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Bernstein-Sato ideals, associated stratifications, and computer-algebraic aspects

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Abstract

Global and local Bernstein-Sato ideals, Bernstein-Sato polynomials and Bernstein-Sato polynomials of varieties are introduced, their basic properties are proven and their algorithmic determination with the method of Briançon/Maisonobe is presented. Stratifications with respect to the local variants of the introduced polynomials and ideals with the methods of Bahloul/Oaku and Levandovskyy/Martín-Morales are treated and the method of Bahloul/Oaku is generalized. Moreover, factors of local Bernstein-Sato ideals for disjoint varieties of components, common factors of components and transversally intersecting varieties of components are given. Furthermore, the connection of multivariate and univariate Bernstein-Sato ideals and polynomials $\mathcal{B}_{(f_1,\ldots,f_r)}$ and $b_{f_1\cdot\ldots\cdot f_r}$ is examined. Budur's approach to determining upper and lower bounds of Bernstein-Sato ideals is presented. Finally, as an application, the computation of $\operatorname{ann}_{D_n}(f^\alpha)$ for $f \in \mathbb{C}[\underline{x}]^r$ and $\alpha \in \mathbb{C}^r$ is described.

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Contents

Introduction

This thesis deals with the topic of Bernstein-Sato ideals which are connected to both algebraic geometry and D-module theory, the theory of modules over rings of differential operators. Many properties will be shown by geometric proofs and interpretations, e.g. through tangent spaces and smoothness of varieties.

After clarifying some notations in Chapter 1, we deal with different definitions and variants of Bernstein-Sato ideals and polynomials in Chapter 2. We define Bernstein-Sato ideals as global objects associated to a tuple $f \in \mathbb{C}[x_1, \ldots, x_n]^r$. Similarly, local Bernstein-Sato ideals in a point $p \in \mathbb{C}^n$ can be defined via localizations. We describe an algorithm that computes the Bernstein-Sato ideal in Section 2.1.

In Section 2.2, we learn about the concept of stratifications and introduce an algorithm to determine a specific stratification with respect to Bernstein-Sato ideals which provides information about the local Bernstein-Sato ideal in $p \in \mathbb{C}^n$, given the stratification. For this stratification, we will use primary decompositions. A byproduct of the stratification is a different proof of a fact about the connection of global and local Bernstein-Sato ideals by intersections.

In Section 2.3, we generalize the concept of local Bernstein-Sato ideals which correspond to points or maximal ideals to local Bernstein-Sato ideals which correspond to varieties or prime ideals.

For Bernstein-Sato polynomials, which are a special case of Bernstein-Sato ideals, more effective stratification algorithms are known than the one for Bernstein-Sato ideals. One of those will be presented in Section 2.4.

A different approach to the generalization of Bernstein-Sato polynomials to the multivariate case $f \in \mathbb{C}[x_1, \ldots, x_n]^r$ other than Bernstein-Sato ideals are Bernstein-Sato polynomials of varieties which are presented in Section 2.5. These can be defined such that they only depend on $\langle f_1, \ldots, f_r \rangle$. However, their definition requires more sophisticated constructions than Bernstein-Sato ideals. In Section 2.6, different variants of Bernstein-Sato polynomials for varieties are compared. Both stratification algorithms presented can be applied to stratifications with respect to Bernstein-Sato polynomials of varieties which will be done in Section 2.7.

All stratifications using primary decompositions presented up to this point have a very similar structure such that we can introduce a generalized type of stratification in Section 2.8.

In Chapter 3, our objective is to develop an understanding of the form of local Bernstein-Sato ideals in certain standard situations. For this, we first examine the role of units for Bernstein-Sato ideals in Section 3.1 which allows us to recapitulate some well-known facts about local Bernstein-Sato ideals. In Section 3.2, statements about the Bernstein-Sato ideals of $f \in \mathbb{C}[x_1, \ldots, x_n]^r$ with disjoint $\mathbb{V}(f_i)$, common factors of f_i, f_j

and some kinds of intersecting $\mathbb{V}(f_i), \mathbb{V}(f_j)$ are shown.

Next, we recall a conjecture about the connection of certain multivariate and univariate annihilators and examine its significance for Bernstein-Sato ideals in Section 3.3.

In Chapter 4, we present a relation that has various computational implications because it allows us to determine Bernstein-Sato ideals up to powers in a more effective way.

The concluding Chapter 5 deals with an application of Bernstein-Sato ideals, the computation of the annihilator of complex powers of polynomials, which is interrelated with the roots of the Bernstein-Sato ideal.

In Appendix A some of the algorithms presented are given in a Singular implementation. Appendix B gives examples of computations in Singular.

1. Notations

We denote the set of positive integers $\{1, 2, \ldots\}$ by $\mathbb{N}$.

By $\delta_{i,j}$ we denote the *Kronecker-Delta* with

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

In the following, we will work over the field of complex numbers $\mathbb{C}$ and consider the polynomial ring in $n \in \mathbb{N}$ variables $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_n]$ and the corresponding *nth Weyl algebra* with polynomial coefficients

$$\begin{aligned} D := D_n := \mathbb{C}\langle & x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \\ & \partial_i x_j = x_j \partial_i + \delta_{i,j}, x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i \text{ for all } 1 \leq i, j \leq n \rangle. \end{aligned}$$

The polynomial ring $\mathbb{C}[\underline{x}]$ becomes a D-module by the interpretation of the elements of D as differential operators

$$\partial_i \bullet \prod_{j=1}^{n} x_j^{\alpha_j} = \alpha_i x_i^{\alpha_i - 1} \prod_{j \neq i} x_j^{\alpha_j}, \quad x_i \bullet \prod_{j=1}^{n} x_j^{\alpha_j} = x_i^{\alpha_i + 1} \prod_{j \neq i} x_j^{\alpha_j} \quad \text{for all } 1 \leq i \leq n, \alpha \in \mathbb{N}_0^n.$$

Remark 1.1. The above definition of D_n implies the Leibniz rule $\partial_i \bullet fg = (\partial_i \bullet f)g + f(\partial_i \bullet g)$ for $1 \leq i \leq n$ and $f, g \in \mathbb{C}[\underline{x}]$.

Furthermore, for $f \in \mathbb{C}[\underline{x}]$ and $i \in \{1, \ldots, n\}$ the action of ∂_i on f corresponds to the ith partial derivative of f, i.e.

$$\partial_i \bullet f = \frac{\partial f}{\partial x_i}.$$

We will denote all module actions considered by $\bullet$.

For $\alpha \in \mathbb{C}^r$ and $\beta \in \mathbb{Z}^n$, $f \in \mathbb{C}[\underline{x}]^r$ we denote powers in multi-index notation by $f^\alpha = f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r}$, $x^\beta = x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}$ and $\partial^\beta = \partial_1^{\beta_1} \cdot \ldots \cdot \partial_n^{\beta_n}$.

In examples, we will often use the polynomial rings $\mathbb{C}[x, y]$ and $\mathbb{C}[x, y, z]$ instead of $\mathbb{C}[x_1, x_2]$ and $\mathbb{C}[x_1, x_2, x_3]$.

We use the *Lie bracket* $[a, b] := ab - ba$ for ring elements a, b.

For a ring R, an R-module M and $m \in M$ we denote the annihilator of m in R by

$$\operatorname{ann}_R(m) := \{r \in R \mid r \bullet m = 0\}.$$

We work with the *Krull dimension* of a commutative ring R and an ideal $I \subseteq R$, defined as

$$\begin{aligned}\operatorname{krdim}(R) &:= \sup\{\ell \mid \exists \mathfrak{p}_0, \ldots, \mathfrak{p}_\ell \text{ prime ideals in } R \text{ with } \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_\ell\}, \\ \operatorname{krdim}(I) &:= \operatorname{krdim}(R/I).\end{aligned}$$

For reasons of space we will often denote column vectors v from $\mathbb{C}^n = \mathbb{C}^{n\times 1}$ or from $\mathbb{C}^r$ as row vectors $v = (v_1, \ldots, v_n)$ or $v = (v_1, \ldots, v_r)$, respectively.

By e_i we denote the ith standard basis vector $(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0)$ of the complex vector spaces $\mathbb{C}^n$ and $\mathbb{C}^r$ respectively.

The quotient $I : h$ for an ideal $I \subseteq R$ and a ring element $h \in R$ is defined as $I : h = \{f \in R \mid fh \in I\}$. We will only use this notation for h that are contained in the center of R.

2. Bernstein-Sato ideals and polynomials

In the following, let $f \in \mathbb{C}[\underline{x}]^r$ for a fixed $r \in \mathbb{N}$ and denote the product of the components of f by $F = \prod_{i=1}^r f_i$.

We work only over the field of complex numbers since it is algebraically closed and has characteristic zero but many of the statements shown can be generalized to other cases.

Furthermore, we will work with symbolic powers $f^s := f_1^{s_1} \cdot \ldots \cdot f_r^{s_r}$ by considering the module $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]f^s$ over the ring $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]$, where $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}] = \mathbb{C}[\underline{x}, \underline{s}]_F = S^{-1}\mathbb{C}[\underline{x}, \underline{s}] \subseteq \mathbb{C}(\underline{x})[\underline{s}]$ denotes the localization of $\mathbb{C}[\underline{x}, \underline{s}]$ at the multiplicatively closed set $S := \{F^j \mid j \in \mathbb{N}_0\}$. Here, the $f_i^{s_i}$ are treated as formal symbols. Only in the following D-module structure of $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]f^s$ do we find the interpretation of f^s as a power, since we set

$$\partial_i \bullet f_j^{s_j} = s_j f_j^{s_j-1}(\partial_i \bullet f_j) := s_j f_j^{-1} f_j^{s_j}(\partial_i \bullet f_j),$$
$$\partial_i \bullet f^s = \left(\sum_{j=1}^{n} s_j \frac{(\partial_i \bullet f_j)}{f_j} \right) f^s$$

and otherwise continue the structure of $\mathbb{C}[\underline{x}, \underline{s}]$ as a D-module with the Leibniz rule.

For terms in the symbolic powers, we use the following intuitive notations:

$$f^{s+1} := F f^s, \quad f_i^{s_i+1} := f_i f_i^{s_i}, \quad f_i^{s_i-1} := \frac{1}{f_i} f_i^s.$$

Working with symbolic powers, we can introduce Bernstein-Sato ideals.

Definition 2.1. The *Bernstein-Sato ideal* of f is defined as

$$\mathcal{B} := \mathcal{B}_f := \left\{ b \in \mathbb{C}[\underline{s}] \mid b(s) f^s = \delta(s) \bullet f^{s+1} \text{ for some } \delta \in D_n[\underline{s}] \right\}.$$

Remark 2.2. For $f \in \mathbb{C}[\underline{x}]^r$, it holds that $\mathcal{B} \neq \{0\}$ (compare e.g. [Sab87] and [Lev15]).

The defining functional equation for Bernstein-Sato ideals can be reformulated by

$$\begin{aligned} b \in \mathcal{B} \quad &\Leftrightarrow \quad \exists \delta \in D_n[\underline{s}] : b f^s = \delta \bullet f^{s+1} \\ &\Leftrightarrow \quad \exists \delta \in D_n[\underline{s}] : (b - \delta F) \bullet f^s = 0 \\ &\Leftrightarrow \quad \exists \delta \in D_n[\underline{s}] : (b - \delta F) \in \operatorname{ann}_{D_n[s]}(f^s) \\ &\Leftrightarrow \quad b \in (\operatorname{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F \rangle) \cap \mathbb{C}[\underline{s}], \end{aligned}$$

so

$$\mathcal{B} = (\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F \rangle) \cap \mathbb{C}[\underline{s}].$$

The differential operators δ from the definition of Bernstein-Sato ideals are also called *Bernstein-Sato operators.*

Other variants of Bernstein-Sato ideals include

$$\mathcal{B}_\Sigma := (\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle f_1, \dots, f_r \rangle) \cap \mathbb{C}[\underline{s}]$$

and

$$\mathcal{B}_{(i)} := (\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle f_i \rangle)) \cap \mathbb{C}[\underline{s}] \text{ for } 1 \leq i \leq r.$$

The elements $b_\Sigma \in \mathcal{B}_\Sigma$ and $b_{(i)} \in \mathcal{B}_{(i)}$ can also be described by the functional equations

$$b_\Sigma f^s = \sum_{i=1}^r \delta_i f_i \bullet f^s \text{ for some } \delta_1, \dots, \delta_r \in D_n[\underline{s}], \quad b_{(i)} f^s = \delta f_i \bullet f^s \text{ for some } \delta \in D_n[\underline{s}].$$

In the following we will mainly work with $\mathcal{B}$.

The three variants are connected by the inclusions $\mathcal{B} \subseteq \mathcal{B}_{(i)} \subseteq \mathcal{B}_\Sigma$, since for $b \in \mathcal{B}$ with $bf^s = \delta \bullet f^{s+1}$ one has $bf^s = \left(\delta \frac{F}{f_i}\right) \bullet f_i f^s$ and for $b \in \mathcal{B}_{(i)}$ with $bf^s = \delta \bullet f_i f^s$ it holds that $bf^s = (\delta \bullet f_i + \sum_{j \neq i} 0 \bullet f_j) f^s$.

Example 2.3. For $f = (x, y) \in \mathbb{C}[x, y]^2$, the Bernstein-Sato ideal is $\mathcal{B} = \langle (s_1 + 1)(s_2 + 1) \rangle$. A Bernstein-Sato operator corresponding to the generator of $\mathcal{B}$ is $\delta(s) = \partial_x \partial_y$. Furthermore, $\mathcal{B}_{(1)} = \langle s_1 + 1 \rangle$ with Bernstein-Sato operator ∂_x and $\mathcal{B}_{(2)} = \langle s_2 + 1 \rangle$ with Bernstein-Sato operator ∂_y. Lastly, $\mathcal{B}_\Sigma = \langle s_1 + 1, s_2 + 1 \rangle$.

Remark 2.4. This example and the other ones presented in this thesis were computed with the help of the library `dmod.lib` ([LM15]) of the computer algebra system Singular/Plural ([DGPS15]/[GLMS15]).

Now we want to consider local Bernstein-Sato ideals, i.e. we want to replace $\mathbb{C}[\underline{x}]$ by $\mathbb{C}[\underline{x}]_p$ for some $p \in \mathbb{C}^n$, where $\mathbb{C}[\underline{x}]_p$ denotes the geometric localization at the point p with denominator set $S_p := \{f \in \mathbb{C}[\underline{x}] \mid f(p) \neq 0\} = \mathbb{C}[\underline{x}] \setminus {}_{\mathbb{C}[\underline{x}]}\langle x_1 - p_1, \dots, x_n - p_n \rangle$:

$$\mathbb{C}[\underline{x}]_p := S_p^{-1} \mathbb{C}[\underline{x}] \subseteq \mathbb{C}(\underline{x}).$$

We also consider the Weyl algebra with coefficients in $\mathbb{C}[\underline{x}]_p$ as a sub-algebra of the nth Weyl algebra with rational coefficients

$$\begin{aligned} D_p :=& D_{n,p} := S_p^{-1} D_n = \left\{ \frac{\delta}{g} \mid g \in \mathbb{C}[\underline{x}], g(p) \neq 0, \delta \in D_n \right\} \subseteq W_n \\ :=& \mathbb{C}\left\langle \tfrac{f}{g}, \partial_i \;\middle|\; 1 \leq i \leq n, f \in \mathbb{C}[\underline{x}], g \in \mathbb{C}[\underline{x}] \setminus \{0\}, \partial_i \tfrac{f}{g} = \tfrac{f}{g} \partial_i + \partial_i \bullet \tfrac{f}{g} \right\rangle, \end{aligned}$$

where $\partial_i \bullet \frac{f}{g}$ denotes the derivative action of ∂_i on $\frac{f}{g}$,

$$\partial_i \bullet \frac{f}{g} = \frac{(\partial_i \bullet f) g - f (\partial_i \bullet g)}{g^2}.$$

Definition 2.5. The local Bernstein-Sato ideal at the point $p \in \mathbb{C}^n$ is defined as

$$\mathcal{B}_p = (\operatorname{ann}_{D_{n,p}[\underline{s}]}(f^s) + {}_{D_{n,p}[\underline{s}]}\langle F \rangle) \cap \mathbb{C}[\underline{s}].$$

Remark 2.6. This definition can again be equivalently rewritten as a functional equation by defining $\mathcal{B}_p$ as the ideal of all $b(s) \in \mathbb{C}[\underline{s}]$ such that

$$b(s) f^s = \delta(s) \bullet f^{s+1} \text{ for some } \delta(s) \in D_{n,p}[\underline{s}].$$

Remark 2.7. Similarly as in Definition 2.5 we may also define

$$\mathcal{B}_{\Sigma,p} = (\operatorname{ann}_{D_{n,p}[\underline{s}]}(f^s) + {}_{D_{n,p}[\underline{s}]}\langle f_1, \dots, f_r \rangle) \cap \mathbb{C}[\underline{s}]$$

and

$$\mathcal{B}_{(i),p} = (\operatorname{ann}_{D_{n,p}[\underline{s}]}(f^s) + {}_{D_{n,p}[\underline{s}]}\langle f_i \rangle) \cap \mathbb{C}[\underline{s}].$$

Lemma 2.8. For a multiplicatively closed set $S \subseteq \mathbb{C}[\underline{x}]$ of the form $S = \mathbb{C}[\underline{x}] \setminus \mathfrak{p}$ for a prime ideal $\mathfrak{p} \subseteq \mathbb{C}[\underline{x}]$ it holds that

$$\operatorname{ann}_{S^{-1}D_n[\underline{s}]}(f^s) = S^{-1} \operatorname{ann}_{D_n[\underline{s}]}(f^s).$$

Proof. In [Lev15, 1.4.10] it is shown that S is also an Ore set suitable for localization in D_n and thus also in $D_n[\underline{s}]$, which makes both terms in the equation well-defined.

