Solution to Exercise 2.1.10

Let $U/N = \mathbf{C}_{\bar{G}}(gN)$ with $U \leq G$. Then

$$U = \{u \in G \mid g^u N = gN\} = \{u \in G \mid [g,u] \in N\} \geq N\,\mathbf{C}_G(g).$$

Thus

$$\begin{aligned} |\,\mathbf{C}_G(g)| &= [N\,\mathbf{C}_G(g) : N] \cdot |\,\mathbf{C}_G(g) \cap N| \leq [U : N] \cdot |\,\mathbf{C}_N(g)| \\ &\leq |\,\mathbf{C}_{\bar{G}}(gN)| \cdot |\,\mathbf{C}_N(g)| \end{aligned}$$

with equality if and only if $U = N\,\mathbf{C}_G(g)$.
(b) Using part (a) we obtain

$$\sum_{g \in G} |\,\mathbf{C}_G(g)| = \sum_{i=1}^{n} \sum_{h \in H} |\,\mathbf{C}_G(g_i h)| \leq \sum_{i=1}^{n} \sum_{h \in H} |\,\mathbf{C}_{\bar{G}}(g_i N)| \cdot |\,\mathbf{C}_N(g_i h)|.$$

(c) Let $\Omega_i := \{(h,u) \in N \times N \mid g_i\,h\,u = u\,g_i\,h\}$. Then

$$|\Omega_i| = \sum_{h \in N} |\,\mathbf{C}_N(g_i h)| = \sum_{u \in N} |\,\mathbf{C}_{g_i N}(u)|.$$

Furthermore

$$\begin{aligned} |\,\mathbf{C}_{g_i N}(u)| &= |\{g_i x \mid x \in N,\ u^{g_i x} = u\}| = |\{x \in N \mid u^{g_i} = u^{x^{-1}}\}| = \\ &= \begin{cases} 0 & \text{if } u^{g_i} \notin u^N \\ |\,\mathbf{C}_N(u)| & \text{if } u^{g_i} \in u^N \end{cases} . \end{aligned}$$

(d) Consider G as a G-set with conjugation as action, thus $\mathrm{Fix}_G(g) = \mathbf{C}_G(g)$. We then obtain from (2.3) and part (b) and (c)

$$\mathrm{k}(G) = \frac{1}{|G|} \sum_{g \in G} |\,\mathbf{C}_G(g)| \leq \frac{1}{|\bar{G}|} \sum_{i=1}^{n} |\,\mathbf{C}_{\bar{G}}(g_i N) \frac{1}{|N|} \sum_{h \in N} |\,\mathbf{C}_N(h)| = \mathrm{k}(\bar{G}) \cdot \mathrm{k}(N).$$