Solution to Exercise 3.8.2

In [170] we find the following matrices for $\boldsymbol{\delta}(a)$ and $\boldsymbol{\delta}(b)$

```
da:=[
[0,1,0,0,0,0,0,0],
[1,0,0,0,0,0,0,0],
[0,0,0,0,1,0,0,0],
[1,0,0,1,1,1,0,0],
[0,0,1,0,0,0,0,0],
[1,1,1,0,1,1,0,0],
[0,0,0,0,0,0,0,1],
[0,0,0,0,0,0,1,0]
]*Z(2);;
db:=[
[0,0,1,0,0,0,0,0],
[0,0,0,1,0,0,0,0],
[0,0,0,0,0,1,0,0],
[0,0,0,0,0,0,1,0],
[0,0,0,1,1,0,1,0],
[1,0,1,0,0,0,0,0],
[0,1,1,1,1,1,1,0],
[1,0,0,1,1,1,1,1]
]*Z(2);;
```

Using GAP we compute the orbits $\mathcal{O}_i$ of $G := \langle \mathtt{da}, \mathtt{db} \rangle$ on $\mathbb{F}_2^8$ and stabilizers T_i of representatices of these orbits:

```
gap> G := Group( da, db );; Orbs := Orbits( G, GF(2)^8 );;
gap> List( Orbs, Length );
[ 1, 135, 120 ]
gap> reps := List( Orbs, Representative );;
gap> Stabs := List( reps, x -> Stabilizer( G, x ) );;
gap> List( Stabs, h -> List( NormalSubgroups(h) , Size ) );
[ [ 1, 1451520 ], [ 1, 8, 64, 10752 ], [ 1, 6048, 12096 ] ]
gap> List( NormalSubgroups( Stabs[2]) , IsElementaryAbelian );
[ true, true, true, false ]
```

We have computed the orders of all normal subgroups of T_1, T_2, T_3 and have verified that T_2 is an extension of an elementary abelian group of order 2^6 by a simple group of order 168, thus is of the form $2^6.\mathrm{L}_2(7)$ in ATLAS notation. Also it is apparent that $T_3 \cong \mathrm{U}_3.2$.