The inclusion '$\supseteq$' obviously holds. For '$\subseteq$', let $\delta = \sum_{\alpha \in \mathbb{N}_0^n} \frac{f_\alpha}{g_\alpha} \partial^\alpha \in \operatorname{ann}_{S^{-1}D_n[\underline{s}]}(f^s)$ with $f_\alpha \in \mathbb{C}[\underline{x}]$ non-zero for only finitely many α and $g_\alpha \in S$. We choose a common denominator $g \in S$ as the product of all g_α with $f_\alpha \neq 0$. Then $g\delta \in \operatorname{ann}_{D_n[\underline{s}]}(f^s)$, so $\delta \in S^{-1} \operatorname{ann}_{D_n[\underline{s}]}(f^s)$. ☐

Corollary 2.9. For the denominator set

$$S_p = \{ f \in \mathbb{C}[\underline{x}] \mid f(p) \neq 0 \} = \mathbb{C}[\underline{x}] \setminus \langle x_1 - p_1, \dots, x_n - p_n \rangle,$$

we obtain

$$\operatorname{ann}_{D_{n,p}[\underline{s}]}(f^s) = S_p^{-1} \operatorname{ann}_{D_n[\underline{s}]}(f^s)$$

and thus

$$\mathcal{B}_p = (S_p^{-1}(\operatorname{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F \rangle)) \cap \mathbb{C}[\underline{s}].$$

This hints at a connection between global and local Bernstein-Sato ideals.

Proposition 2.10 ([BM02])**.** For local and global Bernstein-Sato ideals of $f \in \mathbb{C}[\underline{x}]^r$ it holds true that

$$\mathcal{B} = \bigcap_{p \in \mathbb{C}^n} \mathcal{B}_p.$$

Proof. For $b \in \mathbb{C}[\underline{s}]$ we define the $\mathbb{C}[\underline{s}]$-module

$$M_b := (bD_n[\underline{s}]f^s)/D_n[\underline{s}]f^{s+1}.$$

With this definition, we have $b \in \mathcal{B}$ if and only if $M_b = \{0\}$. Analogously, we define the $\mathbb{C}[\underline{s}]$-module

$$M_{b,p} := (bD_{n,p}[\underline{s}]f^s)/D_{n,p}[\underline{s}]f^{s+1}$$

with $b \in \mathcal{B}_p$ if and only if $M_{b,p} = \{0\}$.

The exactness of the localization functor applied to the exact sequence

$$0 \to D_n[\underline{s}]f^{s+1} \to D_n[\underline{s}]f^s \to D_n[\underline{s}]f^s/D_n[\underline{s}]f^{s+1} \to 0$$

yields $S_p^{-1}M_b \cong M_{b,p}$ for all $b \in \mathbb{C}[\underline{s}], p \in \mathbb{C}^n$.

Now, the claim follows together with the fact that the property '$= \{0\}$' of a module (over a commutative ring) is local, since

$$b \in \mathcal{B} \Leftrightarrow M_b = \{0\} \Leftrightarrow M_{b,p} = \{0\} \text{ for all } p \in \mathbb{C}^n \Leftrightarrow b \in \bigcap_{p\in\mathbb{C}^n} \mathcal{B}_p.$$

□

Remark 2.11. Analogous statements hold for $\mathcal{B}_\Sigma$ and $\mathcal{B}_{(i)}$.

Specializing the theory developed to the univariate case $r = 1$, i.e. $f \in \mathbb{C}[\underline{x}]$, we obtain a principal ideal $\mathcal{B} \subseteq \mathbb{C}[s]$. We denote its monic generator by $b_f(s)$ or $b_{f,p}(s)$, respectively, and call it the *Bernstein-Sato polynomial* of f or the Bernstein-Sato polynomial of f in p. Proposition 2.10 over the principal ideal domain $\mathbb{C}[s]$ becomes

$$\mathrm{lcm}_{p\in\mathbb{C}^n}(b_{f,p}(s)) = b_f(s).$$

In this case, the different concepts $\mathcal{B}, \mathcal{B}_\Sigma, \mathcal{B}_{(1)}$ coincide, so all three of them can be regarded as natural generalizations of the Bernstein-Sato polynomial which (especially in the local variant) is far better researched than Bernstein-Sato ideals.

Remark 2.12. The original object of interest was the Bernstein-Sato polynomial and not the Bernstein-Sato ideal. Bernstein introduced it in order to examine the meromorphic continuation of f^s as a function in $s \in \mathbb{C}$, where f^s is originally only defined for real part $\Re(s) > 0$ (see [Ber72]).

It can also be used to find rational solutions of holonomic systems of differential equations by finding upper bounds of the denominator degree of solutions which can be determined by roots of the Bernstein-Sato polynomial (see [OTT01]).

In Chapter 5, we will see another application of Bernstein-Sato ideals, the computation of $\mathrm{ann}_{D_n}(f^\alpha)$ for a fixed $\alpha \in \mathbb{C}^r$.

2.1. Computer-algebraic aspects

In this subsection we shall briefly recall an algorithm from [BM02] which allows for an algorithmic determination of $\mathcal{B}$. For details on Gröbner bases we refer to [Lev05, Cas84, Gal85] and for details on the algorithm and a comparison with another algorithm to [UC04]. We follow the approach of [UC04].

The equality $\mathcal{B} = (\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F\rangle) \cap \mathbb{C}[\underline{s}]$ hints us at an algorithm for determining $\mathcal{B}$. We can use Gröbner bases in $D_n[\underline{s}]$ and this equality in order to determine $\mathcal{B}$. Here, Gröbner bases are applicable in large parts analogously as in a commutative polynomial ring, in particular because the non-commutative relations $\partial_i x_i = x_i\partial_i + 1$ do not change the leading term with respect to total degree (with ∂_i and x_i both of degree 1).

Two problems arise when calculating $\mathcal{B}$: First, we need to determine $\mathrm{ann}_{D_n[\underline{s}]}(f^s)$ algorithmically and then compute the intersection with the commutative ring $\mathbb{C}[\underline{s}]$.

An algorithm by Briançon and Maisonobe solves these problems by means of elimination orderings.

We introduce additional variables $t_1, \ldots, t_r$ for the computation of $\mathrm{ann}(f^s)$.

Definition 2.13. By $D_n\langle \underline{s}, \underline{t}\rangle$, we denote the ring

$$D\langle \underline{s}, \underline{t}\rangle := D_n\langle \underline{s}, \underline{t}\rangle := (D_n[\underline{s}])\langle \underline{t}\rangle$$

with additional non-commutative relations (besides those of $D_n[\underline{s}]$) given by $s_i t_j = t_j s_i + \delta_{i,j} t_i$.

The $D_n[\underline{s}]$-module $\mathbb{C}\left[\underline{x}, \underline{s}, \frac{1}{F}\right] f^s$ becomes a $D_n\langle \underline{s}, \underline{t}\rangle$-module via

$$t_i \bullet \underbrace{g(s)}_{\in \mathbb{C}\left[\underline{x}, \underline{s}, \frac{1}{F}\right]} f^s = -g(s_1, \ldots, s_{i-1}, s_i - 1, s_{i+1}, \ldots, s_r) s_i \frac{1}{f_i} f^s$$

for $i \in \{1, \ldots, r\}$.

Remark 2.14. For the elements of $D_n\langle \underline{s}, \underline{t}\rangle$ we can assume a standard representation

$$\sum_{\alpha,\beta,\gamma,\delta} c_{\alpha,\beta,\gamma,\delta} x^\alpha \partial^\beta t^\gamma s^\delta$$

with $c_{\alpha,\beta,\gamma,\delta} \in \mathbb{C}$. This can be shown by induction on the total degree using the fact that all non-commutative relations do not change the total degree. Analogously as in the the commutative case, the total degree (with $x_i, \partial_i, t_i, s_j$ all of total degree 1) is well-behaved in the sense that

$$\mathrm{tdeg}(\delta \cdot \gamma) = \mathrm{tdeg}(\delta) + \mathrm{tdeg}(\gamma).$$

Iterated application of the module action yields

$$t^\alpha \bullet g(s) f^s = (-1)^{|\alpha|} g(s - \alpha) \left(\prod_{i=1}^{r} (s_i - \alpha_i - 1) \cdot \ldots \cdot s_i \right) \frac{1}{f^\alpha} f^s$$

for $g(s) \in \mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]$, $\alpha \in \mathbb{N}_0^r$.

Theorem 2.15 ([BM02, UC04])**.** The annihilator of f^s in $D_n\langle \underline{s}, \underline{t}\rangle$ is given by

$$\operatorname{ann}_{D_n\langle \underline{s},\underline{t}\rangle}(f^s) = \left\langle s_j + f_j t_j, \partial_i + \sum_{l=1}^{r}(\partial_i \bullet f_l)t_l \;\middle|\; i \in \{1,\ldots,n\}, j \in \{1,\ldots,r\} \right\rangle. \quad (1)$$

Proof. First, we check '$\supseteq$'. For $j \in \{1,\ldots,r\}$, $i \in \{1,\ldots,n\}$ it holds that

$$(s_j + f_j t_j) \bullet f^s = s_j f^s - f_j s_j \frac{1}{f_j} f^s = 0$$

and

$$\begin{aligned}\left(\partial_i + \sum_{l=1}^{r}(\partial_i \bullet f_l)t_l\right) \bullet f^s &= \left(\sum_{k=1}^{r} s_k \frac{1}{f_k}(\partial_i f_k) + \sum_{l=1}^{r}(\partial_i \bullet f_l)t_l\right) \bullet f^s \\ &= \left(\sum_{k=1}^{r} s_k \frac{1}{f_k}(\partial_i f_k) - \sum_{l=1}^{r}(\partial_i \bullet f_l)s_l \frac{1}{f_l}\right) f^s = 0.\end{aligned}$$

For '$\subseteq$', let $\delta \in \operatorname{ann}_{D_n\langle \underline{s},\underline{t}\rangle}(f^s)$. Denote by J the ideal on the right hand side of (1). Let δ' be the normal form of δ with respect to the ideal J and the lexicographic ordering $<$ with

$$x_i < t_j < \partial_k < s_l \text{ for all } i,k \in \{1,\ldots,n\}, j,l \in \{1,\ldots,r\}.$$

By the form of the generators of J, it follows that $\delta' \in \mathbb{C}[\underline{x}, \underline{t}]$. For all $\alpha \in \mathbb{N}_0^r$, for $\delta' = \sum_{\alpha\in\mathbb{N}_0^n, \beta\in\mathbb{N}_0^r} c_{\alpha,\beta} x^\alpha t^\beta$ we have

$$\begin{aligned}\delta' \bullet f^s &= \sum_{\alpha\in\mathbb{N}_0^n,\beta\in\mathbb{N}_0^r} c_{\alpha,\beta} x^\alpha t^\beta \bullet f^s \\ &= \sum_{\alpha\in\mathbb{N}_0^n,\beta\in\mathbb{N}_0^r} c_{\alpha,\beta} x^\alpha (-1)^{|\beta|} \left(\prod_{i=1}^{r}(s_i - \beta_i - 1)\cdot\ldots\cdot s_i\right) \frac{1}{f^\beta} f^s \\ &= \sum_{\beta\in\mathbb{N}_0^r} (-1)^{|\beta|} \left(\prod_{i=1}^{r}(s_i - \beta_i - 1)\cdot\ldots\cdot s_i\right) \frac{1}{f^\beta} \sum_{\alpha\in\mathbb{N}_0^n} c_{\alpha,\beta} x^\alpha f^s.\end{aligned}$$

By equating the coefficients in s, we conclude that $\operatorname{ann}_{\mathbb{C}[\underline{x},\underline{t}]}(f^s) = 0$, since $c_{\alpha,\beta}$ does not depend on s, so $\delta' = 0$, which implies $\delta \in J$. □

Using this statement, $\operatorname{ann}_{D_n[\underline{s}]}(f^s) = \operatorname{ann}_{D_n\langle \underline{s},\underline{t}\rangle}(f^s) \cap D_n[\underline{s}]$ and the following definition, we obtain an algorithm for computing $\mathcal{B}$.

Definition 2.16. We define $<_t$ to be an elimination ordering on $D_n\langle \underline{s}, \underline{t}\rangle$ with respect to the t_i, i.e. $<_t$ is a monomial ordering and additionally $s_i, x_j, \partial_j <_t t_k$ for all $i,k \in \{1,\ldots,r\}$, $j \in \{1,\ldots,n\}$.

Analogously, $<_s$ shall denote an elimination ordering on $D_n[\underline{s}]$ with respect to the x_i, ∂_i with $s_i <_s x_j, \partial_j$ for all $i \in \{1,\ldots,r\}$, $j \in \{1,\ldots,n\}$.

Remark 2.17. The orderings $<_t$ and $<_s$ are called elimination orderings because from Gröbner bases with respect to these orderings we can easily eliminate the variables t_i and x_i, ∂_i from ideals $I \subseteq D_n\langle \underline{s}, \underline{t}\rangle$ and $J \subseteq D_n[\underline{s}]$, respectively, i.e. we can compute $I \cap D_n[\underline{s}]$ and $J \cap \mathbb{C}[\underline{s}]$.

This classical application of Gröbner bases is due to the fact that if t_i appears in a leading term of an element of a Gröbner basis of I with respect to $<_t$, then this element plays a role as a generator of I only for elements of higher degree with respect to $<_t$ which also have a leading term in a t_j. The analogous procedure works for $<_s$.

For $<_t, <_s$ we may choose block orderings. More precisely, we can extend any monomial ordering $<_{D_n[\underline{s}]}$ on $D_n[\underline{s}]$ and $<_{\mathbb{C}[\underline{t}]}$ on $\mathbb{C}[\underline{t}]$ to an elimination ordering $<_t$ via

$$x^\alpha \partial^\beta s^\gamma t^\varepsilon <_t x^{\alpha'} \partial^{\beta'} s^{\gamma'} t^{\varepsilon'} :\Leftrightarrow \begin{cases} t^\varepsilon <_{\mathbb{C}[\underline{t}]} t^{\varepsilon'} & \text{or} \\ \varepsilon = \varepsilon' \text{ and } x^\alpha \partial^\beta s^\gamma <_{D_n[\underline{s}]} x^{\alpha'} \partial^{\beta'} s^{\gamma'}. \end{cases}$$

Analogously, we can construct $<_s$.

Algorithm 2.18 ([BM02])**.**
Input: $f \in \mathbb{C}[\underline{x}]^r$.
Output: the Bernstein-Sato ideal $\mathcal{B}$ of f.

1: Set $I := \langle s_j + f_j t_j, \partial_i + \sum_{l=1}^r (\partial_i \bullet f_l) t_l \mid i \in \{1, \dots, n\}, j \in \{1, \dots, r\}\rangle$. $\triangleright$ $I = \operatorname{ann}_{D_n\langle \underline{s}, \underline{t}\rangle}(f^s)$.
2: Compute a Gröbner basis G of I with respect to $<_t$.
3: Set $J := {}_{D_n[\underline{s}]}\langle g \in G \cap D_n[\underline{s}]\rangle$. $\triangleright$ $J = \operatorname{ann}_{D_n[\underline{s}]}(f^s)$.
4: Set $J' := J + {}_{D_n[\underline{s}]}\langle F\rangle$. $\triangleright$ $J' = \operatorname{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F\rangle$.
5: Compute a Gröbner basis H of J' with respect to $<_s$.
6: **return** $\langle h \in H \cap \mathbb{C}[\underline{s}]\rangle$. $\triangleright$ $\langle h \in H \cap \mathbb{C}[\underline{s}]\rangle = \mathcal{B}$.

