Solution to Exercise 2.8.2

We use GAP:

```
gap> M := [ [4,6,5,2,2], [ 6,13,7,4,4], [ 5,7,11,2,0],
>           [2,4,2,8,4], [ 2,4,0,4,8] ] ;;
gap> emb6 := OrthogonalEmbeddings( M, 6 );; time;
71680
gap> Mats := List( emb6.solutions, x -> TransposedMat(emb6.vectors{x}) );;
gap> for x in Mats do Display(x); od;
[ [  1,  1,  1,  1,  0 ],
  [  3,  1,  1,  1,  1 ],
  [  1,  3,  1,  0,  0 ],
  [  0,  0,  2,  0,  2 ],
  [  0,  0,  0,  2,  2 ] ]
[ [   1,   1,   1,   1,   0,   0 ],
  [   2,   2,   2,   0,   0,   1 ],
  [   2,   1,   0,   2,   1,   1 ],
  [   2,   0,   0,   0,  -2,   0 ],
  [   0,   2,   0,   0,  -2,   0 ] ]
gap> List( [1,2] ,  i -> Mats[i] * TransposedMat(Mats[i]) = M );
[ true, true ]
```

For $n \leq 6$ we have displayed the matrices $X \in \mathbb{Z}^{5\times n}$ with $X \cdot X^{\mathrm{T}} = M$. There is one solution (up to equivalence) for $n = 5$ and one for $n = 6$. Only the first solution is in $\mathbb{Z}^{5\times n}_{\geq 0}$. Observe that the computation took about 72 seconds (on a regular Linux desktop of 2009). Computing all 10254 solutions (for any n) takes about twice as long:

```
gap> emb := OrthogonalEmbeddings( M );;  Length( emb.solutions );
10254
gap> lengths := List( emb.solutions, Length );; Collected( lengths );
[ [ 5, 1 ], [ 6, 1 ], [ 7, 34 ], [ 8, 180 ], [ 9, 861 ], [ 10, 1830 ],
  [ 11, 1486 ], [ 12, 2121 ], [ 13, 2039 ], [ 14, 961 ], [ 15, 574 ],
  [ 16, 121 ], [ 17, 41 ], [ 18, 3 ], [ 19, 1 ] ]
```

The number of solutions drops drastically if we restrict ourselves to non-negative matrices $X \in \mathbb{Z}^{5\times n}_{\geq 0}$:

```
gap> emb := OrthogonalEmbeddings(M, "positive" );; time;
5717
gap> Length( emb.solutions );
42
gap> lengths:=List( emb.solutions, Length );; Collected( lengths );
[ [ 5, 1 ], [ 7, 2 ], [ 8, 1 ], [ 9, 3 ], [ 10, 5 ], [ 11, 4 ], [ 12, 6 ],
  [ 13, 8 ], [ 14, 4 ], [ 15, 6 ], [ 16, 1 ], [ 17, 1 ] ]
```