Solution to Exercise 1.1.12

(a) Since

$$\gamma(\, i \cdot \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}) = \begin{bmatrix} -i & 0 \\ -1 & -i \end{bmatrix} \neq \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix} = i \cdot \gamma(\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix})$$

γ is not a $\mathbb{C}$-algebra homomorphism.

We write $\operatorname{Inf} V = (V, \star)$ with $a \star v := \gamma(a) \cdot v$ for $a \in A$ and $v \in V$, where $\cdot$ is the usual matrix multiplication. Observe that for $\alpha \in \mathbb{C}$ and $v \in \operatorname{Inf} V$ we have $\alpha \star v = (\alpha 1_A) \star v = \overline{\alpha}\, v$. Let $a := \begin{bmatrix} \alpha_1 & 0 \\ \beta & \alpha_2 \end{bmatrix} \in A$ and $B = (v_1, v_2)$ be the standard basis of V. Then

$$a \star v_1 = \overline{\alpha}_1 v_1 + \overline{\beta} v_2 = \alpha_1 \star v_1 + \beta \star v_2. \quad a \star v_2 = \overline{\alpha}_2 v_2 = \alpha_2 \star v_2.$$

Thus V and $\operatorname{Inf} V$ afford the same matrix representation $\boldsymbol{\delta}$, although e.g. $a \star v_2 \neq a \cdot v_2$ if $\alpha_2 \notin \mathbb{R}$.

(b) Again write $\operatorname{Inf} V = (V, \star)$ with $a \star v = \gamma(a) \cdot v$ for $a \in \mathbb{C}G$ and $v \in V$ where $\cdot \colon \mathbb{C}G \times V \to V$ is the $\mathbb{C}G$-module operation in V. Again $\alpha \star v = \overline{\alpha}\, v$. It follows that

$$g \star v = g \cdot v = i \cdot v = (-i) \star v \qquad \text{for} \qquad v \in V.$$