Remark 2.19. The correctness of the algorithm follows from Theorem 2.15, which justifies the assignment in the first step, and the choice of elimination orderings $<_s$, $<_t$, which make sure that in the third and sixth step we really compute the desired intersections.

Analogously, we can compute $\mathcal{B}_{(i)}$ and $\mathcal{B}_\Sigma$ by altering the fourth step to $J := J + {}_{D_n[\underline{s}]}\langle f_i\rangle$ and $J := J + {}_{D_n[\underline{s}]}\langle f_1, \dots, f_r\rangle$, respectively.

2.2. Stratifications with respect to local Bernstein-Sato ideals

In order to develop the concept of a stratification with respect to local Bernstein-Sato ideals, first we are concerned with a primary decomposition of $\mathcal{B}$.

Definition 2.20. Let R be a commutative ring with 1 and $I \subseteq R$ be an ideal. We call a decomposition into primary components Q_i (i.e. for $fg \in Q_i$ we have $f \in Q_i$ or $g^j \in Q_i$ for some $j \in \mathbb{N}$) of the form

$$I = \bigcap_{i=1}^{l} Q_i$$

a *primary decomposition* of I.

Remark 2.21 (compare e.g. [Eis95])**.** Over a Noetherian ring, any ideal has a finite primary decomposition. It can be algorithmically computed.

Instead of choosing the direct way of determining a primary decomposition of $\mathcal{B}$, we instead decompose $Q := (\operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle F\rangle) \cap \mathbb{C}[\underline{x}, \underline{s}]$, following the approach of [BO10]. In the following, we fix a primary decomposition of Q as

$$Q = \bigcap_{i=1}^{\ell} Q_i.$$

In this primary decomposition, we still have both sets of variables $\{x_1, \ldots, x_n\}$ and $\{s_1, \ldots, s_r\}$. We denote the intersections with the corresponding subrings by $I_i := Q_i \cap \mathbb{C}[\underline{x}]$ and $\mathcal{B}_i := Q_i \cap \mathbb{C}[\underline{s}]$. Indeed, $\bigcap_i \mathcal{B}_i$ and $\bigcap_i I_i$ are primary decompositions of $\mathcal{B} = Q \cap \mathbb{C}[\underline{x}, \underline{s}]$ and $Q \cap \mathbb{C}[\underline{x}]$, respectively, which is a consequence of the following lemma.

Lemma 2.22. Let $S \subseteq R$ be an extension of commutative rings with 1, $I \subseteq R$ be an ideal and $I = \bigcap_{i=1}^{l} Q_i$ a primary decomposition. A primary decomposition of $I \cap S$ is given by $I \cap S = \bigcap_{i=1}^{l}(Q_i \cap S)$.

Proof. Obviously, $\bigcap_{i=1}^{l}(Q_i \cap S) = \left(\bigcap_{i=1}^{l} Q_i\right) \cap S = I \cap S$. Furthermore, for Q_i primary and $fg \in Q_i \cap S$ with $f, g \in S$ we have $fg \in Q_i$, so $f \in Q_i \cap S$ or $g^j \in Q_i \cap S$ for some j. □

Remark 2.23. If we choose an irredundant primary decomposition $I = \bigcap_{i=1}^{l} Q_i$ with $Q_i \neq Q_j$ for all $i \neq j$ in Lemma 2.22, we do not necessarily obtain an irredundant primary decomposition $I \cap S = \bigcap_{i=1}^{l}(Q_i \cap S)$, which can be seen in the example $\langle x, xy, xz, yz\rangle = \langle x, y\rangle \cap \langle x, z\rangle \subseteq \mathbb{C}[x, y, z]$ which after intersection with $\mathbb{C}[x]$ becomes $\langle x\rangle = \langle x\rangle \cap \langle x\rangle$.

We include the following result since it is applicable in a more general setting than for the primary decompositions we are interested in.

Lemma 2.24. Let $B \subseteq A$ and $C \subseteq A$ be extensions of commutative rings. Furthermore, let $Q \subseteq A$ be a primary ideal and $\mathfrak{p}$ be a prime ideal in B. We define the multiplicatively closed set $S := B \setminus \mathfrak{p}$.

- If $\mathfrak{p} \supseteq Q \cap B$, the equality $(S^{-1}Q) \cap C = Q \cap C$ holds.
- If $\mathfrak{p} \not\supseteq Q \cap B$, the equality $(S^{-1}Q) \cap C = C$ holds.

Proof. We show the first claim.

The inclusion '$\supseteq$' obviously holds. For '$\subseteq$', let $\frac{f}{g} = h \in (S^{-1}Q) \cap C$ with $f \in Q$ and $g \in S \subseteq B$. We have to show that $h \in Q$.

Assume towards a contradiction that $h \notin Q$. Since $hg = f \in Q$ and Q is primary, we conclude that $g^i \in Q$ for some $i \in \mathbb{N}$. But then $g^i \in Q \cap B \subseteq \mathfrak{p}$, so also $g \in \mathfrak{p}$, which contradicts $g \in S = B \setminus \mathfrak{p}$.

For the second claim, let $q \in (Q \cap B) \setminus \mathfrak{p} \subseteq S$. Then $1 = \frac{q}{q} \in S^{-1}Q$, which implies the claim. □

The following proposition will be the basis of a stratification associated to Bernstein-Sato ideals.

Proposition 2.25 ([BO10])**.** For any $p \in \mathbb{C}^n$,

$$\mathcal{B}_p = \bigcap_{i : p \in \mathbb{V}(I_i)} \mathcal{B}_i.$$

Proof. We have

$$\mathcal{B}_p = (S_p^{-1}Q) \cap \mathbb{C}[\underline{s}] = \left(S_p^{-1} \bigcap_{i=1}^{\ell} Q_i \right) \cap \mathbb{C}[\underline{s}] = \left(\bigcap_{i=1}^{\ell} S_p^{-1} Q_i \right) \cap \mathbb{C}[\underline{s}]$$

and

$$S_p^{-1}Q_i = S_p^{-1}\mathbb{C}[\underline{x}, \underline{s}] \Leftrightarrow S_p^{-1} I_i = \mathbb{C}[\underline{x}]_p \Leftrightarrow p \notin \mathbb{V}(I_i),$$

which implies

$$\mathcal{B}_p = \left(\bigcap_{i : p \in \mathbb{V}(I_i)} S_p^{-1} Q_i \right) \cap \mathbb{C}[\underline{s}].$$

From Lemma 2.24 with $A = \mathbb{C}[\underline{x}, \underline{s}]$, $B = \mathbb{C}[\underline{x}]$, $C = \mathbb{C}[\underline{s}]$, $Q = Q_i$ and $\mathfrak{p} = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$ it follows that $(S_p^{-1}Q_i) \cap \mathbb{C}[\underline{s}] = Q_i \cap \mathbb{C}[\underline{s}]$.

Thus, the generation over the ring $\mathbb{C}[\underline{x}]_p$ does not contribute anything to $\mathcal{B}_p$ and we conclude

$$\mathcal{B}_p = \left(\bigcap_{i : p \in \mathbb{V}(I_i)} Q_i \right) \cap \mathbb{C}[\underline{s}] = \bigcap_{i : p \in \mathbb{V}(I_i)} \mathcal{B}_i.$$

□

Remark 2.26. In fact, Proposition 2.25 gives another proof of Proposition 2.10 ($\mathcal{B} = \bigcap_{p \in \mathbb{C}^n} \mathcal{B}_p$). It follows directly from the former proposition that $\mathcal{B} \subseteq \mathcal{B}_p$ for all $p \in \mathbb{C}^n$ and thus $\mathcal{B} \subseteq \bigcap_{p \in \mathbb{C}^n} \mathcal{B}_p$. On the other hand by Hilbert's Nullstellensatz we can find $p_i \in \mathbb{V}(I_i)$ for all $1 \leq i \leq \ell$ and for these p_i it holds that $\mathcal{B}_{p_i} \subseteq \mathcal{B}_i$, which implies $\mathcal{B} \supseteq \bigcap_i \mathcal{B}_{p_i}$. If, on the other hand, $p \notin \mathbb{V}(I_i)$ for all i, this implies $\mathcal{B}_p = \langle 1 \rangle$ or equivalently $p \notin \mathbb{V}(f)$.

Proposition 2.25 induces the following partition of $\mathbb{C}^n$.

Theorem 2.27 ([BO10])**.** For $J \subseteq \{1, \dots, \ell\}$ we set

$$W_J = \left(\bigcap_{j\in J} \mathbb{V}(I_j)\right) \setminus \left(\bigcup_{j\notin J} \mathbb{V}(I_j)\right).$$

Then, $\{W_J \mid J \subseteq \{1, \dots, \ell\}\}$ is a partition of $\mathbb{C}^n$ and $\mathcal{B}_p = \mathcal{B}_q$ for all $p, q \in W_J$.

Proof. The claim follows from Proposition 2.25. In particular, $W_\varnothing = \mathbb{C}^n \setminus \bigcup_{J\neq\varnothing} W_J$ by the definition of the W_J and $W_{J_1} \cap W_{J_2} = \varnothing$ for $J_1 \neq J_2$, which shows that the W_J define a partition. □

The W_J have a structure that can be described by the following definition.

Definition 2.28 ([Gor76])**.** A finite *stratification* of a closed subset M of a topological space is a decomposition

$$M = \bigcup_{j\in J} W_j$$

with a finite index set J and $W_j \subseteq M$ which fulfill the following conditions:

- All W_j are *locally closed*, i.e. $W_j = U \cap A$ for an open set U and a closed set A.
- The W_j are *pairwise disjoint*, i.e. for $j \neq i$ it holds that $W_i \cap W_j = \varnothing$.
- For all $j \neq i$ the *condition of the frontier* holds: If $W_i \cap \overline{W_j} \neq \varnothing$, then $W_i \subseteq \overline{W_j}$. Here $\overline{\cdot}$ denotes the closure with respect to the topological space.

The W_j are called *strata*.

Remark 2.29. We will only work with $\mathbb{C}^n$, the affine space of dimension n over the complex numbers, and the Zariski topology (compare e.g. [Har77]).

If for all $j \in J$ a map $P(\cdot)$ is constant on W_j, i.e. $|\{P(x) \mid x \in W_j\}| = 1$, we call the stratification a stratification with respect to P.

Lemma 2.30. The set $\{W_J \mid J \subseteq \{1, \dots, \ell\}\}$ with the W_J from Theorem 2.27 defines a finite stratification of $\mathbb{V}(F)$ with respect to the local Bernstein-Sato ideal. Here, $\mathcal{B}_p$ is regarded as a mapping of p,

$$\mathcal{B}_\cdot : \ \mathbb{C}^n \to \{I \subseteq \mathbb{C}[\underline{s}] \mid I \text{ ideal}\}\,; \ p \mapsto \mathcal{B}_p.$$

Proof. Obviously, $\bigcup_J W_J = \mathbb{C}^n$, since $W_\varnothing = \mathbb{C}^n \setminus \bigcup_{J\neq\varnothing} W_J$.

As a set difference of two finite intersections of Zariski-closed sets, the W_J are locally closed.

Let $J_1 \neq J_2$, e.g. $i \in J_1 \setminus J_2$. Then $W_{J_1} \subseteq \mathbb{V}(I_i)$ and $\mathbb{V}(I_i) \cap W_{J_2} = \varnothing$, so the W_J are pairwise disjoint.

Now let $J_1 \neq J_2$ such that $W_{J_1} \cap \overline{W_{J_2}} \neq \varnothing$. By the irreducibility of the $\mathbb{V}(I_i)$ (as varieties of primary ideals) and the properties of the Zariski topology (compare e.g. [Har77]), we conclude that

$$\overline{W_{J_2}} = \overline{\left(\bigcap_{j\in J_2} \mathbb{V}(I_j)\right) \setminus \left(\bigcup_{j\notin J_2} \mathbb{V}(I_j)\right)} = \bigcap_{j\in J_2} \mathbb{V}(I_j).$$

Together with

$$W_{J_1} = \left(\bigcap_{j\in J_1} \mathbb{V}(I_j)\right) \setminus \left(\bigcup_{j\notin J_1} \mathbb{V}(I_j)\right)$$

we conclude that $J_2 \supseteq J_1$ and thus $W_{J_1} \subseteq \overline{W_{J_2}}$.

It follows that $\{W_J \mid J \subseteq \{1,\dots,r\}\}$ defines a stratification. Theorem 2.27 shows that the stratification is indeed a stratification with respect to local Bernstein-Sato ideals. □

Example 2.31. Consider $f = (x^2 - y, y^2) \in \mathbb{C}[x,y]^2$. We obtain the following primary components of the primary decomposition of $(\mathrm{ann}(f^s) + \langle F\rangle) \cap \mathbb{C}[\underline{x},\underline{s}]$:

Q_i	$\mathcal{B}_i = Q_i \cap \mathbb{C}[\underline{s}]$	$I_i = Q_i \cap \mathbb{C}[\underline{x}]$
$Q_1 = \langle s_1+1, x^2-y\rangle$	$\mathcal{B}_1 = \langle s_1+1\rangle$	$I_1 = \langle x^2-y\rangle$
$Q_2 = \langle s_2+1, y^2\rangle$	$\mathcal{B}_2 = \langle s_2+1\rangle$	$I_2 = \langle y^2\rangle$
$Q_3 = \langle 2s_2+1, y\rangle$	$\mathcal{B}_3 = \langle 2s_2+1\rangle$	$I_3 = \langle y\rangle$
$Q_4 = \langle 2s_1+4s_2+5, y^2, xy, 4x^2s_2+2x^2+y, x^3\rangle$	$\mathcal{B}_4 = \langle 2s_1+4s_2+5\rangle$	$I_4 = \langle y^2, xy, x^3\rangle$
$Q_5 = \langle 2s_1+4s_2+3, y, x\rangle$	$\mathcal{B}_5 = \langle 2s_1+4s_2+3\rangle$	$I_5 = \langle y, x\rangle$
$Q_6 = \langle 2s_1+4s_2+7, y^3, xy^2, 4x^2ys_2+4x^2y+y^2, 4x^3s_2+2x^3+3xy, x^3y, x^5\rangle$	$\mathcal{B}_6 = \langle 2s_1+4s_2+7\rangle$	$I_6 = \langle y^3, xy^2, x^3y, x^5\rangle$

We remark several points about the possible structure of the primary components: There may be primary components that are not prime, see e.g. Q_2 with radical $\sqrt{Q_2} = \langle s_2+1, y\rangle \neq Q_2$. Here, $\sqrt{I_2} \neq I_2$. There are also other examples in which $\mathcal{B}_i \neq \sqrt{\mathcal{B}_i}$. We may also find $i \neq j$ with $\sqrt{I_i} = \sqrt{I_j}$ (and consequently $\mathbb{V}(I_i) = \mathbb{V}(I_j)$), but $\sqrt{\mathcal{B}_i} \neq \sqrt{\mathcal{B}_j}$, see e.g. I_2, I_3 or I_4, I_5, I_6. There may be cases in which $Q_i \neq \mathbb{C}[\underline{x},\underline{s}]\mathcal{B}_i + \mathbb{C}[\underline{x},\underline{s}]I_i$, see e.g. $i = 4$.

The stratification obtained consists of the strata $\mathbb{C} \setminus \mathbb{V}(F)$, $\mathbb{V}(x^2-y) \setminus \{(0,0)\}$, $\mathbb{V}(y) \setminus \{(0,0)\}$ and $\{(0,0)\}$.

Remark 2.32. Absolutely analogously, we can construct stratifications with respect to the local Bernstein-Sato ideals $\mathcal{B}_{\Sigma,p}$, $\mathcal{B}_{(i),p}$ by computing primary decompositions of the ideals

$$Q_\Sigma := (\mathrm{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f_1,\dots,f_r\rangle) \cap \mathbb{C}[\underline{x},\underline{s}]$$

(here, we allow the trivial decomposition $Q_\Sigma = \mathbb{C}[\underline{x},\underline{s}]$ with one component) and

$$Q_{(i)} := (\mathrm{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f_i\rangle) \cap \mathbb{C}[\underline{x},\underline{s}].$$

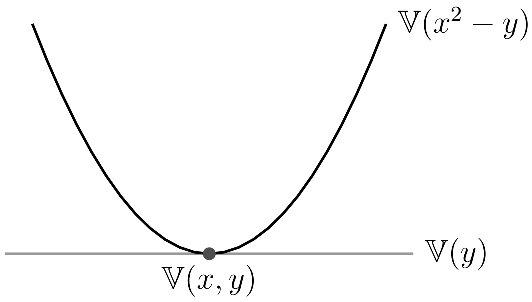

Figure 2.1.: The stratification from Example 2.31 and Example 2.33.