The computation of the character tables of the T_i does not present any difficulty and can be done using the GAP-command `CharacterTable`. Of course, the character table of T_2 can also be computed using the method of Clifford matrices and, in fact, could serve as a good example for this method. We display the character table of T_2 after sorting characters and classes in such a way that not only the character tables of the factor groups $\mathrm{L}_2(7)$ and $2^3.\mathrm{L}_2(7)$ become visible, but also almost the Clifford matrices (compare Example 3.8.4):

```
gap> cts := List( Stabs, CharacterTable );;
gap> ct2 := CharacterTableWithSortedClasses( cts[2] );;
gap> ct2 := SortedCharacterTable( ct2,
>                   Set( List( Irr(ct2), ClassPositionsOfKernel )) );;
gap> ct2 := SortedCharacterTable( ct2, CharacterTable("L2(7)"), [1..5] );;
gap> ct2 := CharacterTableWithSortedCharacters( ct2, (16,17,18,19,20)(21,22,23) );;
gap> Display(ct2);


CT2

      2  9  9   9  9  7  7  7  6  7  7  6  6  6  5  2  2  2  2  4  4  4  4  .  .
      3  1  1   1  .  1  .  .  .  .  .  .  .  .  .  1  1  1  1  .  .  .  .  .  .
      7  1  .   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  1  1

        1a 2a  2b 2c 2d 2e 2f 2g 4a 4b 4c 4d 4e 4f 3a 6a 6b 6c 4g 4h 8a 8b 7a 7b
     2P 1a 1a  1a 1a 1a 1a 1a 1a 2a 2a 2c 2c 2a 2c 3a 3a 3a 3a 2f 2e 4a 4b 7a 7b
     3P 1a 2a  2b 2c 2d 2e 2f 2g 4a 4b 4c 4d 4e 4f 1a 2a 2b 2d 4g 4h 8a 8b 7b 7a
     7P 1a 2a  2b 2c 2d 2e 2f 2g 4a 4b 4c 4d 4e 4f 3a 6a 6b 6c 4g 4h 8a 8b 1a 1a

X.1      1  1   1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  1
X.2      3  3   3  3  3 -1 -1 -1 -1 -1 -1 -1 -1 -1  .  .  .  .  1  1  1  1  A /A
X.3      3  3   3  3  3 -1 -1 -1 -1 -1 -1 -1 -1 -1  .  .  .  .  1  1  1  1 /A  A
X.4      6  6   6  6  6  2  2  2  2  2  2  2  2  2  .  .  .  .  .  .  .  . -1 -1
X.5      7  7   7  7  7 -1 -1 -1 -1 -1 -1 -1 -1 -1  1  1  1  1 -1 -1 -1 -1  .  .
X.6      8  8   8  8  8  .  .  .  .  .  .  .  .  . -1 -1 -1 -1  .  .  .  .  1  1
X.7      7  7  -1 -1 -1  3  3 -1 -1 -1 -1 -1  3 -1  1  1 -1 -1  1  1 -1 -1  .  .
X.8      7  7  -1 -1 -1 -1 -1  3  3  3 -1 -1 -1 -1  1  1 -1 -1 -1 -1  1  1  .  .
X.9     14 14  -2 -2 -2  2  2  2  2  2 -2 -2  2 -2 -1 -1  1  1  .  .  .  .  .  .
X.10    21 21  -3 -3 -3  1  1 -3 -3 -3  1  1  1  1  .  .  .  . -1 -1  1  1  .  .
X.11    21 21  -3 -3 -3 -3 -3  1  1  1  1  1 -3  1  .  .  .  .  1  1 -1 -1  .  .
X.12     7 -1  -5  3 -1 -1  3 -1  3 -1 -3  1 -1  1  1  1 -1  1 -1  1 -1 -1 -1  .  .
X.13     7 -1  -5  3 -1  3 -1 -1 -1  3  1 -3 -1  1  1 -1  1 -1  1 -1 -1  1  1 -1  .  .
X.14    14 -2 -10  6 -2  2  2 -2  2  2 -2 -2 -2  2 -1  1 -1  1  .  .  .  .  .  .
X.15    21 -3 -15  9 -3 -3  1  1  1 -3 -1  3  1 -1  .  .  .  . -1  1  1 -1  .  .
X.16    21 -3 -15  9 -3  1 -3  1 -3  1  3 -1  1 -1  .  .  .  .  1 -1 -1  1  .  .
X.17    21 -3   9  1 -3  1  5  1  1 -3  3 -1 -3 -1  .  .  .  .  1 -1  1 -1  .  .
X.18    21 -3   9  1 -3  1 -3 -3  1  5 -1  3  1 -1  .  .  .  .  1 -1  1 -1  .  .
X.19    21 -3   9  1 -3  5  1  1 -3  1 -1  3 -3 -1  .  .  .  . -1  1 -1  1  .  .
X.20    21 -3   9  1 -3 -3  1 -3  5  1  3 -1  1 -1  .  .  .  . -1  1 -1  1  .  .
X.21    42 -6  18  2 -6 -2  2 -2 -2 -2 -2 -2  2  2  .  .  .  .  .  .  .  .  .  .
X.22    28 -4  -4 -4  4  4 -4  .  4 -4  .  .  .  .  1 -1 -1  1  .  .  .  .  .  .
X.23    28 -4  -4 -4  4 -4  4  . -4  4  .  .  .  .  1 -1 -1  1  .  .  .  .  .  .
X.24    56 -8  -8 -8  8  .  .  .  .  .  .  .  .  . -1  1  1 -1  .  .  .  .  .  .

A = E(7)+E(7)^2+E(7)^4
  = (-1+ER(-7))/2 = b7
```