Solution to Exercise 3.7.6

Let λ be the non-trivial linear character of $\mathbf{Z}(H)$ and $\alpha \in Z^2(\mathrm{L}_2(11), \mathbb{C})$ be in a cohomology class of order two. By Lemma 3.7.28 all classes of $G := \mathrm{L}_2(11)$ are α-regular except possibly `2a` and `6a` and in Example 3.7.29 it was shown that $\mathtt{2a} \in \mathrm{cl}(G)$ is not α-regular. Hence from Theorem 3.7.31 we conclude that $|\operatorname{Irr}(H \mid \lambda)| \in \{6, 7\}$.

From Exercise 3.7.5 and Theorem 3.7.32 we find $\chi(1) = 2n$ with $n \in \{1, 2, 3, 5, 6, 10, 11\}$ for all $\chi \in \operatorname{Irr}(H \mid \lambda)$. Observe that for $\chi \in \operatorname{Irr}(H \mid \lambda)$ $\chi^2 = \chi^{[1^2]} + \chi^{[2]}$ is the inflation of a character of G and

$$(\chi^2, \mathbf{1}_H)_H = \begin{cases} 1 & \text{if } \overline{\chi} = \chi \text{ and} \\ 0 & \text{else.} \end{cases}$$

Let ψ be a character of G of degree m^2 with $(\psi, \mathbf{1}_G)_G \leq 1$ which is a sum of two characters of degrees $\frac{m(m-1)}{2}$ and $\frac{m(m+1)}{2}$. Then the character table of G shows that $\psi(1) \neq 2^2$ and if $\psi(1) = 4^2$ then $\overline{\psi} \neq \psi$ and $(\psi, \mathbf{1}_G)_G = 1$. Finally, if $\psi(1) = 6^2$ then also $\overline{\psi} \neq \psi$. We conclude that for $\chi \in \operatorname{Irr}(H \mid \lambda)$ we have $\chi(1) = 2n$ with $n \in \{3, 5, 6, 10, 11\}$ and $n \geq 5$ if χ is real. Since

$$\sum_{\chi \in \operatorname{Irr}(H|\lambda)} \left(\frac{\chi(1)}{2}\right)^2 = \frac{|G|}{4} = 165$$

we quickly see that $\chi(1) \in \{6, 10, 12\}$ for $\chi \in \operatorname{Irr}(H \mid \lambda)$ and that we must have $\operatorname{Irr}(H \mid \lambda) = \{\chi_9, \ldots \chi_{15}\}$ with $\overline{\chi_9} = \chi_{10}$ of degree 6, $\chi_{11}(1) = \chi_{12}(1) = \chi_{13}(1) = 10$ and $\chi_{14}(1) = \chi_{15}(1) = 12$. We will denote the inflations of the irreducible characters of G as displayed on page 180 by $\chi_1, \ldots, \chi_8$.

Observe that the number of involutions of H is 110, twice the number of involutions of G if the preimages of the elements of $\mathtt{2a} \in \mathrm{cl}(G)$ in H are involutions and one otherwise. But since $\sum_{i=3}^{7} \chi(1) = 54 < 55$ the second alternative must hold and $\nu_2(\chi_i) = -1$ for $11 \leq i \leq 15$ (see Theorem 2.9.9). We also conclude that the preimages of the elements of $\mathtt{6a} \in \mathrm{cl}(G)$ in H have order 12. The power maps of H are now determined. We see that $\{\chi_9^{[1^2]}, \chi_{10}^{[1^2]}\} = \{\chi_3 + \chi_5, \chi_2 + \chi_5\}$ and $\chi_9^{[2]} = \chi_{10}^{[2]} = \chi_4 + \chi_6$. From this we obtain the values of $\chi_9,\ \chi_{10}$ up so signs, which are determined by the congruence relations.

By tensoring $\chi_9,\ \chi_{10}$ with $\chi_2, \ldots, \chi_8$, reducing with $\chi_9,\ \chi_{10}$ and using the LLL-algorithm we obtain $\chi_{14},\ \chi_{15}$. Using also symmetrizations (up to degree 3) we also get χ_{11} and $\chi_{11} + \chi_{12}$ and the character table is easily finished.

Alternatively, using the fact the characters of degree 10 must vanish on elements of order 5 (see Theorem 3.10.10) we see that

$$\{\chi_{11}^{[1^2]}, \chi_{12}^{[1^2]}, \chi_{13}^{[1^2]}\} = \{\chi_1 + 2\,\chi_5 + \chi_7 + \chi_8,\ \chi_1 + \chi_2 + \chi_3 + \chi_7 + \chi_8\}$$

and we may obtain the values of χ_i for $i = 11, 12, 13$ as above. We then find $\chi_{14} = \chi_9 \cdot (\chi_7 - \chi_6) + \chi_{10}$ and $\chi_{14} = \chi_9 \cdot (\chi_8 - \chi_6) + \chi_{10}$.

We finish by displaying the full character table of H:

$\lvert \mathbf{C}_H(g) \rvert$	$\lvert H \rvert$	$\lvert H \rvert$	12	12	12	10	10	10	10	12	12	22	22	22	22
g^H	1a	2a	4a	3a	6a	5a	10a	5b	10b	12a	12b	11a	22a	11b	22b
$(g^2)^H$	1a	1a	2a	3a	3a	5b	5b	5a	5a	6a	6a	11b	11b	11a	11a
$(g^3)^H$	1a	2a	4a	1a	2a	5b	10b	5a	10a	4a	4a	11a	22a	11b	22b
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	5	5	1	-1	-1	0	0	0	0	1	1	α	α	$\overline{\alpha}$	$\overline{\alpha}$
χ_3	5	5	1	-1	-1	0	0	0	0	1	1	$\overline{\alpha}$	$\overline{\alpha}$	α	α
χ_4	10	10	-2	1	1	0	0	0	0	1	1	-1	-1	-1	-1
χ_5	10	10	2	1	1	0	0	0	0	-1	-1	-1	-1	-1	-1
χ_6	11	11	-1	-1	-1	1	1	1	1	-1	-1	0	0	0	0
χ_7	12	12	0	0	0	β	β	β'	β'	0	0	1	1	1	1
χ_8	12	12	0	0	0	β'	β'	β	β	0	0	1	1	1	1
χ_9	6	-6	0	0	0	1	-1	1	-1	0	0	$-\alpha$	α	$-\overline{\alpha}$	$\overline{\alpha}$
χ_{10}	6	-6	0	0	0	1	-1	1	-1	0	0	$-\overline{\alpha}$	$\overline{\alpha}$	$-\alpha$	α
χ_{11}	10	-10	0	-2	2	0	0	0	0	0	0	-1	1	-1	1
χ_{12}	10	-10	0	1	-1	0	0	0	0	γ	$-\gamma$	-1	1	-1	1
χ_{13}	10	-10	0	1	-1	0	0	0	0	$-\gamma$	γ	-1	1	-1	1
χ_{14}	12	-12	0	0	0	β	$-\beta$	β'	$-\beta'$	0	0	1	-1	1	-1
χ_{15}	12	-12	0	0	0	β'	$-\beta'$	β	$-\beta$	0	0	1	-1	1	-1

with $\alpha := \frac{-1+\sqrt{-11}}{2}$, $\beta, \beta' := \frac{-1\pm\sqrt{5}}{2}$ and $\gamma := \sqrt{3}$.

Character table of $2 \cdot \mathrm{L}_2(11)$