Example 2.33. Consider again $f = (x^2 - y, y^2) \in \mathbb{C}[x,y]^2$. The primary components obtained for $\mathcal{B}_\Sigma$ are:

Q_i	$\mathcal{B}_i = Q_i \cap \mathbb{C}[\underline{s}]$	$I_i = Q_i \cap \mathbb{C}[\underline{x}]$
$Q_1 = \langle 2s_1 + 4s_2 + 3, x, y\rangle$	$\mathcal{B}_1 = \langle 2s_1 + 4s_2 + 3\rangle$	$I_1 = \langle x, y\rangle$
$Q_2 = \langle 2s_2 + 1, s_1 + 1, x^2, y\rangle$	$\mathcal{B}_2 = \langle 2s_2 + 1, s_1 + 1\rangle$	$I_2 = \langle x^2, y\rangle$
$Q_3 = \langle s_2 + 1, s_1 + 1, y^2, x^2 - y\rangle$	$\mathcal{B}_3 = \langle s_2 + 1, s_1 + 1\rangle$	$I_3 = \langle y^2, x^2 - y\rangle$
$Q_4 = \langle 4s_2 + 3, s_1 + 1, y^2, xy, x^2 - y\rangle$	$\mathcal{B}_4 = \langle 4s_2 + 3, s_1 + 1\rangle$	$I_4 = \langle y^2, xy, x^2 - y\rangle$

Although these primary components differ a lot from those obtained for $\mathcal{B}$ in Example 2.31, we remark that we can find the inclusion $\mathcal{B} \subseteq \mathcal{B}_\Sigma$ in the primary components. Yet, we do not have a relation of the form $\mathcal{B}_\Sigma \mid \mathcal{B}$, i.e. we cannot write $\mathcal{B}$ in the form $\mathcal{B} = \mathcal{B}_\Sigma I$ for some ideal I. We notice that the strata obtained are those from Example 2.31.

The primary components obtained for $\mathcal{B}_{(1)}$ are:

Q_i	$\mathcal{B}_i = Q_i \cap \mathbb{C}[\underline{s}]$	$I_i = Q_i \cap \mathbb{C}[\underline{x}]$
$Q_1 = \langle s_1 + 1, x^2 - y\rangle$	$\mathcal{B}_1 = \langle s_1 + 1\rangle$	$I_1 = \langle x^2 - y\rangle$
$Q_2 = \langle 2s_1 + 4s_2 + 3, x, y\rangle$	$\mathcal{B}_2 = \langle 2s_1 + 4s_2 + 3\rangle$	$I_2 = \langle x, y\rangle$

For $\mathcal{B}_{(2)}$ we get:

Q_i	$\mathcal{B}_i = Q_i \cap \mathbb{C}[\underline{s}]$	$I_i = Q_i \cap \mathbb{C}[\underline{x}]$
$Q_1 = \langle s_2 + 1, y^2\rangle$	$\mathcal{B}_1 = \langle s_2 + 1\rangle$	$I_1 = \langle y^2\rangle$
$Q_2 = \langle 2s_2 + 1, y\rangle$	$\mathcal{B}_2 = \langle 2s_2 + 1\rangle$	$I_2 = \langle y\rangle$
$Q_3 = \langle 2s_1 + 4s_2 + 5, y^2, xy, 4x^2s(2) + 2x^2 + y, x^3\rangle$	$\mathcal{B}_3 = \langle 2s_1 + 4s_2 + 5\rangle$	$I_3 = \langle x^3, xy, y^2\rangle$
$Q_3 = \langle 2s_1 + 4s_2 + 3, y, x\rangle$	$\mathcal{B}_3 = \langle 2s_1 + 4s_2 + 3\rangle$	$I_3 = \langle x, y\rangle$

We remark that $\mathcal{B} \subseteq \mathcal{B}_{(i)} \subseteq \mathcal{B}_\Sigma$. In contrast to $\mathcal{B}_\Sigma$, the $\mathcal{B}_{(i)}$ are principal in this example. The corresponding strata differ from the ones from Example 2.33, since for the $\mathcal{B}_{(i)}$ one of the components plays a more important role than the other one.

We can give an algorithm for determining the primary components $\mathcal{B}_i, I_i$ by modifying Algorithm 2.18 in the appropriate places. We need three additional elimination orderings because we are now also interested in intersections with $\mathbb{C}[\underline{x}, \underline{s}]$ and $\mathbb{C}[\underline{x}]$.

Definition 2.34. We define $<_{x,s}$ to be an elimination ordering on $D_n[\underline{s}]$ with respect to the ∂_i, i.e. $x_i, s_j <_{x,s} \partial_k$ for all $i, k \in \{1, \ldots, n\}$, $j \in \{1, \ldots, r\}$.

By $<_s$ and $<_x$ we denote elimination orderings on $\mathbb{C}[\underline{x}, \underline{s}]$ with respect to the s_i and x_i, respectively, i.e. $x_i <_s s_j$, $s_j <_x x_i$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, r\}$.

For the computation of primary decompositions (which is also algorithmic in $\mathbb{C}[\underline{x}, \underline{s}]$), we refer to [DGP99].

Now, we can adapt Algorithm 2.18 to our requirements.

Algorithm 2.35 (see A.1)**.**
Input: $f \in \mathbb{C}[\underline{x}]^r$.
Output: compatible primary decompositions of $(\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle f^s \rangle) \cap R$ for $R \in \{\mathbb{C}[\underline{x}, \underline{s}], \mathbb{C}[\underline{x}], \mathbb{C}[\underline{s}]\}$.

Step 1-4: as in Algorithm 2.18; set $J := \mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F \rangle$.
5: Compute a Gröbner basis G of J with respect to $<_{x,s}$.
6: Set $Q := {}_{\mathbb{C}[\underline{x},\underline{s}]}\langle g \in G \cap \mathbb{C}[\underline{x}, \underline{s}] \rangle$. $\triangleright$ $Q = (\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle f \rangle) \cap \mathbb{C}[\underline{x}, \underline{s}]$.
7: Determine a primary decomposition $Q = \bigcap_{i=1}^{\ell} Q_i$.
8: Compute Gröbner bases $H_{x,i}$ and $H_{s,i}$ of Q_i with respect to $<_s$ and $<_x$, respectively, for $i = 1, \ldots, \ell$.
9: Set $\mathcal{B}_i := {}_{\mathbb{C}[\underline{s}]}\langle h \in H_{s,i} \cap \mathbb{C}[\underline{s}] \rangle$, $I_i := {}_{\mathbb{C}[\underline{x}]}\langle h \in H_{x,i} \cap \mathbb{C}[\underline{x}] \rangle$ for all $1 \leq i \leq \ell$.
10: **return** $(Q_1, \ldots, Q_\ell)$, $(\mathcal{B}_1, \ldots, \mathcal{B}_\ell)$ and $(I_1, \ldots, I_\ell)$.

Remark 2.36. The algorithm relies on the elimination orders and the algorithm for primary decomposition. A typical bottleneck of computational complexity is the computation of a primary decomposition.

Now, we examine the Krull dimension of the primary components through which we obtained a stratification. As we work with ideals in $\mathbb{C}[\underline{x}, \underline{s}]$, $\mathbb{C}[\underline{x}]$ and $\mathbb{C}[\underline{s}]$, respectively, we can work with the Krull dimension of ideals in a commutative ring. A possible expectation that we could have is equidimensionality of the primary components, but this does not hold in general, neither for the Q_i nor for the $\mathcal{B}_i$ nor for the I_i, as we can see in the following example.

Example 2.37. Consider the example $f = (z, x^4 + y^4 + 2zx^2y^2) \in \mathbb{C}[x, y, z]^2 =: \mathbb{C}[\underline{x}]^2$, taken from [BO10]. In the primary decomposition of $Q = (\mathrm{ann}(f^s) + \langle F \rangle) \cap \mathbb{C}[\underline{x}, \underline{s}]$, we find primary components $\langle s_1 + 1, z \rangle \subseteq \mathbb{C}[\underline{x}, \underline{s}]$ of Krull dimension 3 and $\langle 2s_2 + 1, y, x \rangle$ of Krull dimension 2. After intersecting with $\mathbb{C}[\underline{x}]$, two components are $\langle z \rangle \subseteq \mathbb{C}[\underline{x}]$ of dimension 2 and $\langle x, y \rangle$ of dimension 1. After intersecting with $\mathbb{C}[\underline{s}]$, we find that two components are $\langle s_1 + 1 \rangle \subseteq \mathbb{C}[\underline{s}]$ of dimension 1 and $\langle 2s_2 + 3, s_1 + 2 \rangle$ of dimension 0.

We conclude that none of the three primary decompositions are equidimensional. On the levels $\mathbb{C}[\underline{x}]$ and $\mathbb{C}[\underline{s}]$ we can find intuitive reasons for this. Both principal and non-principal components appear in the $\mathcal{B}_i$ which implies a difference of dimensions. Furthermore, some of the I_i describe $\mathbb{V}(f)$ whereas others describe the singular locus $\mathrm{Sing}(f)$ (cf. Definition 3.22) which has a strictly smaller dimension than $\mathbb{V}(f)$.

2.3. Localizations at prime ideals

So far we have only dealt with localizations at maximal ideals with respect to multiplicatively closed denominator set

$$S_p = \{f \in \mathbb{C}[\underline{x}] \mid f(p) \neq 0\} = \mathbb{C}[\underline{x}] \setminus \langle x_1 - p_1, \dots, x_n - p_n\rangle =: \mathbb{C}[\underline{x}] \setminus \mathfrak{m}_p.$$

However, we can localize at $S_\mathfrak{p} := \mathbb{C}[\underline{x}] \setminus \mathfrak{p}$ for any prime ideal $\mathfrak{p}$. Similarly as in the definition of $D_{n,p}$ we can define $D_{n,\mathfrak{p}} := S_\mathfrak{p}^{-1} D_n \subseteq W_n$, since $S_\mathfrak{p}$ is an Ore set in D_n as well. We are interested in prime ideals with $f \in \mathfrak{p}$ in particular, because these correspond to irreducible components of a decomposition of $\mathbb{V}(F)$ into varieties.

Similarly as in the case of local Bernstein-Sato ideals at a point, we can define those at a prime ideal or at the corresponding variety.

Definition 2.38. The local Bernstein-Sato ideal of f at the prime ideal $\mathfrak{p} \subseteq \mathbb{C}[\underline{x}]$ or at the corresponding variety $\mathbb{V}(\mathfrak{p})$ is defined as

$$\mathcal{B}_\mathfrak{p} = (\operatorname{ann}_{D_{n,\mathfrak{p}}[\underline{s}]}(f^s) + {}_{D_{n,\mathfrak{p}}[\underline{s}]}\langle F\rangle) \cap \mathbb{C}[\underline{s}].$$

Remark 2.39. Applying Lemma 2.8, we conclude that

$$\mathcal{B}_\mathfrak{p} = (S_\mathfrak{p}^{-1}(\operatorname{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F\rangle)) \cap \mathbb{C}[\underline{s}].$$

The following lemma can be seen as a variant of Proposition 2.25, since it reduces the computation of local Bernstein-Sato ideals to primary decompositions and the computation of global Bernstein-Sato ideals. For notations compare Section 2.2.

Lemma 2.40. For a prime ideal $\mathfrak{p} \subseteq \mathbb{C}[\underline{x}]$,

$$\mathcal{B}_\mathfrak{p} = \bigcap_{i:\mathfrak{p} \supseteq \sqrt{I_i}} \mathcal{B}_i.$$

Proof. For the primary ideals I_i and the prime ideal $\mathfrak{p}$ the following equivalence holds:

$$I_i \subseteq \mathfrak{p} \quad \Longleftrightarrow \quad \sqrt{I_i} \subseteq \mathfrak{p}.$$

Now the claim follows completely analogously as in the proof of Proposition 2.25. $\square$

A conclusion from Proposition 2.10 and the fact that $S_\mathfrak{p} = \mathbb{C}[\underline{x}] \setminus \mathfrak{p} \supseteq \mathbb{C}[\underline{x}] \setminus \mathfrak{m} =: S_\mathfrak{m}$ for all prime ideals $\mathfrak{p}$ and maximal ideals $\mathfrak{m}$ with $\mathfrak{p} \subseteq \mathfrak{m}$ is that

$$\mathcal{B} = \bigcap_{\mathfrak{p}:F\in\mathfrak{p}} \mathcal{B}_\mathfrak{p}.$$

Moreover, we can now show the following corollary.

Corollary 2.41. It holds that

$$\mathcal{B}_\mathfrak{p} = \bigcap_{p\in\mathbb{V}(\mathfrak{p})} \mathcal{B}_p.$$

Proof. The claim follows from Proposition 2.25 and Lemma 2.40, since both use the same primary decompositions and it holds that

$$\langle x_1 - p_1, \dots, x_n - p_n \rangle \supseteq \mathfrak{p} \supseteq \sqrt{I_i} \quad \Longleftrightarrow \quad p \in \mathbb{V}(\mathfrak{p}) \subseteq \mathbb{V}(I_i),$$

which implies that

$$\mathcal{B}_{\mathfrak{p}} = \bigcap_{i: \sqrt{I_i} \subseteq \mathfrak{p}} \mathcal{B}_i = \bigcap_{p \in \mathbb{V}(\mathfrak{p})} \mathcal{B}_p.$$

□

This corollary gives us an interpretation of the localized Bernstein-Sato ideals at prime ideals that we can use in the following example.

Example 2.42. Consider again $f = (x, y, x+1) \in \mathbb{C}[x,y]^3$ and the prime (but not maximal) ideals $\mathfrak{p} = \langle y \rangle$ and $\mathfrak{q} = \langle x \rangle$. We obtain

$$\mathcal{B}_{\mathfrak{p}} = \langle (s_1+1)(s_2+1)(s_3+1) \rangle,$$
$$\mathcal{B}_{\mathfrak{q}} = \langle (s_1+1)(s_2+1) \rangle.$$

2.4. Stratifications with respect to local Bernstein-Sato polynomials

For Bernstein-Sato polynomials, more effective algorithms for stratifications are known which do not rely on primary decompositions. Here, we want to examine the approach of [LM12] (a similar approach has been used in [NN10]). In this subsection, let $f \in \mathbb{C}[\underline{x}]$, i.e. $r = 1$.

The general procedure here is to find an upper bound of the local Bernstein-Sato polynomial $b_{f,p}$, e.g. the global Bernstein-Sato polynomial b_f (see Proposition 2.10), factorize it and then to check whether the roots of the upper bound are roots of the local Bernstein-Sato polynomial as well and, if so, which multiplicity these roots have. In practice, this method is more effective because the typical bottleneck here, finding the roots of the upper bound, is oftentimes less expensive than the computation of a primary decomposition of $(\operatorname{ann}(f^s) + \langle f \rangle) \cap \mathbb{C}[\underline{x}, s]$.

The practical approach is due to the following theorem.

Theorem 2.43 ([LM12, 2.1])**.** Let $q(s) \in \mathbb{C}[s]$, R be a $\mathbb{C}$-algebra whose center contains $\mathbb{C}[s]$ (e.g. $R \in \{D_n[s], D_{n,p}[s]\}$), and I a left ideal in R with $I \cap \mathbb{C}[s] \neq \{0\}$. It holds that $(I + {}_R\langle q(s) \rangle) \cap \mathbb{C}[s] = I \cap \mathbb{C}[s] + {}_{\mathbb{C}[s]}\langle q(s) \rangle$.

Proof. Obviously, '$\supseteq$' holds. For '$\subseteq$', let $I \cap \mathbb{C}[s] = {}_{\mathbb{C}[s]}\langle b(s) \rangle$ and $f + gq(s) \in (I + {}_R\langle q(s) \rangle) \cap \mathbb{C}[s]$ with $f \in I$, $g \in R$. Multiplying $f + gq(s)$ by $d := \frac{b(s)}{\gcd(b(s),q(s))} \in \mathbb{C}[s]$, we get

$$d \cdot (f + gq(s)) = df + g \underbrace{dq}_{\in I \cap \mathbb{C}[s] = \langle b(s) \rangle} \in I \cap \mathbb{C}[s] = \langle b(s) \rangle.$$

This implies $f + gq(s) \in \langle \frac{b(s)}{d} \rangle = \langle \gcd(b(s), q(s)) \rangle = I \cap \mathbb{C}[s] + {}_{\mathbb{C}[s]}\langle q(s) \rangle$. □

Remark 2.44. Although this result is very intuitive, it is unclear whether it can be extended to the multivariate case for $\mathbb{C}[\underline{s}]$, since the proof relies heavily on the work over a principal ideal domain. At least in this generality, a counterexample is given by $I = {}_{D_n[s_1,s_2]}\langle \partial_1 s_1 + 1, s_2\rangle$, $q(s) = s_1$, where

$$(I + {}_{D_n}\langle q(s)\rangle) \cap \mathbb{C}[\underline{s}] = \mathbb{C}[\underline{s}] \neq {}_{\mathbb{C}[\underline{s}]}\langle s_1, s_2\rangle = I \cap \mathbb{C}[\underline{s}] + {}_{\mathbb{C}[\underline{s}]}\langle q(s)\rangle.$$

Applying Theorem 2.43 to $I = \operatorname{ann}_R(f^s) + {}_R\langle f\rangle$ and a factor $q(s)$ of $b_f(s)$ yields the following corollary.

