Solution to Exercise 1.3.3

(a) Clearly $c_\varphi = c_{\varphi-\lambda\cdot\mathrm{id}_V}(X-\lambda) \in K[X]$ for $\lambda \in K$. Hence $m_\lambda(\varphi) = 1$ implies $m_0(\varphi - \lambda \cdot \mathrm{id}_V) = 1$ and hence $\dim_K \ker p(\varphi) = 1$.

(b) The argument of (a) shows that $n(\lambda, \lambda) = n(0,0) = n_{d,q}$ for any $\lambda \in K$. Let $B := (v_1, \ldots, v_d)$ be a basis of V. Let $\varphi \in \mathrm{End}_K V$ and $m_0(\varphi) = 1$, so that in particular $\dim_K \ker\varphi = 1$. Then $\ker\varphi = \langle v_1 \rangle$ if and only if $[\varphi]_B = \begin{bmatrix} 0 & a \\ \mathbf{0} & A \end{bmatrix}$ with $a \in K^{1\times(q-1)}$ and $A \in \mathrm{GL}_{d-1}(q)$. In fact, $A = [\bar\varphi]_{\bar B}$, where $\bar\varphi$ is as in the hint to the exercise and $\bar B = (v_2 + \ker\varphi, \ldots, v_d + \ker\varphi)$. Since the number of 1-dimensional subspaces of V is $(q^d-1)/(q-1)$ and $|\mathrm{GL}_d(q)| = \prod_{i=0}^{d-1}(q^d - q^i)$, we get

$$n_{d,q} = \frac{q^d-1}{q-1}\, q^{d-1}\, |\mathrm{GL}_{d-1}(q)| = \frac{q^d-1}{q-1} \prod_{i=1}^{d-1}(q^d - q^i) = \frac{|\mathrm{GL}_d(q)|}{q-1}. \qquad (1)$$

Clearly $n(\lambda_1, \lambda_2) = n(0, \lambda)$ if $\lambda_1 \neq \lambda_2$ and $\lambda \neq 0$. If B, φ and $\bar\varphi$ are as above and, in addition $m_\lambda(\varphi) = 1$, then $[\varphi]_B = \begin{bmatrix} 0 & a \\ \mathbf{0} & A \end{bmatrix}$ with $a \in K^{1\times(q-1)}$ and $A \in \mathrm{GL}_{d-1}(q)$ with $m_\lambda(\bar\varphi) = 1$. Obviously the number of such A (or $\bar\varphi$) is $\leq n_{d-1,q}$, hence

$$n(\lambda_1, \lambda_2) \leq \frac{q^d-1}{q-1}\, q^{d-1}\, n_{d-1,q}.$$

For $\lambda \in K$ let $M_\lambda := \{\varphi \in \mathrm{End}_K V \mid m_\lambda(\varphi) = 1\}$. Then $|M_{\lambda_1} \cap M_{\lambda_2}| = n(\lambda_1, \lambda_2)$. Hence

$$\begin{aligned}
m_{d,q} &= |\bigcup_{\lambda\in K} M_\lambda| \geq \sum_{\lambda\in K} |M_\lambda| - \sum_{|\{(\lambda_1,\lambda_2\}|=2} n(\lambda_1,\lambda_2) & (2)\\
&\geq q\, n_{d,q} - \frac{q\,(q-1)}{2}\,\frac{q^d-1}{q-1}\, q^{d-1}\, n_{d-1,q} & (3)\\
&\geq q\, n_{d,q} - \frac{1}{2}\,(q^d-1)\, q\, n_{d-1,q} = \frac{1}{2}\, q\, n_{d,q}. & (4)
\end{aligned}$$

Observe, that (1) implies that $n_{d,q}/n_{d-1,q} = (q^d-1)\,q^{d-1}$.

From (4) and (1) we conclude, since $q \geq 2$

$$\begin{aligned}
\frac{m_{d,q}}{|\mathrm{End}_K V|} &\geq \frac{|\mathrm{GL}_d(q)|}{2\, q^{d-1}\,(q-1)} = \frac{1}{2}(1-q^{-2})(1-q^{-3})\cdots(1-q^{-d})\\
&\geq \prod_{i=1}^{d}(1-2^{-i}) =: p_d.
\end{aligned}$$

For $n, k \in \mathbb{N}$ we put

$$p_{n,k} := (1-2^{-(n+1)})\cdots(1-2^{-(n+k)})$$

and using induction on k (and $p_{n,k+1} = p_{n,1} \cdot p_{n+1,k}$) we immediately see that

$$1 - p_{n,k} < \frac{1}{2^n} \qquad \text{for all} \qquad n, k \in \mathbb{N}. \tag{5}$$

We compute $p_4 = \frac{315}{1024} > 0.3$ and for $d > 4$ we obtain

$$p_d = p_4 \cdot p_{4,d-4} > \frac{315}{1024} \cdot \frac{15}{16} > 0.288.$$

Note: It follows from (5) that the infinite product $\prod_{i=1}^{\infty}(1 - 2^{-i})$ is convergent. The first 100 decimal digits of

$$Q := \prod_{i=1}^{\infty}(1 - 2^{-i}) = 0.28878809508\ldots$$

can be found in Sloane's On-Line Encyclopedia of Integer Sequences, see http://www.research.att.com/ njas/sequences/?q=id%3aA048651&p=1 &n=10&sort=1 . The constant Q plays also a role in digital tree searching, see http://mathworld.wolfram.com/TreeSearching.html .