Solution to Exercise 1.4.5

We compute the radicals of the group algebras of $G := \mathrm{A}_5$ over $\mathbb{F}_2$ and $\mathbb{F}_4$ and their dimensions:

```
gap> G := AlternatingGroup(5);;
gap> K2G :=  GroupRing( GF(2), G );;
gap> K4G :=  GroupRing( GF(4), G );;
gap> R2 :=  RadicalOfAlgebra(K2G);; Dimension(R2);
35
gap> R4 :=  RadicalOfAlgebra(K4G);; Dimension(R4);
35
```

Actually, the last computation is unnecessary, because in general for any finite group G and any field extension $L \supseteq K$ one has $\mathrm{J}(LG) = L \otimes \mathrm{J}(KG)$ (see the remark after Lemma 2.1.5) and hence $\dim_K \mathrm{J}(KG) = \dim_L \mathrm{J}(LG)$.

Thus for $K \in \{\mathbb{F}_2, \mathbb{F}_4\}$ the dimension of $KG/\mathrm{J}(KG)$ is $60 - 35 = 25$. On the other hand, we have seen in Exercise 1.3.5 that there are simple KG-modules $V_1.V_2, V_3$ with dimension $1, 4, 4$ with $\mathrm{End}_{KG} V_1 \cong \mathrm{End}_{KG} V_2 \cong K$ and $\mathrm{End}_{KG} V_3 = D > K$ for $K = \mathbb{F}_2$ and absolutely simple modules of dimension $1, 2, 2, 4$ for $K = \mathbb{F}_4$. Thus Theorem 1.4.6 shows that $D \cong \mathbb{F}_4$ (clearly $D \cong \mathbb{F}_{2^4}$ is impossible, since $G' = G$) and

$$\mathbb{F}_2G/\mathrm{J}(\mathbb{F}_2G) \cong \mathbb{F}_2 \oplus \mathbb{F}_4^{2\times 2} \oplus \mathbb{F}_2^{4\times 4} \quad \text{and} \quad \mathbb{F}_4G/\mathrm{J}(\mathbb{F}_4G) \cong \mathbb{F}_4 \oplus \mathbb{F}_4^{2\times 2} \oplus \mathbb{F}_4^{2\times 2} \oplus \mathbb{F}_4^{4\times 4}.$$

One can also compute the dimensions of the simple summands of $KG/\mathrm{J}(KG)$ using GAP:

```
gap> List( DirectSumDecomposition(K2G/R2), Dimension );
[ 16, 8, 1 ]
gap> List( DirectSumDecomposition(K4G/R4), Dimension );
[ 16, 1, 4, 4 ]
```

We finally compute the dimensions of the powers of $\mathrm{J}(KG)$:

```
gap> I := ShallowCopy( K2G );; d := Dimension( I );;
gap>  while d > 0 do I := ProductSpace( R2, I );
>      d := Dimension( I ); Print( d, " , " );
>     od;
35 , 27 , 17 , 9 , 0 , gap>
gap> I := ShallowCopy( K4G );; d := Dimension( I );;
gap>  while d > 0 do I := ProductSpace( R4, I );
>      d := Dimension( I ); Print( d, " , " );
>     od;
35 , 27 , 17 , 9 , 0 , gap>
```

Thus $(\dim(\mathrm{J}(KG))^n)_{n=1}^4 = (35, 27, 17, 9)$ and $(\mathrm{J}(KG))^n = \{0\}$ for $n > 4$.