Corollary 2.45 ([LM12, 2.4])**.** Let $\alpha \in \mathbb{C}$ be a root of $b_f(-s)$ of multiplicity m_α, $0 \leq i \leq n$. The following are equivalent:

(i) $m_\alpha > i$.

(ii) $J_i := (\operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f, (s+\alpha)^{i+1}\rangle) \cap \mathbb{C}[s] = {}_{\mathbb{C}[s]}\langle (s+\alpha)^{i+1}\rangle$.

(iii) $(s+\alpha)^i \notin \operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f, (s+\alpha)^{i+1}\rangle$.

Proof. Rewriting (ii) with Theorem 2.43 ($I = \operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f\rangle$, $q = (s+\alpha)^i$), this equality now reads

$${}_{\mathbb{C}[s]}\langle (s+\alpha)^{i+1}\rangle = (\operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f, (s+\alpha)^{i+1}\rangle) \cap \mathbb{C}[s] = {}_{\mathbb{C}[s]}\langle b_f(s)\rangle + {}_{\mathbb{C}[s]}\langle (s+\alpha)^{i+1}\rangle,$$

which is obviously equivalent to (i).

We may reformulate the condition in (iii) with Theorem 2.43 as

$$(s+\alpha)^i \notin (\operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f, (s+\alpha)^{i+1}\rangle) \cap \mathbb{C}[s] = {}_{\mathbb{C}[s]}\langle b_f(s)\rangle + {}_{\mathbb{C}[s]}\langle (s+\alpha)^{i+1}\rangle,$$

which directly implies '(i)⇒(iii)', since $(s+\alpha)^{i+1}$ divides the right hand side of this reformulation if (i) holds. The implication '(iii)⇒(ii)' can be concluded from the fact that the generator of J_i must have the form $(s+\alpha)^j$ for $j \leq i+1$ and (iii) implying $j = i+1$, i.e. (ii). ☐

These results allow us to algorithmically check for candidate roots of the Bernstein-Sato polynomial and their multiplicity by means of Gröbner bases.

The following theorem hints toward a stratification, as it determines vanishing sets on which the local Bernstein-Sato polynomial does not vary.

Theorem 2.46 ([LM12, 2.14])**.** Let $L := \operatorname{ann}_{D_n[s]}(f^s) + {}_{D_n[s]}\langle f\rangle$ and α be a root of $b_f(-s)$ of multiplicity m_α. Denote by $m_{\alpha,p}$ the multiplicity of α as a root of $b_{f,p}(-s)$.

For $1 \leq i < m_\alpha$ we define $I_{\alpha,i} := (L : (s+\alpha)^i) + {}_{D_n[s]}\langle s+\alpha\rangle$. It holds that

- $(s+\alpha) \mid b_{f,p}(s) \Leftrightarrow p \in \mathbb{V}((L + {}_{D_n[s]}\langle s+\alpha\rangle) \cap \mathbb{C}[\underline{x}])$,
- $m_{\alpha,p} > i \Leftrightarrow p \in \mathbb{V}(I_{\alpha,i} \cap \mathbb{C}[\underline{x}])$.

Proof. By Corollary 2.9, we know that $\langle b_{f,p}(s)\rangle = (S_p^{-1}L)\cap \mathbb{C}[\underline{s}]$.

Now for the first claim.

For $p\in\mathbb{C}^n$ it holds that $(s+\alpha)\nmid b_{f,p}(s)$ if and only if $\gcd(s+\alpha, b_{f,p}(s)) = 1$ or equivalently

$$1 = \gcd(s+\alpha, b_{f,p}(s)) \in \left(S_p^{-1}L + {}_{D_{n,p}[s]}\langle s+\alpha\rangle\right).$$

In this case we can equivalently state that $1\in\left(S_p^{-1}L + {}_{D_{n,p}[s]}\langle s+\alpha\rangle\right)\cap\mathbb{C}[\underline{x}]$, which is equivalent to

$$\left(L + {}_{D_n[s]}\langle s+\alpha\rangle\right)\cap S_p \neq \{0\}$$

by choosing a common denominator. This is equivalent to $p\notin\mathbb{V}\left(L + {}_{D_n[s]}\langle s+\alpha\rangle\right)$, which shows the first claim.

For the second claim, we proceed analogously. We have $m_{\alpha,p} > i$ if and only if $\gcd(b_{f,p} : (s+\alpha)^i, s+\alpha)\neq 1$. With analogous steps as in the proof of the first claim, this holds if and only if $1\notin S_p^{-1}I_{\alpha,i}$ or, equivalently, $S_p\cap I_{\alpha,i} = \varnothing$ or $p\in\mathbb{V}(I_{\alpha,i}\cap\mathbb{C}[\underline{x}])$. □

Based on the varieties $V_{\alpha,i} := \mathbb{V}(I_{\alpha,i})$, we obtain a stratification with respect to Bernstein-Sato polynomials.

Corollary 2.47 ([LM12, 2.14])**.** Let $b_f(s) = \prod_{i=1}^{\ell}(s-\alpha_i)^{m_{\alpha_i}}$. We set $I_{\alpha,k} := \varnothing$ for all roots α and $k > m_\alpha$. Then

$$\mathbb{C}^n = \bigcup_{j_{\alpha_1}=0}^{m_{\alpha_1}-1}\cdots\bigcup_{j_{\alpha_\ell}=0}^{m_{\alpha_\ell}-1}\underbrace{\left(\bigcap_{1\leq i\leq\ell}\mathbb{V}(I_{\alpha_i,j_{\alpha_i}})\setminus\mathbb{V}(I_{\alpha_i,j_{\alpha_i}+1})\right)}_{=:W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})}},$$

and the $W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})}$ fulfill the first two conditions of Definition 2.28 of a stratification with respect to local Bernstein-Sato polynomials $b_{f,p}(\cdot)$.

Proof. Obviously, $\bigcup_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})}W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})} = \mathbb{C}^n$, since

$$W_{(0,\ldots,0)} = \mathbb{C}^n\setminus\bigcup_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})\neq(0,\ldots,0)}W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})}.$$

As finite intersection of set differences of two Zariski-closed sets, the $W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})}$ are locally closed.

Let $(j_{\alpha_1},\ldots,j_{\alpha_\ell})\neq(k_{\alpha_1},\ldots,k_{\alpha_\ell})$, e.g. $j_{\alpha_1} < k_{\alpha_1}$. Then $W_{(k_{\alpha_1},\ldots,k_{\alpha_\ell})}\subseteq\mathbb{V}(I_{\alpha_1,k_{\alpha_1}})$ and $\mathbb{V}(I_{\alpha_1,k_{\alpha_1}})\cap W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})} = \varnothing$, so the $W_{(j_{\alpha_1},\ldots,j_{\alpha_\ell})}$ are pairwise disjoint. □

We do not consider the condition of frontier here because it is much more intertwined with the properties of the Bernstein-Sato polynomial and those of the singular locus, but does not contribute to the properties we are actually interested in.

2.5. Bernstein-Sato polynomials of varieties

Let again $f \in \mathbb{C}[\underline{x}]^r$, $F = \prod_{i=1}^r f_i$ and $f^s = \prod_{i=1}^r f_i^{s_i}$. At this point, we consider another generalization of the Bernstein-Sato polynomial introduced in [BMS06] which preserves the principality of the ideals associated to polynomials.

We need some preparations in order to define this generalization.

Definition 2.48 ([ALM09])**.** We define

$$\mathbb{C}\langle S\rangle := \mathbb{C}\langle s_{i,j} \mid i,j \in \{1,\dots,r\}\,, [s_{i,j}, s_{k,l}] = \delta_{j,k}s_{i,l} - \delta_{i,l}s_{k,j}\rangle$$

and

$$D\langle S\rangle := D_n\langle S\rangle := D_n \otimes_{\mathbb{C}} \mathbb{C}\langle S\rangle.$$

Remark 2.49 ([ALM09, BMS06])**.** The ring $\mathbb{C}\langle S\rangle$ is the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.

It can also be obtained as a subring of

$$\mathbb{C}\langle s_i, t_i, t_i^{-1} \mid t_j s_i = (s_i + \delta_{i,j})t_j, s_i s_j = s_j s_i, t_i t_j = t_j t_i \text{ for } i,j \in \{1,\dots,r\}\rangle$$

generated by $s_{i,j} := s_i t_i^{-1} t_j$ for $i,j \in \{1,\dots,r\}$. With this construction,

$$\begin{aligned}
s_{i,j}\cdot s_{k,l} - s_{k,l}\cdot s_{i,j} &= s_i t_i^{-1} t_j s_k t_k^{-1} t_l - s_k t_k^{-1} t_l s_i t_i^{-1} t_j \\
&= s_i t_i^{-1}(s_k + \delta_{k,j}) t_j t_k^{-1} t_l - s_k t_k^{-1}(s_i + \delta_{i,l}) t_l t_i^{-1} t_j \\
&= s_i(s_k + \delta_{k,j} - \delta_{i,k}) t_i^{-1} t_j t_k^{-1} t_l - s_k(s_i + \delta_{i,l} - \delta_{i,k}) t_k^{-1} t_l t_i^{-1} t_j \\
&= \delta_{k,j} s_i t_i^{-1} t_j t_k^{-1} t_l - \delta_{i,l} s_k t_k^{-1} t_l t_i^{-1} t_j \\
&= \delta_{k,j} s_i t_i^{-1} t_l - \delta_{i,l} s_k t_k^{-1} t_j = \delta_{j,k} s_{i,l} - \delta_{i,l} s_{k,j},
\end{aligned}$$

as desired.

Now we want to extend the D-module structure of $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]f^s$ to the structure of a $D\langle S\rangle$-module.

Definition 2.50 ([ALM09])**.** The D-module $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]f^s$ becomes a $D\langle S\rangle$-module with the application of differential operators and

$$s_{i,j} \bullet \underbrace{g(s)}_{\in\mathbb{C}[\underline{x},\underline{s},\frac{1}{F}]} f^s := s_i g(t_i^{-1} t_j \bullet s)\frac{f_j}{f_i} f^s,$$

for $i,j \in \{1,\dots,r\}$, where $t_i \bullet s_j = s_j + \delta_{i,j}$ for $i,j \in \{1\dots r\}$.

Remark 2.51 ([ALM09])**.** The module action of $s_{i,i}$ is given by

$$s_{i,i} \bullet \underbrace{g(s)}_{\in\mathbb{C}[\underline{x},\underline{s},\frac{1}{F}]} f^s = s_i g(t_i^{-1} t_i \bullet s)\frac{f_i}{f_i} f^s = s_i g(s) f^s,$$

so it coincides with the multiplication with s_i and we can identify $s_{i,i}$ and s_i by formally working over $D_n\langle S\rangle/\langle s_{i,i} - s_i \mid i \in \{1, \ldots, r\}\rangle$.

Using the construction from Remark 2.49 with $t_i s_j = (s_j + \delta_{i,j})t_j$ and $t_i \bullet s_j = s_j + \delta_{i,j}$, we can again develop a more intuitive understanding of this statement, since

$$s_{i,i} \bullet g(s)f^s = s_i t_i^{-1} t_i \bullet g(s)f^s = s_i \bullet g(s)f^s = s_i g(s)f^s.$$

Finally, we can define the Bernstein-Sato polynomial of f.

Definition 2.52. We define the *(generalized) Bernstein-Sato polynomial* $b_f(s)$ of f such that after substituting $s \mapsto s_{1,1} + \ldots + s_{r,r}$ the polynomial $b_f(s_{1,1} + \ldots + s_{r,r})$ is the monic generator of

$$(\operatorname{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_1, \ldots, f_r\rangle) \cap \mathbb{C}[s_{1,1} + \ldots + s_{r,r}].$$

Remark 2.53. The subring $\mathbb{C}[s_{1,1} + \ldots + s_{r,r}]$ is central in $D_n\langle S\rangle$ which makes the definition also computationally implementable.

We find a monic generator $b_f(s)$ because $\mathbb{C}[s_{1,1} + \ldots + s_{r,r}]$ is a principal ideal domain, which makes $b_f(s)$ well-defined.

Again, we can reformulate the definition as a functional equation of the form

$$b_f(s_1 + \ldots + s_r)f^s = (\sum_{i=1}^{r} \delta_i f_i) \bullet f^s$$

for some $\delta_i \in D\langle S\rangle$.

It is due to [BMS06] that we know that $b_f(s) \neq 0$.

For $r = 1$, i.e. $f \in \mathbb{C}[\underline{x}]$, this definition of the Bernstein-Sato polynomial coincides with the classical Bernstein-Sato polynomial of f as a polynomial, since in this case

$$\mathbb{C}\langle S\rangle/\langle s_{i,i} - s_i \mid i \in \{1, \ldots, r\}\rangle \cong \mathbb{C}[s_{1,1}] \cong \mathbb{C}[s_1].$$

While the Bernstein-Sato polynomial for $f \in \mathbb{C}[\underline{x}]^r$ resembles the Bernstein-Sato polynomial by this connection and the principality of ideals, it also carries resemblance with $\mathcal{B}_\Sigma$ since we add the ideal ${}_{D\langle S\rangle}\langle f_1, \ldots, f_r\rangle$ before intersecting with the principal ideal domain. Later, we will see that $\mathcal{B}_\Sigma$ displays rather untypical behaviour in comparison with $\mathcal{B}$ and $\mathcal{B}_{(i)}$.

In order to motivate the term 'Bernstein-Sato polynomial *of a variety*', we need a preliminary definition.

Definition 2.54. Let $f \in \mathbb{C}[\underline{x}]$. The *codimension* of $\mathbb{V}(f) \subseteq \mathbb{C}^n$ is defined as

$$\operatorname{codim}_{\mathbb{C}^n}(\mathbb{V}(f)) := n - \operatorname{krdim}(\langle f_1, \ldots, f_r\rangle).$$

The following definition allows for a deeper insight into the nature of the polynomials that we defined.

Definition 2.55 ([BMS06]). We define the *(generalized) Bernstein-Sato polynomial of* $\langle f\rangle$ as

$$b_{\langle f\rangle}(s) := b_{\langle f_1,\ldots,f_r\rangle}(s) := b_f(s-\operatorname{codim}_{\mathbb{C}^n}(\mathbb{V}(f))).$$

Remark 2.56. In [BMS06, 2.5], it is shown that the Bernstein-Sato polynomial $b_{\langle f\rangle}$ is independent of the generators of $\langle f\rangle := {}_{\mathbb{C}[\underline{x}]}\langle f_1,\ldots,f_r\rangle$. This justifies the term chosen to describe the polynomial.

For algorithmic aspects, this result has severe implications, since we can arbitrarily choose generators of $\langle f\rangle$. Here, we find a similarity to $\mathcal{B}_\Sigma$, since for $1 \in {}_{D_n[\underline{s}]}\langle f\rangle$, we have $\mathcal{B}_\Sigma = \langle b_{\langle f\rangle}(s)\rangle = \langle 1\rangle$. This trivializes the determination of both $\mathcal{B}_\Sigma$ and $b_{\langle f\rangle}(s)$ for examples like $f = (1-x+y, x-y)$.

However, the optimal choice of generators is not always clear, but one goal in such a choice can be the minimization of the number of generators.

Remark 2.57. Another important computer-algebraic aspect is the computation of $\operatorname{ann}_{D_n\langle S\rangle}(f^s)$ and of the intersection of $\operatorname{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_1,\ldots,f_r\rangle$ with $\mathbb{C}[s_{1,1}+\ldots+s_{r,r}]$. For a solution to the first problem we refer to [ALM09], where Algorithm 2.18 is generalized, and for the latter problem we recapitulate their approach here.

The necessary algorithms are implemented in the Plural ([GLMS15]) library `dmodvar.lib` ([ALM15]).

The problem of intersection with $\mathbb{C}[\sum_i s_i]$ can be tackled with the following algorithm. Here $\operatorname{NF}(a,G)$ denotes the normal form of the element a with respect to the Gröbner basis G.

Algorithm 2.58 ([ALM09, 4.11]).
Input: $h \in D\langle S\rangle$ and an ideal $J \subseteq D\langle S\rangle$ with $J \cap \mathbb{C}[h] \neq \{0\}$.
Output: a generator of $J\cap\mathbb{C}[h]$ as an ideal in $\mathbb{C}[h]$.

