[5,3,2]$	384
	$[5,3,1^2]$	567
	$[5,2^2,1]$	525
	$[5,2,1^3]$	448
	$[4^2,2]$	252
	$[4,3^2]$	210
11	$[6,5]$	131
	$[6,4,1]$	693
	$[6,3,2]$	485
	$[6,3,1^2]$	1232
	$[6,2^2,1]$	626
	$[6,2,1^3]$	924
	$[6,1^5]$	252
	$[5^2,1]$	199
	$[5,4,2]$	835
	$[5,4,1^2]$	1155
	$[5,3^2]$	594
	$[4^2,3]$	462

n	μ	d^μ
12	$[7,5]$	286
	$[7,4,1]$	1353
	$[7,2,1^3]$	1398
	$[7,1^5]$	462
	$[6^2]$	131
	$[6,5,1]$	1155
	$[6,4,2]$	1320
	$[6,3^2]$	1079
	$[5^2,2]$	1034
	$[5^2,1^2]$	1354
	$[4^3]$	462
13	$[8,5]$	507
	$[8,4,1]$	2145
	$[8,1^5]$	792
	$[7,6]$	417
	$[7,3^2]$	1079
	$[7,1^6]$	924
	$[6^2,1]$	1286

n	μ	d^μ
14	$[9,5]$	728
	$[9,4,1]$	2366
	$[9,1^5]$	792
	$[8,6]$	924
	$[8,1^6]$	924
	$[7^2]$	417
	$[6^2,2]$	2354
15	$[10,5]$	728
	$[10,1^5]$	2002
	$[9,6]$	1652
	$[9,1^6]$	3003
	$[8,7]$	1341
16	$[11,5]$	2548
	$[11,1^5]$	3003
	$[10,6]$	2380
	$[9,7]$	2993
	$[8^2]$	1341
17	$[12,5]$	3808
	$[12,1^5]$	4368
	$[11,6]$	2380
	$[9,8]$	4334
18	$[13,5]$	5507
	$[9^2]$	4334

(6.8) Remark. For the sake of completeness, we remark that to classify the cases $d^\mu \leq n$ and $d^\mu \leq n^2$ it turns out that we may just use the functions

$$f_2 \colon \mathbb{Z} \to \mathbb{Z} \colon n \mapsto \frac{n^2-5n+2}{2} \quad \text{and} \quad f_3 \colon \mathbb{Z} \to \mathbb{Z} \colon n \mapsto \frac{n^3-9n^2+14n}{6}$$

from (6.1), respectively. Proceeding as in (6.2) we find that the conditions of (5.2) and (5.6) are fulfilled for $n_0 = 8$ and $n_0 = 13$, respectively. Since $f_2(n) > n$ for all $n \geq 8$, and $f_3(n) > n^2$ for all $n \geq 15$, it only remains to consider the cases $n \leq 14$ explicitly. In this range, all decomposition matrices of $\mathcal{S}_n$ are known, so that the strategy in (6.3) simplifies considerably. Here are the results, where for $p \geq 3$ we only record the lexicographically larger of μ and μ^M, and for all representations mentioned degree formulae are given in (5.5) and (4.3), in (3.3), and in [7, La.1.21] for $p = 2$, $p = 3$ and $p \geq 5$, respectively, or are accessible in the databases mentioned in Section 1.

• As for $d^\mu \leq n$, apart from the generic cases $\mu \in \mathcal{P}_n^{p\text{-reg}}(\leq 1)$ for $n \in \mathbb{N}_0$, we have the following 'exceptional' cases $\mu, \mu^M \in \mathcal{P}_n^{p\text{-reg}} \setminus \mathcal{P}_n^{p\text{-reg}}(\leq 1)$: For $p = 2$

we get $\{[3,2],[4,2],[5,3]\}$, which are basic spin representations; for $p=3$ we get $\{[4,1^2]\}$; and for $p \geq 5$ we get $\{[2^2],[3,2],[3,3]\}$.

• As for $d^\mu \leq n^2$, apart from the generic cases $\mu \in \mathcal{P}_n^{p\text{-reg}}(\leq 2)$ for $n \in \mathbb{N}_0$, we have the following cases where $\mu, \mu^M \in \mathcal{P}_n^{p\text{-reg}} \setminus \mathcal{P}_n^{p\text{-reg}}(\leq 2)$:

○ For $p=2$, we get the basic spin and second basic spin cases $\mu_{\text{bs}}(n)$ and $\mu_{\text{bbs}}(n)$, and the 'almost generic' cases $\mu \in \mathcal{P}_n^{2\text{-reg}}(3) \setminus \{\mu_{\text{bs}}(n), \mu_{\text{bbs}}(n)\}$ as follows:

μ	n
$\mu_{\text{bs}}(n)$	$\{7,\dots,18\}$
$\mu_{\text{bbs}}(n)$	$\{6,\dots,9\}$

$\overline{\mu}$	n
(3)	$\{9,10,11,12,\}$
$(2,1)$	$\{8,9\}$

Note that $\mu_{\text{bs}}(n)$ for $n \leq 6$ yields generic cases, and that $\mu \in \mathcal{P}_n^{2\text{-reg}}(3)$ for small n yields basic spin or second basic spin cases: $[4,3]=\mu_{\text{bs}}(7)$ and $[5,3]=\mu_{\text{bs}}(8)$, and $[3,2,1]=\mu_{\text{bbs}}(6)$ and $[4,2,1]=\mu_{\text{bbs}}(7)$. There are no 'exceptional' cases.

○ For $p=3$, the 'almost generic' cases $\mu \in \mathcal{P}_n^{3\text{-reg}}(3)$ such that $\mu^M \notin \mathcal{P}_n^{3\text{-reg}}(\leq 2)$, and the 'exceptional' cases $\mu, \mu^M \notin \mathcal{P}_n^{3\text{-reg}}(\leq 3)$ are, respectively:

$\overline{\mu}$	n
(3)	$\{8,\dots,13\}$
$(2,1)$	$\{7,\dots,12\}$

n	μ	d^μ
10	$[6,4]$	90
12	$[8,4]$	131

○ For $p=5$, similarly, the 'almost generic' and 'exceptional' cases are:

$\overline{\mu}$	n
(3)	$\{6,\dots,11\} \dot\cup \{14\}$
$(2,1)$	$\{7,8\}$
(1^3)	$\{7,\dots,11\}$

n	μ	d^μ
8	$[4^2]$	13
9	$[5,4]$	34
	$[5,1^4]$	70
10	$[6,4]$	55
	$[6,1^4]$	70
	$[5^2]$	34

n	μ	d^μ
11	$[7,4]$	55
	$[6,5]$	89
12	$[7,5]$	144
	$[6^2]$	89
13	$[8,5]$	144

○ For $p=7$, similarly, the 'almost generic' and 'exceptional' cases are:

$\overline{\mu}$	n
(3)	$\{6,\dots,11\}$
$(2,1)$	$\{6,7,8\}$
(1^3)	$\{8,\dots,11\}$

n	μ	d^μ
7	$[3^2,1]$	21
8	$[4^2]$	14
	$[4,2^2]$	56
	$[3^2,2]$	42

n	μ	d^μ
9	$[5,4]$	42
	$[5,1^4]$	70
	$[3^3]$	42
10	$[6,4]$	89
	$[5^2]$	42
12	$[6^2]$	131

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Institut für Mathematik
Friedrich-Schiller-Universität Jena
Ernst-Abbe-Platz 2, 07737 Jena, Germany
juergen.manfred.mueller@uni-jena.de