Solution to Exercise 4.3.3

We calculate the 5-Brauer character table of $G = \mathrm{A}_5$. Since we have three conjugacy classes in A_5 that are 5-regular, we are looking for three irreducible Brauer characters $\varphi_1, \varphi_2, \varphi_3$ and the corresponding indecomposable projective characters $\Phi_{\varphi_1}, \Phi_{\varphi_2}, \Phi_{\varphi_3}$. As in Example 4.2.24 we abbreviate $\chi_i' := \chi_i|_{G_{5'}}$ and we use the same numbering of the $\chi_i \in \mathrm{Irr}(G)$ as there. As always $\varphi_1 := \chi_1'$. Now χ_5 is a defect zero character for $p = 5$ and using Theorem 4.4.14 we infer that $\varphi_3 := \chi_5'$ is an irreducible Brauer character. Note that A_5 has a maximal subgroup $H \cong A_4$. Since $5 \nmid |H|$ every character of H is 5-projective and in particular by Lemma 4.3.6 $(\mathbf{1}_H)^G = \chi_1 + \chi_4$ is projective, and in fact, a projective indecomposable, hence equal to Φ_{φ_1}. Moreover, A_5 also has a maximal subgroup $U \cong S_3$ and inducing the sign character ψ_2 of U to A_5 gives a projective character since $3 \nmid |U|$. Moreover, $\psi_2^G = \chi_2 + \chi_3 + \chi_4$ and so in fact is a projective indecomposable character, namely Φ_{φ_2}. All together this shows that the decomposition matrix for A_5 in characteristic 5 looks as follows:

	φ_1	φ_2	φ_3
χ_1	1	.	.
χ_2	.	1	.
χ_3	.	1	.
χ_4	1	1	.
χ_5	.	.	1

We conclude that $\varphi_2 = \chi_2' = \chi_3'$. Hence the irreducible Brauer characters are all restrictions of ordinary characters of A_5 to the 5-regular classes.