1: Set $i := 1$ and choose a Gröbner basis G of J.
2: **while** $\{0\} = \ker_{\mathbb{C}}(\operatorname{NF}(h^i,G),\ldots,\operatorname{NF}(h,G),\operatorname{NF}(1,G)) \subseteq \mathbb{C}^{i+1}$ **do**
3: Set $i := i+1$.
4: **end while**
5: **return** $h^i + \sum_{j=0}^{i-1} \frac{a_j}{a_i} h^j$ for some

$$(a_i,\ldots,a_0) \in \ker_{\mathbb{C}}(\operatorname{NF}(h^i,G),\ldots,\operatorname{NF}(h,G),\operatorname{NF}(1,G)) \setminus \{0\}\,.$$

Remark 2.59. The algorithm can be generalized to arbitrary fields k instead of $\mathbb{C}$ and other associative k-algebras instead of $D\langle S\rangle$, as long as there is a k-linear, algorithmically treatable normal form.

The correctness of the algorithm follows from the iterative procedure such that the element found is of minimal degree in h and the fact that $g \in \langle G\rangle \Leftrightarrow \operatorname{NF}(g,G) = 0$ together with $\mathbb{C}$-linearity of $\operatorname{NF}(\cdot,G)$ up to elements of $\langle G\rangle$.

Applying the algorithm to $h = \sum_{i=1}^r s_i$ and $J = \operatorname{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_1,\ldots,f_r\rangle$ allows to compute the intersection we are interested in.

Since the non-commutative structure of $\mathbb{C}\langle S\rangle$ does not interfere with the x_i, i.e. $x_i s_{j,k} = s_{j,k} x_i$ for all $1 \leq i \leq n, 1 \leq j,k \leq r$, we can completely analogously define local versions of the structures used.

Definition 2.60. For $p \in \mathbb{C}^n$ we define $D_{n,p}\langle S\rangle := \mathbb{C}[\underline{x}]_p \otimes_{\mathbb{C}[\underline{x}]} D\langle S\rangle$.

With this, we can also define local Bernstein-Sato polynomials of f and $\langle f\rangle$.

Definition 2.61. We define $b_{f,p}(s) \in \mathbb{C}[s]$ such that

$$(\mathrm{ann}_{D_{n,p}\langle S\rangle}(f^s) + {}_{D_{n,p}\langle S\rangle}\langle f_1, \ldots, f_r\rangle) \cap \mathbb{C}[s_{1,1} + \ldots + s_{r,r}] = \langle b_{f,p}(s_1 + \ldots + s_r)\rangle$$

and $b_{\langle f\rangle,p}(s) \in \mathbb{C}[s]$ by $b_{\langle f\rangle,p}(s) := b_{f,p}(s - \mathrm{codim}_{\mathbb{C}^n}(\mathbb{V}_{\mathbb{C}[\underline{x}]}(f)))$.

Remark 2.62. In the definition of $b_{\langle f\rangle,p}(s)$ we use the same shift as in the global definition. We do this in order to maintain the connection between local and global polynomials by the least common multiple.

2.6. Other variants of Bernstein-Sato polynomials for varieties

In the definition of Bernstein-Sato polynomials for varieties we have considered

$$(\mathrm{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_1, \ldots, f_r\rangle) \cap \mathbb{C}[s_{1,1} + \ldots + s_{r,r}],$$

which is a construction in analogy to $\mathcal{B}_\Sigma$. We will now consider variations that rather resemble $\mathcal{B}$ and $\mathcal{B}_{(i)}$ by defining $b_{f,\Pi}(s), b_{f,(i)}(s) \in \mathbb{C}[s]$ for $i \in \{1, \ldots, r\}$ such that

$$\langle b_{f,\Pi}(s_1 + \ldots + s_r)\rangle = (\mathrm{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_1 \cdot \ldots \cdot f_r\rangle) \cap \mathbb{C}[s_{1,1} + \ldots + s_{r,r}] \quad \text{and}$$
$$\langle b_{f,(i)}(s_1 + \ldots + s_r)\rangle = (\mathrm{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_i\rangle) \cap \mathbb{C}[s_{1,1} + \ldots + s_{r,r}].$$

Analogously as in the definition of $b_{\langle f\rangle}$, we define $b_{\langle f\rangle,\Pi}(s) := b_{f,\Pi}(s - \mathrm{codim}_{\mathbb{C}^n}(\mathbb{V}(f)))$ and $b_{\langle f\rangle,(i)}(s) := b_{f,(i)}(s - \mathrm{codim}_{\mathbb{C}^n}(\mathbb{V}(f)))$.

We see the strengths of the definition of b_f in the weaknesses of these constructions. The ideals we deal with are principal ideals, but the independence of generators of $\langle f\rangle$ does not hold any longer.

Example 2.63. Consider $f = (1-x, x) \in \mathbb{C}[x]^2$ and $g = (1) \in \mathbb{C}[x]$. It holds that $\langle f\rangle = \langle g\rangle$ but $b_{\langle f\rangle,\Pi}(s) = s + 1 \neq 1 = b_{\langle g\rangle,\Pi}$.

In the example $f = (1-x^2, x^2) \in \mathbb{C}[x]^2$ we have $b_{f,(1)} = s+1 \neq (s+1)(s+\frac{3}{2}) = b_{f,(2)}$, especially the $b_{f,(i)}$ are now dependent on the generators and even on their order.

In [BMS06], another variation of the Bernstein-Sato polynomial for varieties was introduced which incorporates a different polynomial $g \in \mathbb{C}[\underline{x}]$.

Definition 2.64 ([BMS06]). The *Bernstein-Sato polynomial of* $f \in \mathbb{C}[\underline{x}]^r$ *and* $g \in \mathbb{C}[\underline{x}]$ is defined as the monic polynomial $0 \neq b_{f,g}(s) \in \mathbb{C}[s]$ of minimal degree such that

$$b_{f,g}(s_1 + \ldots + s_r) g f^s = \left(\sum_{i=1}^{r} \delta_i g f_i \right) \bullet f^s,$$

where $\delta_i \in D\langle S \rangle$ for all $i \in \{1, \ldots, r\}$.

Remark 2.65. In [BMS06], the existence of $b_{f,g} \neq 0$ was shown.

Similarly as for the other constructions we introduced so far, we can reformulate the problem of finding $b_{f,g}$ as

$$\big((\operatorname{ann}_{D\langle S \rangle}(f^s) + {}_{D\langle S \rangle}\langle g f_1, \ldots, g f_r \rangle) \cap g\mathbb{C}[s_1 + \ldots + s_r] \big) : g = {}_{\mathbb{C}[s_1 + \ldots + s_r]}\langle b_{f,g}(\sum_{i=1}^{r} s_i) \rangle.$$

However, it is not clear how we can compute the intersection with $g\mathbb{C}[s_1 + \ldots + s_r]$. We modify Algorithm 2.58 in order to solve this problem.

Algorithm 2.66.
Input: $h \in D\langle S \rangle$, $g \in \mathbb{C}[\underline{x}]$ and an ideal $J \subseteq D\langle S \rangle$ with $J \cap g\mathbb{C}[h] \neq \{0\}$.
Output: a generator of $(J \cap g\mathbb{C}[h]) : g$ as an ideal in $\mathbb{C}[h]$.

1: Set $i := 1$ and choose a Gröbner basis G of J.
2: **while** $\{0\} = \ker_{\mathbb{C}}(\mathrm{NF}(gh^i, G), \ldots, \mathrm{NF}(gh, G), \mathrm{NF}(g, G)) \subseteq \mathbb{C}^{i+1}$ **do**
3: Set $i := i + 1$.
4: **end while**
5: **return** $h^i + \sum_{j=0}^{i-1} \frac{a_j}{a_i} h^j$ for some

$$(a_i, \ldots, a_0) \in \ker_{\mathbb{C}}(\mathrm{NF}(gh^i, G), \ldots, \mathrm{NF}(gh, G), \mathrm{NF}(g, G)) \setminus \{0\} .$$

Remark 2.67. The correctness of this algorithm follows analogously as that of Algorithm 2.58 by using that

$$\mathrm{NF}(gh^i + \sum_{j=0}^{i-1} \frac{a_j}{a_i} g h^j, G) = \mathrm{NF}(gh^i, G) + \sum_{j=0}^{i-1} \frac{a_j}{a_i} \mathrm{NF}(gh^j, G) = 0.$$

The application of this algorithm to $J = \operatorname{ann}_{D\langle S \rangle}(f^s) + {}_{D\langle S \rangle}\langle g f_1, \ldots, g f_r \rangle$, $h = s_1 + \ldots + s_r$ and the given g solves our problem of computing the intersection for determining $b_{f,g}$ and at the same time allows us to compute the quotient.

For determining $\operatorname{ann}_{D\langle S \rangle}(f^s)$, we can again use the methods from [ALM09].

Example 2.68. We consider $f = x^2 \in \mathbb{C}[x]$ and $g = x \in \mathbb{C}[\underline{x}]$. The Bernstein-Sato polynomial of f is given by $b_f(s) = (s+1)(s+2)$ with corresponding operator ∂_x^2, whereas with the same operator we obtain the Bernstein-Sato polynomial $b_{f,g}(s) = (s+2)(s+3)$.

2.7. Stratifications with respect to local Bernstein-Sato polynomials of varieties

In [LM12], the applicability of the results from Section 2.4 to the case of Bernstein-Sato polynomials of varieties, b_f, was shown. From the upper bound b_f or $b_{\langle f\rangle}$, one can find factors of the respective local Bernstein-Sato polynomials.

The approach via primary decomposition can also be used to find a stratification with respect to local Bernstein-Sato varieties by decomposing

$$(\operatorname{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle f_1, \ldots, f_r\rangle) \cap \mathbb{C}[\underline{x}, s_1 + \ldots + s_r]$$

and intersecting the primary components with $\mathbb{C}[\underline{x}]$ and $\mathbb{C}[s_1 + \ldots + s_r]$.

However, it is not clear how those two methods can be applied to $b_{f,g}$. If we want to use primary decompositions for this task, we cannot directly use the defining equation

$$\left(\underbrace{(\operatorname{ann}_{D\langle S\rangle}(f^s) + {}_{D\langle S\rangle}\langle gf_1, \ldots, gf_r\rangle)}_{=:L} \cap g\mathbb{C}[s_1 + \ldots + s_r]\right) : g =: {}_{\mathbb{C}[s_1+\ldots+s_r]}\langle b_{f,g}(\sum_{i=1}^{r} s_i)\rangle$$

but have to reformulate it as

$$\underbrace{((L \cap g\mathbb{C}[\underline{x}, s_1 + \ldots + s_r]) : g)}_{=:Q} \cap \mathbb{C}[s_1 + \ldots + s_r] =: {}_{\mathbb{C}[s_1+\ldots+s_r]}\langle b_{f,g}(s_1 + \ldots + s_r)\rangle. \quad (2)$$

The intersection needed here to determine Q can be computed by adding an additional variable s with the relation $s = s_1 + \ldots + s_r$. Then, we can compute the intersection with an elimination ordering.

At this point, we have a similar situation as for the stratification with respect to to Bernstein-Sato ideals. We can decompose $Q = \bigcap_i Q_i$ into primary components Q_i and then use the intersections of the Q_i with $\mathbb{C}[\underline{x}]$ and $\mathbb{C}[s_1 + \ldots + s_r]$:

$$\langle b_f^{(i)}\rangle := Q_i \cap \mathbb{C}[s_1 + \ldots + s_r] \text{ and } I_i := Q_i \cap \mathbb{C}[\underline{x}].$$

With analogous arguments as in the case of Bernstein-Sato ideals we obtain

$${}_{\mathbb{C}[s]}\langle b_{f,p}(s)\rangle = \bigcap_{i:p\in\mathbb{V}(I_i)} {}_{\mathbb{C}[s]}\langle b_f^{(i)}\rangle \quad \text{i.e. } b_{f,p}(s) = \operatorname{lcm}(b_f^{(i)} \mid p \in \mathbb{V}(I_i)).$$

for $p \in \mathbb{C}^n$, since

$$b_f^{(i)} \mid b_{f,p} \Leftrightarrow 1 \notin S_p^{-1}Q_i \Leftrightarrow p \in \mathbb{V}(I_i).$$

The approach by [LM12] is still feasible. This is due to the fact that $\mathbb{C}[s_1 + \ldots + s_r]$ is contained in the center of $D\langle S\rangle$, since

$$s_{i,j}\left(\sum_{k=1}^{r} s_k\right) = \sum_{k=1}^{r} s_{i,j}s_{k,k} = \sum_{k\notin\{i,j\}} s_{k,k}s_{i,j} + (s_{j,j}s_{i,j} + s_{i,j}) + (s_{i,i}s_{i,j} - s_{i,j}) = \left(\sum_{k=1}^{r} s_k\right) s_{i,j}$$

and the x_i, ∂_i commute with the s_k anyways. Now, Theorem 2.43 and the resulting algorithms can be applied to Q from (2). Especially, the definitions from Theorem 2.46 can be adapted as follows in analogy to [LM12, 2.14]:

Theorem 2.69. Let Q as in (2) and α a root of $b_{f,g}(-s)$ of multiplicity m_α. Denote by $m_{\alpha,p}$ the multiplicity of α as root of $b_{f,g,p}(-s)$.

For $1 \leq i < m_\alpha$ we define $I_{\alpha,i} := (Q : (\sum_{i=1}^r s_i + \alpha)^i) +_{\mathbb{C}[\underline{x},\sum_{i=1}^r s_i]} \langle s+\alpha\rangle$. It holds that

- $(s+\alpha) \mid b_{f,g,p}(-s) \Leftrightarrow p \in \mathbb{V}((Q +_{\mathbb{C}[\underline{x},\sum_{i=1}^r s_i]} \langle \sum_{i=1}^r s_i + \alpha\rangle) \cap \mathbb{C}[\underline{x}])$,
- $m_{\alpha,p} > i \Leftrightarrow p \in \mathbb{V}(I_{\alpha,i} \cap \mathbb{C}[\underline{x}])$.

This gives the tools for a stratification with respect to local Bernstein-Sato polynomials $b_{f,g}$ constructed analogously as in Corollary 2.47.

Example 2.70. Continuing Example 2.68 with $f = x^2 \in \mathbb{C}[x]$ and $g = x \in \mathbb{C}[x]$, we may give a stratification with respect to the local Bernstein-Sato polynomials $b_{f,g,p}$, since

$$b_{f,g,p} = \begin{cases} (s+2)(s+3) & \text{if } p = 0, \\ 1 & \text{otherwise.} \end{cases}$$

2.8. Generalized stratifications by primary decomposition

In all cases where stratifications through primary decompositions have been constructed so far, we could proceed in analogous constructions. In this section, we want to find out how these constructions can be generalized.

For this approach, we will in the following consider a Noetherian commutative $\mathbb{C}$-algebra A and the (commutative) $\mathbb{C}$-algebra $R := \mathbb{C}[\underline{x}] \otimes_{\mathbb{C}} A$. We start off with a global ideal $Q \subseteq R$ and want to stratify $\mathbb{C}^n$ with respect to the localized intersections $B_p := (S_p^{-1}Q) \cap A$, where $S_p := \{f \in \mathbb{C}[\underline{x}] \mid f(p) \neq 0\}$ for $p \in \mathbb{C}^n$.

Since Lemma 2.24 is applicable in this situation, we conclude that for primary $\tilde{Q} \subseteq R$ it holds that $(S_p^{-1}\tilde{Q}) \cap A = \tilde{Q} \cap A$ for $p \in \mathbb{V}(\tilde{Q} \cap \mathbb{C}[\underline{x}])$ and $(S_p^{-1}\tilde{Q}) \cap A = A$ otherwise.

In order to be able to work with primary ideals, we again fix a primary decomposition of Q as $Q = \bigcap_{i=1}^\ell Q_i$.

This allows us to generalize Proposition 2.25 to our situation.

Lemma 2.71. For $p \in \mathbb{C}^n$,

$$B_p = \bigcap_{i:p\in\mathbb{V}(Q_i\cap\mathbb{C}[\underline{x}])} (Q_i \cap A).$$

Proof. We proceed analogously as in the proof of Proposition 2.25.

From

$$B_p = \left(\bigcap_{i=1}^{\ell} S_p^{-1}Q_i\right) \cap \mathbb{C}[\underline{s}] \text{ and } S_p^{-1}Q_i = S_p^{-1}R \Leftrightarrow p \notin \mathbb{V}(Q_i \cap \mathbb{C}[\underline{x}])$$

we conclude

$$B_p = \left(\bigcap_{i:p \in \mathbb{V}(I_i)} S_p^{-1} Q_i \right) \cap A.$$

Now, the claim follows from Lemma 2.24. □

In conclusion, we can construct a stratification of $\mathbb{C}^n$ with respect to B_p in analogy to Theorem 2.27 and Lemma 2.30.

