Solution to Exercise 4.4.2

Let $G = \mathrm{S}_6$. We first list the character table and the central characters of S_6 using GAP

```
gap> s6:=CharacterTable("S6");
CharacterTable( "A6.2_1" )
gap> Display(s6);
A6.2_1

      2  4  4  1  1  3  .  4  4  3  1  1
      3  2  .  2  2  .  .  1  1  .  1  1
      5  1  .  .  .  .  1  .  .  .  .  .

        1a 2a 3a 3b 4a 5a 2b 2c 4b 6a 6b
     2P 1a 1a 3a 3b 2a 5a 1a 1a 2a 3a 3b
     3P 1a 2a 1a 1a 4a 5a 2b 2c 4b 2b 2c
     5P 1a 2a 3a 3b 4a 1a 2b 2c 4b 6a 6b

X.1      1  1  1  1  1  1  1  1  1  1  1
X.2      1  1  1  1  1  1 -1 -1 -1 -1 -1
X.3      5  1  2 -1 -1  .  3 -1  1  . -1
X.4      5  1  2 -1 -1  . -3  1 -1  .  1
X.5      5  1 -1  2 -1  . -1  3  1 -1  .
X.6      5  1 -1  2 -1  .  1 -3 -1  1  .
X.7     16  . -2 -2  .  1  .  .  .  .  .
X.8      9  1  .  .  1 -1  3  3 -1  .  .
X.9      9  1  .  .  1 -1 -3 -3  1  .  .
X.10    10 -2  1  1  .  .  2 -2  . -1  1
X.11    10 -2  1  1  .  . -2  2  .  1 -1

gap> c:=List(Irr(s6),x->CentralCharacter(s6,x));;
gap> Display(s6,c);
A6.2_1

      2  4  4  1  1   3   .   4   4   3    1    1
      3  2  .  2  2   .   .   1   1   .    1    1
      5  1  .  .  .   .   1   .   .   .    .    .

        1a 2a 3a 3b  4a  5a  2b  2c  4b   6a   6b
     2P 1a 1a 3a 3b  2a  5a  1a  1a  2a   3a   3b
     3P 1a 2a 1a 1a  4a  5a  2b  2c  4b   2b   2c
     5P 1a 2a 3a 3b  4a  1a  2b  2c  4b   6a   6b

Y.1      1 45 40 40  90 144  15  15  90  120  120
Y.2      1 45 40 40  90 144 -15 -15 -90 -120 -120
```

```
Y.3        1  9 16 -8 -18    .   9  -3  18     .  -24
Y.4        1  9 16 -8 -18    .  -9   3 -18     .   24
Y.5        1  9 -8 16 -18    .  -3   9  18   -24    .
Y.6        1  9 -8 16 -18    .   3  -9 -18    24    .
Y.7        1  . -5 -5   .    9   .   .   .     .    .
Y.8        1  5  .  .  10  -16   5   5 -10     .    .
Y.9        1  5  .  .  10  -16  -5  -5  10     .    .
Y.10       1 -9  4  4   .    .   3  -3   .   -12   12
Y.11       1 -9  4  4   .    .  -3   3   .    12  -12
```

We choose (R, K, F) to be a standard 2-modular splitting system (F being the field with 4 elements) for both S_6 and A_6. By reducing the central characters modulo 2 we see that there are exactly two 2-blocks, a block B_2 of defect zero containing the ordinary irreducible character χ_7 and the principal block B_1 containing all the other irreducible characters. The following table lists the central characters and block idempotents (in $\mathbf{Z}(FG)$):

$\mathbf{G} = \mathrm{S}_6$	1a	2a	3a	3b	4a	5a	2b	2c	4b	6a	6b
$\hat{\omega}_{B_1}$	1	1	0	0	0	0	1	1	0	0	0
$\hat{\omega}_{B_2}$	1	0	1	1	0	1	0	0	0	0	0
$\hat{\epsilon}_{B_1}$	1	0	0	0	0	1	0	0	0	0	0
$\hat{\epsilon}_{B_2}$	0	0	0	0	0	1	0	0	0	0	0

Here the block idempotents are given by their coefficients, that is

$$\hat{\epsilon}_{B_i} = \sum_{C \in \mathrm{cl}(G)} \hat{\epsilon}_{B_i}(C)\ C^+.$$

Note that $\hat{\epsilon}_{B_i}(C) = 0$ for all classes C with $C \not\subseteq G_{p'}$, as follows from Theorem 4.4.7.

