Solution to Exercise 2.1.5

(a) Let $\delta\colon G \to \mathrm{GL}(V)$ be the representation corresponding to the KG-module V. Then $\mathrm{Fix}_V(g) = \ker(\delta(g) - \mathrm{id}_V)$. Since $\ker\varphi^{\mathrm{T}} = (\ker\varphi)^o$ for any $\varphi \in \mathrm{End}_K V$ and hence (by Theorem 1.1.34) $\dim\ker\varphi = \dim\ker\varphi^{\mathrm{T}}$ we get $|\,\mathrm{Fix}_V(g)| = |\,\mathrm{Fix}_{V^\star}(g^{-1})|$. By (2.3) on page 93 G has the same number of orbits on V and $V^\star$.

(b) Let $V_4 \cong G = \langle g_1,\, g_2\rangle$ and define the matrix representation

$$\boldsymbol{\delta}\colon G \to \mathbb{F}_2^3, \quad g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_1 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then clearly the orbits of $\boldsymbol{\delta}$ on $\mathbb{F}_2^3$ have lengths $1, 1, 1, 1, 4$, whereas the orbits of $\boldsymbol{\delta}^\star$ have lengths $1, 2, 2, 2, 1$.

One way to find such examples is to look at the groups G of order 32 having an elementary abelian normal subgroup N of order 8 with elementary abelian factor group and compare the orbits of G on the subgroups of N of order two and on those of order four.