Theorem 2.72. For $J \subseteq \{1, \ldots, \ell\}$ we set

$$W_J = \left(\bigcap_{j \in J} \mathbb{V}(Q_j \cap \mathbb{C}[\underline{x}]) \right) \setminus \left(\bigcup_{j \notin J} \mathbb{V}(Q_j \cap \mathbb{C}[\underline{x}]) \right).$$

The set $\{W_J \mid J \subseteq \{1, \ldots, \ell\}\}$ defines a finite stratification of $\mathbb{V}(F)$ with respect to B_p. Here, B_p is regarded as mapping of p,

$$B_. : \ \mathbb{C}^n \to \{I \subseteq A \mid I \text{ ideal}\}\,; \ p \mapsto B_p.$$

Proof. The claim that the B_p are constant on W_J follows from Lemma 2.71.

It remains to be shown that the W_J fulfill the requirements of strata. This can be shown as in the proof of Lemma 2.30, because here we deal with irreducible varieties and their differences as well. □

3. Local Bernstein-Sato ideals

In this section, we want to find out more about the structure of Bernstein-Sato ideals. In particular, we are interested in factors q of $\mathcal{B}$ with $qI = \mathcal{B}$ for some ideal $I \subseteq \mathbb{C}[\underline{s}]$.

3.1. A tool for undesired factors

Again, we consider $f \in \mathbb{C}[\underline{x}]^r$ and $F = \prod_{i=1}^r f_i$ and want to find polynomials $b(s) \in \mathbb{C}[\underline{s}]$ with the property $b(s)f^s = \delta(s) \bullet f^{s+1}$ for some $\delta(s) \in D_n[\underline{s}]$ or $\delta(s) \in D_{n,p}[\underline{s}]$ for global or local Bernstein-Sato ideals, respectively.

A classical result about Bernstein-Sato ideals is that for $p \notin \bigcup_{i=1}^r \mathbb{V}(f_i)$ the local Bernstein-Sato ideal is given by $\mathcal{B}_p = \langle 1 \rangle$, which can be seen with the Bernstein-Sato operator $\delta = F^{-1}$. In $\mathbb{C}[\underline{x}]_p$, F is a unit since $F(0) \neq 0$. We will now introduce a tool that generalizes this result and allows to omit factors that do not vanish in a point for the construction of the local Bernstein-Sato ideal in that point. This gives another proof of the result that for units $u_1, \ldots, u_r$ it holds that $\mathcal{B}_f = \mathcal{B}_{(u_1 f_1, \ldots, u_r f_r)}$, as we will see in Theorem 3.5 (see [BO10] for the case of $u_1, \ldots, u_r, f_1, \ldots, f_r \in \mathbb{C}[[\underline{x}]]$).

Unlike for the definition of Bernstein-Sato ideals, where we could work with the $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]$-module $\mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]f^s$, we now need a more sophisticated structure in order to formalize the application of differential operators to f^s. Consider the finitely generated module over the ring $R := \mathbb{C}[\underline{x}, \underline{s}, \frac{1}{F}]$ defined by

$$M = \bigoplus_{k=0}^{r} \bigoplus_{1 \leq i_1 < \ldots < i_k \leq r} R \prod_{j=1}^{k} f_{i_j}^{s_{i_j}}.$$

This module can be regarded as an R-submodule of the R-algebra $R[f_1^{s_1}, \ldots, f_r^{s_r}]$ with the natural R-module structure given by $\partial_k \bullet f_i^{s_i} = s_i f_i^{s_i - 1}(\partial_k \bullet f_i)$ and the Leibniz rule. We do not work with $R[f_1^{s_1}, \ldots, f_r^{s_r}]$ because this polynomial ring is not finitely generated as an R-module.

The module M has an additional structure induced by $R[f_1^{s_1}, \ldots, f_r^{s_r}]$: For $\alpha, \beta \in \{0,1\}^r$ with $\alpha_i \beta_i = 0$ for all $1 \leq i \leq r$, we define $f^{\alpha s} \cdot f^{\beta s} = f^{(\alpha+\beta)s}$.

Remark 3.1. For this structure, it is not necessary to have pairwise distinct s_{i_j}, since the $f_i^{s_i}$ are treated as formal symbols, so we may choose $s_{i_{j_1}} = s_{i_{j_2}}$ for $j_1 \neq j_2$ as well, which can be used to factorize f_i. For example, we could consider $f = f_{1,1}^{s_1} f_{1,2}^{s_1} \in \mathbb{C}[\underline{x}]$ and the corresponding module $M = R \oplus R f_{1,1}^{s_1} \oplus R f_{1,2}^{s_1} \oplus R f_{1,1}^{s_1} f_{1,2}^{s_1}$.

Proposition 3.2. For $g \in \mathbb{C}[\underline{x}] \setminus \{0\}$ and $1 \le i \le r$ we define the isomorphism of rings $\phi_{g,s_i} : D_n\left[\underline{s}, \frac{1}{g}\right] \to D_n\left[\underline{s}, \frac{1}{g}\right]$ (abbreviated as ϕ) by

$$\partial_k \mapsto \partial_k + \frac{s_i}{g} \cdot (\partial_k \bullet g), \quad x_i \mapsto x_i, \quad s_i \mapsto s_i.$$

For $\delta \in D_n[\underline{s}]$ and $h = \prod_{1 \le j \le r} f_j^{s_j}$ (we may choose $f_j = 1$) it holds that

$$\delta \bullet (h \cdot g^{s_i}) = g^{s_i} \cdot (\phi(\delta) \bullet h) \in M.$$

Proof. We show the claim for $\delta = \partial_k$, $1 \le k \le n$, which implies the general case by iterative application. With Leibniz rule and chain rule we obtain

$$\begin{aligned} g^{s_i} \cdot (\phi(\partial_k) \bullet h) &= g^{s_i} \cdot \left(\partial_k + \frac{s_i}{g} \cdot (\partial_k \bullet g)\right) \bullet h = h \cdot s_i g^{s_i - 1} \cdot (\partial_k \bullet g) + g^{s_i} \cdot (\partial_k \bullet h) \\ &= h \cdot (\partial_k \bullet g^{s_i}) + g^{s_i} \cdot (\partial_k \bullet h) = \partial_k \bullet (h \cdot g^{s_i}). \end{aligned}$$

It remains to be shown that ϕ is a bijective homomorphism. We have to show that it is compatible with the non-commutative relations of $D_n\left[\underline{s}, \frac{1}{g}\right]$. It holds that

$$\phi(x_k\partial_k + 1) = x_k\partial_k + x_k\frac{s_i}{g} \cdot (\partial_k \bullet g) + 1 = \partial_k x_k + \frac{s_i}{g} \cdot (\partial_k \bullet g)x_k = \phi(\partial_k x_k),$$

and with $[a,b] = ab - ba$ and

$$\partial_k \frac{s_i}{g} \cdot (\partial_m \bullet g) = s_i \partial_k \frac{(\partial_m \bullet g)}{g} = s_i \left(\frac{(\partial_m \bullet g)}{g}\partial_k + \frac{(\partial_k\partial_m \bullet g)g - (\partial_k \bullet g)(\partial_m \bullet g)}{g^2}\right)$$

we obtain

$$\begin{aligned} [\phi(\partial_k), \phi(\partial_m)] =& [\partial_k + \frac{s_i}{g} \cdot (\partial_k \bullet g), \partial_m + \frac{s_i}{g} \cdot (\partial_m \bullet g)] \\ \stackrel{\text{bilinearity}}{=}& \underbrace{[\partial_k, \partial_m]}_{=0} + [\partial_k, \frac{s_i}{g} \cdot (\partial_m \bullet g)] \\ &+ [\frac{s_i}{g} \cdot (\partial_k \bullet g), \partial_m] + \underbrace{[\underbrace{\frac{s_i}{g} \cdot (\partial_k \bullet g)}_{\in \mathbb{C}[\underline{x},\underline{s}]}, \underbrace{\frac{s_i}{g} \cdot (\partial_m \bullet g)}_{\in \mathbb{C}[\underline{x},\underline{s}]}]}_{=0} \\ =& \partial_k \frac{s_i}{g} \cdot (\partial_m \bullet g) - \frac{s_i}{g} \cdot (\partial_m \bullet g)\partial_k - \partial_m \frac{s_i}{g} \cdot (\partial_k \bullet g) + \frac{s_i}{g} \cdot (\partial_k \bullet g)\partial_m \\ =& s_i \left(\frac{(\partial_m \bullet g)}{g}\partial_k + \frac{(\partial_k\partial_m \bullet g)g - (\partial_k \bullet g)(\partial_m \bullet g)}{g^2}\right) + \frac{s_i}{g} \cdot (\partial_k \bullet g)\partial_m - \\ & s_i \left(\frac{(\partial_k \bullet g)}{g}\partial_m + \frac{(\partial_m\partial_k \bullet g)g - (\partial_m \bullet g)(\partial_k \bullet g)}{g^2}\right) - \frac{s_i}{g} \cdot (\partial_m \bullet g)\partial_k \\ =& s_i \left(\frac{(\partial_k\partial_m \bullet g)g - (\partial_k \bullet g)(\partial_m \bullet g)}{g^2}\right) \\ &- s_i \left(\frac{(\partial_m\partial_k \bullet g)g - (\partial_m \bullet g)(\partial_k \bullet g)}{g^2}\right) = 0. \end{aligned}$$

Furthermore

$$\begin{aligned}[\phi(\partial_k), x_m] =& [\partial_k + \frac{s_i}{g} \cdot (\partial_k \bullet g), x_m] \\ =& \partial_k x_m + \frac{s_i}{g} \cdot (\partial_k \bullet g) x_m - x_m \partial_k + x_m \frac{s_i}{g} \cdot (\partial_k \bullet g) = 0\end{aligned}$$

for $k \neq m$.

The bijectivity of ϕ follows with the inverse that maps ∂_k to $\partial_k - \frac{s_i}{g} \cdot (\partial_k \bullet g)$, which is a homomorphism as well. $\square$

Remark 3.3. Proposition 3.2 can be generalized to rings $D_{n,p}[\underline{s}]$.

We now examine how the structural properties of Proposition 3.2 can be generalized.

Lemma 3.4. We define the mapping $\phi : D_n\left[\underline{s}, \frac{1}{g}\right] \to D_n\left[\underline{s}, \frac{1}{g}\right]$ by

$$\partial_k \mapsto \partial_k + w_k, \quad x_i \mapsto x_i, \quad s_i \mapsto s_i$$

for $w_1, \ldots, w_n \in D_n[\underline{s}, \frac{1}{g}]$.

The mapping ϕ is a homomorphism of rings if and only if $w_k \in \mathbb{C}[\underline{s}, \underline{x}, \frac{1}{g}]$ and $\partial_k \bullet w_m - \partial_m \bullet w_k = 0$ for all $1 \leq k, m \leq n$.

Proof. Three properties need to be fulfilled for all $1 \leq k, m, l \leq n$, $k \neq m$ to make ϕ a homomorphism of rings (and these properties are sufficient):

$$\begin{aligned}0 &= [\phi(\partial_k), \phi(x_k)] - 1 = [\phi(\partial_k), x_k] - 1 = [\partial_k + w_k, x_k] - 1 &(3)\\ &\overset{\text{bilinearity}}{=} [w_k, x_k] = w_k x_k - x_k w_k, \\ 0 &= [\phi(\partial_k), \phi(x_m)] = [\phi(\partial_k), x_m] = [\partial_k + w_k, x_m] &(4)\\ &\overset{\text{bilinearity}}{=} [w_k, x_m] = w_k x_m - x_m w_k, \\ 0 &= [\phi(\partial_k), \phi(\partial_l)] = [\partial_k + w_k, \partial_l + w_l] &(5)\\ &= \partial_k \partial_l + \partial_k w_l + w_k \partial_l + w_k w_l - \partial_l \partial_k - \partial_l w_k - w_l \partial_k - w_l w_k \\ &= \partial_k w_l + w_k \partial_l + w_k w_l - \partial_l w_k - w_l \partial_k - w_l w_k\end{aligned}$$

The equalities (3) and (4) are equivalent to $w_k \in \mathbb{C}[\underline{x}, \underline{s}, \frac{1}{g}]$. With this knowledge, we can further simplify (5) as

$$\begin{aligned}0 &= \partial_k w_l + w_k \partial_l - \partial_l w_k - w_l \partial_k = w_l \partial_k + \partial_k \bullet w_l + w_k \partial_l - w_k \partial_l - \partial_l \bullet w_k - w_l \partial_k \\ &= \partial_k \bullet w_l - \partial_l \bullet w_k,\end{aligned}$$

the second condition.

On the other hand, if $w_k \in \mathbb{C}[\underline{s}, \underline{x}, \frac{1}{g}]$ and $\partial_k \bullet w_m - \partial_m \bullet w_k = 0$, by the same arguments, ϕ is a homomorphism of rings. $\square$

Theorem 3.5 (see also [BO10, Lemma 10])**.** For g and h as in Proposition 3.2 and $1 \leq i \leq r$ with $g(p) \neq 0$ it holds that

$$\phi_{g,s_i}\left(\operatorname{ann}_{D_{n,p}[\underline{s}]}(g^{s_i} \cdot h)\right) = \operatorname{ann}_{D_{n,p}[\underline{s}]}(h).$$

In this case, $\mathcal{B}_{(f_1,\ldots,f_{i-1},f_i \cdot g,f_{i+1},\ldots,f_r),p} = \mathcal{B}_{(f_1,\ldots,f_{i-1},f_i,f_{i+1},\ldots,f_r),p}$.

Proof. Consider the first claim. For '$\subseteq$' let $\delta \in \operatorname{ann}_{D_{n,p}[\underline{s}]}(g^{s_i} \cdot h)$. Then $0 = \delta \bullet (g^{s_i} \cdot h) = g^{s_i}(\phi(\delta) \bullet h)$ by Proposition 3.2, so $\phi(\delta) \in \operatorname{ann}_{D_{n,p}[\underline{s}]}(h)$.

The other inclusion follows analogously by using the inverse ϕ^{-1}.

For the second claim let $b \in \mathcal{B}_{(f_1,\ldots,f_{i-1},f_i \cdot g,f_{i+1},\ldots,f_r),p}$, e.g. $bg^{s_i}f^s = \delta \bullet g^{s_i+1}f^{s+1}$ for $b \in \mathbb{C}[\underline{s}]$ and $\delta \in D_n[\underline{s}]$, or equivalently

$$b - \delta g F \in \operatorname{ann}_{D_n[\underline{s}]}(f^s g^{s_i}).$$

Applying $\phi = \phi_{g,s_i}$ yields

$$b - \phi(\delta) g F \in \phi\left(\operatorname{ann}_{D_n[\underline{s}]}(f^s g^{s_i})\right) = \operatorname{ann}_{D_{n,p}[\underline{s}]}(f^s)$$

because $\phi(b) = b$ and $\phi(gF) = gF$. It follows that $b \in \mathcal{B}_{(f_1,\ldots,f_{i-1},f_i,f_{i+1},\ldots,f_r),p}$. The other inclusion follows analogously. $\square$

Remark 3.6. This theorem allows us to omit all those $g \mid F$ which do not vanish at p when determining $\mathcal{B}_p$ or, in other words, assuming w.l.o.g. that all f considered fulfill $f_i(p) = 0$ and even $g(p) = 0$ for all $g \mid f_i$, $1 \leq i \leq r$.

More precisely, for $(u_1 f_1, \ldots, u_r f_r)$ with units $u_i \in \mathbb{C}[\underline{x}]_p$ with $u_i(p) \neq 0$ and non-units f_i we can apply $\phi_{u_1,s_1} \circ \ldots \circ \phi_{u_r,s_r}$ to obtain $\mathcal{B}_{(u_1 f_1,\ldots,u_r f_r)} = \mathcal{B}_{(f_1,\ldots,f_r)}$.

3.2. Common factors of generators of local Bernstein-Sato ideals

In this subsection we are concerned with applying the previously developed tool in order to obtain partial information of local Bernstein-Sato ideals. In most cases we will show that for certain polynomials $q(s) \in \mathbb{C}[\underline{s}]$ it holds that $q(s) \mid \mathcal{B}_f$, i.e. $q(s) \mid b$ for all $b \in \mathcal{B}$. We start off with a generalization of the fact that $(s+1) \mid b_f$ for $f \in \mathbb{C}[\underline{x}] \setminus \mathbb{C}$.

Lemma 3.7. Let $1 \leq i \leq r$ with $p \in \mathbb{V}(f_i) \setminus \mathbb{V}(\prod_{j \neq i} f_j)$. Then $(s_i + 1) \mid \mathcal{B}_p$.