The character table of $H = A_6$ as given in GAP is:

```
gap> a6:=CharacterTable("a6");
CharacterTable( "A6" )
gap> Display(a6);
A6

     2  3  3  .  .  2  .  .
     3  2  .  2  2  .  .  .
     5  1  .  .  .  .  1  1

       1a 2a 3a 3b 4a 5a 5b
    2P 1a 1a 3a 3b 2a 5b 5a
    3P 1a 2a 1a 1a 4a 5b 5a
    5P 1a 2a 3a 3b 4a 1a 1a

X.1     1  1  1  1  1  1  1
X.2     5  1  2 -1 -1  .  .
```

```
X.3      5  1 -1  2 -1  .  .
X.4      8  . -1 -1  .  A *A
X.5      8  . -1 -1  . *A  A
X.6      9  1  .  .  1 -1 -1
X.7     10 -2  1  1  .  .  .

A = -E(5)-E(5)^4
  = (1-ER(5))/2 = -b5
```

and the central characters are given by:

```
gap> c:=List(Irr(a6),x->CentralCharacter(a6,x));;
gap> Display(a6,c);
A6

     2  3  3  .  .   2  .  .
     3  2  .  2  2   .  .  .
     5  1  .  .  .   .  1  1

       1a 2a 3a 3b  4a 5a 5b
    2P 1a 1a 3a 3b  2a 5b 5a
    3P 1a 2a 1a 1a  4a 5b 5a
    5P 1a 2a 3a 3b  4a 1a 1a

Y.1      1 45 40 40  90 72 72
Y.2      1  9 16 -8 -18  .  .
Y.3      1  9 -8 16 -18  .  .
Y.4      1  . -5 -5   .  A *A
Y.5      1  . -5 -5   . *A  A
Y.6      1  5  .  .  10 -8 -8
Y.7      1 -9  4  4   .  .  .

A = -9*E(5)-9*E(5)^4
  = (9-9*ER(5))/2 = -9b5
```

In the following we denote the ordinary irreducible characters of H by $\psi_1, \ldots, \psi_7$. Since the irrationality `b5` reduced modulo 2 gives the element $(X+1)+(c)$, where $c := X^2 + X + 1$ is the Conway polynomial for the field of 4 elements, whereas `b5*` is $X + (c)$, we get exactly three distinct central characters of $\mathbf{Z}(FH)$ and also three distinct central primitive idempotents in $\mathbf{Z}(FH)$.

$\mathbf{G = A_6}$	1a	2a	3a	3b	4a	5a	5b
$\hat{\omega}_{b_1}$	1	1	0	0	0	0	0
$\hat{\omega}_{b_2}$	1	0	1	1	0	$X+1+(c)$	$X+(c)$
$\hat{\omega}_{b_2}$	1	0	1	1	0	$X+(c)$	$X+1+(c)$
$\hat{\epsilon}_{b_1}$	1	0	0	0	0	1	1
$\hat{\epsilon}_{b_2}$	0	0	1	1	0	$X+1+(c)$	$X+(c)$
$\hat{\epsilon}_{b_3}$	0	0	1	1	0	$X+(c)$	$X+1+(c)$

Hence there are three 2-blocks for H: two blocks of defect zero b_2 respectively b_3 containing the ordinary irreducible characters ψ_4 respectively ψ_5 and the principal block b_1 containing all the other irreducible characters. From the table given above we conclude that the central idempotent $\hat{\epsilon}^{\cdot}_{B_2}$ of G can be written as $\hat{\epsilon}^{\cdot}_{B_2} = \hat{\epsilon}^{\cdot}_{b_2} + \hat{\epsilon}^{\cdot}_{b_3}$.