Solution to Exercise 4.1.4

For any G-orbit $\mathcal{O}$ on $\Omega_2 \times \Omega_1$ define the R-linear map

$$\theta_{\mathcal{O}}^R \colon R\Omega_1 \to R\Omega_2 \quad \text{by} \quad \theta_{\mathcal{O}}^R(x) := \sum_{\substack{y \in \Omega_2 \\ (y,x) \in \mathcal{O}}} y \qquad (x \in \Omega_1).$$

Then $\{\theta_{\mathcal{O}}^R \mid \mathcal{O} \text{ a } G\text{-orbit on } Y \times X\}$ is an R-basis of $\mathrm{Hom}_{RG}(R\Omega_1, R\Omega_2)$ by Lemma 1.2.15. If $\eta \colon R \to F$ is the canonical epimorphism then

$$\begin{aligned} \eta' \colon \mathrm{Hom}_{RG}(R\Omega_1, R\Omega_2) &\to \mathrm{Hom}_{FG}(F\Omega_1, F\Omega_2), \\ \sum_{\mathcal{O}} a_{\mathcal{O}}\, \theta_{\mathcal{O}}^R &\mapsto \sum_{\mathcal{O}} \eta(a_{\mathcal{O}})\, \theta_{\mathcal{O}}^F \end{aligned}$$

is an R-linear epimorphism with kernel $\pi \,\mathrm{Hom}_{RG}(R\Omega_1, R\Omega_2)$. Observe that $\mathrm{Hom}_{FG}(F\Omega_1, F\Omega_2)$ is an R-module by inflation (using η). From this (a) follows.

Let $\Omega := \Omega_1 = \Omega_2$. Then $\theta_{\mathcal{O}}^R \mapsto \theta_{\mathcal{O}}^K$ yields an embedding of the R-order $\mathrm{End}_{RG}(R\Omega)$ into $\mathrm{End}_{KG}(K\Omega)$. In this case η' is an R-algebra-homomorphism and hence

$$\mathrm{End}_{FG}(F\Omega) \cong \mathrm{End}_{RG}(R\Omega)/\pi \,\mathrm{End}_{RG}(R\Omega)$$

as R-algebras and also as F-algebras, since πR acts trivially on both sides. So (b) holds.