Proof. Let $b \in \mathcal{B}_p$ and $\delta \in D_{n,p}[\underline{s}]$ such that $bf^s = \delta \bullet f^{s+1}$. We choose $s_i := -1$ and leave s_j symbolic. In this case, with $\hat{f} := \prod_{j \neq i} f_j$ and $\hat{s} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_r)$, the defining equation of b becomes

$$\frac{b(\hat{s}, s_i = -1)}{f_i} \hat{f}^{\hat{s}} = \delta(\hat{s}, s_i = -1) \bullet \hat{f}^{\hat{s}+1}$$

for some $\delta \in D_{n,p}[\underline{s}]$. Equating the coefficients of $\hat{f}^{\hat{s}}$, the right hand side of the equation is contained in $S_p^{-1}\mathbb{C}[\underline{x}, \underline{s}]$. On the other hand, a factor of f_i appears in the denominator of the left hand side, so it follows that $b(\hat{s}, s_i = -1) = 0$ and thus $(s_i + 1) \mid b$. $\square$

Observation 3.8. The claim of Lemma 3.7 can be transferred to $\mathcal{B}_{(i)}$ in the sense that under the conditions of Lemma 3.7, $(s_i+1) \mid \mathcal{B}_{(i),p}$.

Proof. For $\mathcal{B}_{(i)}$ and $s_i = -1$ the equation considered becomes

$$\frac{b(\hat{s}, s_i = -1)}{f_i} \hat{f}^{\hat{s}} = \delta(\hat{s}, s_i = -1) \bullet f_i^{1-1} \hat{f}^{\hat{s}} = \delta(\hat{s}, s_i = -1) \bullet \hat{f}^{\hat{s}}.$$

□

For $\mathcal{B}_{(j)}$ and $\mathcal{B}_{\Sigma}$ with $f_j(p) \neq 0$, the situation becomes even more comfortable.

Lemma 3.9. Let $1 \leq j \leq r$ such that $p \notin \mathbb{V}(f_j)$. Then $\mathcal{B}_{\Sigma,p} = \mathcal{B}_{(j),p} = \langle 1 \rangle$.

Proof. For the claim about $\mathcal{B}_{(j)}$, set $\delta(s) := f_j^{-1}$. With this

$$\delta(s) \bullet f_j f^s = f^s,$$

so $\mathcal{B}_{(j)} = \langle 1 \rangle$. For the claim about $\mathcal{B}_{\Sigma}$, we can use the functional equation

$$f^s = (f_j^{-1} \bullet f_j + \sum_{i \neq j} 0 \bullet f_i) f^s$$

to obtain $\mathcal{B}_{\Sigma} = \langle 1 \rangle$. □

The proof of Lemma 3.7 followed the classical structure of the proof of the fact that $(s+1) \mid b_f$ for $f \in \mathbb{C}[\underline{x}]$, but with the use of ϕ from Proposition 3.2 we can show an even stronger result.

Lemma 3.10. For $1 \leq i \leq r$ with $p \in \mathbb{V}(f_i) \setminus \mathbb{V}(\prod_{j \neq i} f_j)$, we have $\mathcal{B}_p = \langle b_{f_i,p}(s_i) \rangle$, where $b_{f_i,p}(s)$ denotes the Bernstein-Sato polynomial of f_i in p.

Proof. Remark 3.6 tells us that $\mathcal{B}_{f,p} = \mathcal{B}_{(1,\ldots,1,f_i,1,\ldots,1),p}$. The functional equation that needs to be fulfilled for membership on the right hand side is of the form

$$b(s) f_i^{s_i} = \delta(s) \bullet f_i^{s_i+1},$$

which directly implies the claim. □

Remark 3.11. The analogous result holds for $\mathcal{B}_{(i),p}$ and $p \in \mathbb{V}(f_i) \setminus \mathbb{V}(\prod_{j \neq i} f_j)$, but not necessarily for $\mathcal{B}_{\Sigma}$, see Remark 3.25 below.

Next, we want to find out how common factors of several of the f_i influence the Bernstein-Sato ideal.

Lemma 3.12. Let $f_i = f_{i,1}^{\alpha_{i,1}} \cdot \ldots \cdot f_{i,l_i}^{\alpha_{i,l_i}}$ for all $1 \leq i \leq r$ with $f_{i,j}$ irreducible for all $1 \leq i \leq r, 1 \leq j \leq l_i$. Furthermore, let $1 \leq i_0 \leq r, 1 \leq j_0 \leq l_{i_0}$ such that the factor f_{i_0,j_0} appears in this factorization only as $f_{i_0,j_0} = f_{i_1,j_1} = \ldots = f_{i_\ell,j_\ell}$, i.e. $f_{i_0,j_0} \mid f_i$ for $i \in \{i_0, \ldots, i_\ell\}$ and $f_{i_0,j_0} \nmid f_i$ for $i \notin \{i_0, \ldots, i_\ell\}$. Moreover let

$$p \in \mathbb{V}(f_{i_0,j_0}) \setminus \mathbb{V}\left(\prod_{(i,j) \notin \{(i_0,j_0),\ldots,(i_\ell,j_\ell)\}} f_{i,j} \right).$$

Then $\left(\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k} s_{i_k} \right) + m \right) \mid \mathcal{B}_p$ for all $m \in \mathbb{N}$ with $1 \leq m \leq \sum_{k=0}^{\ell} \alpha_{i_k,j_k}$.

Proof. Let m be as described and $b \in \mathcal{B}_p$. We set

$$s_{i_0} := -\frac{\left(\sum_{k=1}^{\ell} \alpha_{i_k,j_k} s_{i_k}\right) + m}{\alpha_{i_0,j_0}}.$$

With this, $\tilde{f}_{i_k} := \frac{f_{i_k}}{f_{i_k,j_k}^{\alpha_{i_k,j_k}}}$ and $\hat{f} := \prod_{i \notin \{i_1,\dots,i_\ell\}} f_i$ the functional equation of b becomes

$$\begin{aligned}
b(s) f_{i_0,j_0}^{-m} \hat{f}^{\hat{s}} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}} &= b(s) f_{i_0,j_0}^{-\left(\left(\sum_{k=1}^{\ell} \alpha_{i_k,j_k} s_{i_k}\right)+m\right)} \prod_{k=1}^{\ell} f_{i_k,j_k}^{\alpha_{i_k,j_k} s_{i_k}} \hat{f}^{\hat{s}} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}} \\
&= b(s) f_{i_0,j_0}^{\alpha_{i_0,j_0} s_{i_0}} \prod_{k=1}^{\ell} f_{i_k,j_k}^{\alpha_{i_k,j_k} s_{i_k}} \hat{f}^{\hat{s}} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}} \\
&= b(s) \left(\prod_{k=0}^{\ell} f_{i_k,j_k}^{\alpha_{i_k,j_k} s_{i_k}} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}} \right) \hat{f}^{\hat{s}} \\
&= b(s) f^s = \delta(s) \bullet f^{s+1} \\
&= \delta(s) \bullet f_{i_0,j_0}^{-\left(\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k}\right)+m\right)+\alpha_{i_0,j_0}} \hat{f}^{\hat{s}+1} \prod_{k=0}^{\ell} f_{i_k}^{s_{i_k}+1} \\
&= \delta(s) \bullet f_{i_0,j_0}^{-\left(\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k}\right)+m\right)+\alpha_{i_0,j_0}} \hat{f}^{\hat{s}+1} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}+1} \prod_{k=0}^{\ell} f_{i_0,j_0}^{\alpha_{i_k,j_k} s_{i_k}+\alpha_{i_k,j_k}} \\
&= \delta(s) \bullet f_{i_0,j_0}^{\overbrace{-m+\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k}\right)}^{\geq 0}} \hat{f}^{\hat{s}+1} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}+1}.
\end{aligned}$$

Now we equate the coefficients of $\hat{f}^{\hat{s}} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}}$. The important point here is that f_{i_0,j_0} appears in the denominator on the left hand side but not on the right hand side and by the choice of the s_i there is no different denominator on the right hand side. Thus, the polynomial $b(s)$ vanishes for this s_{i_0}, which implies the claim. $\square$

Remark 3.13. When considering $\mathcal{B}_{(i_k),p}$ for some $1 \leq k \leq \ell$, we can show the analogous result with an analogous proof, but in this case, we may choose m only such that $1 \leq m \leq \alpha_{i_k,j_k}$.

For $\mathcal{B}_\Sigma$, the result does not hold, for which we again refer to Remark 3.25 below.

We can obtain even more information about the primary components of the ideal $Q := (\operatorname{ann}(f^s) + \langle F \rangle) \cap \mathbb{C}[\underline{x}, \underline{s}]$ from the following proposition.

Proposition 3.14. For $g \in Q$ in the situation of Lemma 3.12 and $1 \leq m \leq \sum_{k=0}^{\ell} \alpha_{i_k,j_k}$, it holds that

$$\left(\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k} s_{i_k} \right) + m \right) \mid g \quad \text{or} \quad f_{i_0,j_0}^m \mid g.$$

Proof. Let $g \in Q$ with $P := \left(\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k} s_{i_k}\right) + m\right) \nmid g$. As $g \in Q$, it holds that $g \bullet f^s = gf^s \in \langle F \rangle f^s$, e.g. $gf^s = \delta \bullet f^{s+1}$.

We apply the restriction from Lemma 3.12 ($s_{i_0} := -\frac{\left(\sum_{k=1}^{\ell} \alpha_{i_k,j_k} s_{i_k}\right)+m}{a_{i_0,j_0}}$, $s_j \in \mathbb{N}$) and obtain completely analogously

$$g f_{i_0,j_0}^{-m} \hat{f}^{\hat{s}} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}} = \delta(s) \bullet f_{i_0,j_0}^{\left(\sum_{k=0}^{\ell} \alpha_{i_k,j_k}\right)-m} \hat{f}^{\hat{s}+1} \prod_{k=0}^{\ell} \tilde{f}_{i_k}^{s_{i_k}+1},$$

where f_{i_0,j_0}^m appears in the denominator of the left hand side but not of the right hand side. As, by assumption, $P \nmid g$, by the substitution it either holds that $f_{i_0,j_0}^m \mid g$, the desired statement, or that $\mathbb{V}(P) \supseteq \mathbb{V}(g)$, a contradiction since then $P \mid g$. □

Remark 3.15. With an analogous proof we can show the analogous result for $\mathcal{B}_{(i_k)}$ in the sense of Remark 3.13.

The following lemma specifies the relation of the primary ideals $\mathcal{B}_i$ and I_i.

Lemma 3.16. Let $1 \leq i \leq r$ and $\mathcal{B}_m = Q_m \cap \mathbb{C}[\underline{s}]$ be a primary component such that there exists $b \in \mathcal{B}_m$ of the form $b = b_1 b_2$ with $b_1 \in \mathbb{C}[s_i] \setminus \mathbb{C}$ and $b_2 \notin \mathcal{B}_m$ (i.e. $\mathcal{B}_m$ is not saturated at $\mathbb{C}[s_i]$). Then $\sqrt{I_m} \supseteq \sqrt{\langle f_i \rangle}$.

Proof. We show the claim by a proof by contrapositive.

Assume that $\sqrt{I_m} \not\supseteq \sqrt{\langle f_i \rangle}$. Let $p \in \mathbb{V}(I_m) \setminus \mathbb{V}(f_i)$.

As f_i is invertible in $\mathbb{C}[x]_p$, we can w.l.o.g. assume a functional equation of the form

$$b \prod_{j \neq i} f_j^{s_j} f_i^{s_i} = \delta \bullet f_i^{s_i} \prod_{j \neq i} f_j^{s_j+1}$$

and thanks to the ϕ_{f_j,s_j} from Proposition 3.2 even of the form

$$b \prod_{j \neq i} f_j^{s_j} f_i^{s_i} = f_i^{s_i} \delta \bullet \prod_{j \neq i} f_j^{s_j+1} \quad \Leftrightarrow \quad b \prod_{j \neq i} f_j^{s_j} = \delta \bullet \prod_{j \neq i} f_j^{s_j+1}.$$

In this form it is obvious that b and δ can only depend on s_i through common factors which can be left out, which implies that $\mathcal{B}$ is saturated at $\mathbb{C}[s_i]$. In particular, $\mathcal{B}_m$ is saturated at $\mathbb{C}[s_i]$. □

Corollary 3.17. Let $1 \leq i \leq r$ such that $p \notin \mathbb{V}(f_i)$. Then $\mathcal{B}_p$ is saturated at $\mathbb{C}[s_i]$.

Proof. Let p be as described. Then, for every primary component Q_m that appears non-trivially in the primary decomposition of $\left(\operatorname{ann}_{D_{n,p}[\underline{s}]}(f^s) + {}_{D_{n,p}[\underline{s}]}\langle F \rangle\right) \cap \mathbb{C}[\underline{x}]$, it holds that $\sqrt{I_m} \not\supseteq \sqrt{\langle f_i \rangle}$, because otherwise we would have $1 \in I_i \subseteq Q_i$. From Lemma 3.16 we conclude that $\mathcal{B}_m$ is saturated at $\mathbb{C}[s_i]$. Since m was chosen arbitrarily, this also holds for $\mathcal{B}_p$. □

Using Theorem 3.5, we can prove an even stronger result.

Lemma 3.18. Let $1 \leq i \leq r$ and $p \notin \mathbb{V}(f_i)$. Then $\mathcal{B}_p = {}_{\mathbb{C}[\underline{s}]}\langle G_1, \dots, G_e\rangle$ with $G_1, \dots, G_e \in \mathbb{C}[s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_r]$.

Proof. By Remark 3.6, we know that $\mathcal{B}_{f,p} = \mathcal{B}_{(f_1,\dots,f_{i-1},f_{i+1},\dots,f_r)}$. For a polynomial $b \in \mathcal{B}_{(f_1,\dots,f_{i-1},f_{i+1},\dots,f_r)}$, it is obvious that s_i can only appear in b by multiplication of another element of $\mathcal{B}_{(f_1,\dots,f_{i-1},f_{i+1},\dots,f_r)}$ with a polynomial that contains s_i. $\square$

Proposition 3.19. Let $f \in \mathbb{C}[\underline{x}]^r$ and $M \subseteq \{1, \dots, r\}$ such that

$$\underbrace{\left(\bigcup_{i\in M} \mathbb{V}(f_i)\right)}_{=:V_1} \cap \underbrace{\left(\bigcup_{i\notin M} \mathbb{V}(f_i)\right)}_{=:V_2} = \varnothing.$$

Then it holds that $\mathcal{B} = \mathcal{B}_1 \cdot \mathcal{B}_2$, where $\mathcal{B}_j$ denotes the Bernstein-Sato ideal of F_j for

$$(F_1)_i = \begin{cases} f_i, & i \in M, \\ 1, & i \notin M, \end{cases} \qquad (F_2)_i = \begin{cases} 1, & i \in M, \\ f_i, & i \notin M. \end{cases}$$

Proof. Let $p \in V_1 = V_1 \setminus V_2$. Due to Lemma 3.18 we can choose a generating set $G_p \subseteq \mathbb{C}[s_i \mid i \in M]$ of $\mathcal{B}_p$. On the other hand we can analogously choose a generating set $G_q \subseteq \mathbb{C}[s_i \mid i \notin M]$ of $\mathcal{B}_q$ for $q \in V_2 = V_2 \setminus V_1$. With this

$$\begin{aligned}\mathcal{B} &= \bigcap_{p\in\mathbb{V}(f)} \mathcal{B}_p = \bigcap_{p\in V_1} \mathcal{B}_p \cap \bigcap_{q\in V_2} \mathcal{B}_q \\ &= \underbrace{\left(\mathbb{C}[\underline{s}] \bigcap_{p\in V_1} {}_{\mathbb{C}[s_i|i\in M]}\langle G_p\rangle\right)}_{=:\tilde{\mathcal{B}}_1} \cap \underbrace{\left(\mathbb{C}[\underline{s}] \bigcap_{q\in V_2} {}_{\mathbb{C}[s_i|i\notin M]}\langle G_q\rangle\right)}_{=:\tilde{\mathcal{B}}_2} = \tilde{\mathcal{B}}_1 \cdot \tilde{\mathcal{B}}_2.\end{aligned}$$

It holds that $\tilde{\mathcal{B}}_1 = \mathcal{B}_1$, because for $p \in V_1$ the functional equation

$$b \prod_{i\in M} f_i^{s_i} \prod_{i\notin M} f_i^{s_i} = \delta \bullet \prod_{i\in M} f_i^{s_i+1} \prod_{i\notin M} f_i^{s_i+1}$$

can be transfered through application of the ϕ_{f_i,s_i} from Proposition 3.2 for $i \notin M$ and right multiplication of δ with $\prod_{i\notin M} f_i^{-1}$ to

$$b \prod_{i\in M} f_i^{s_i} \prod_{i\notin M} f_i^{s_i} = \prod_{i\notin M} f_i^{s_i} \delta \bullet \prod_{i\in M} f_i^{s_i+1} \quad \Leftrightarrow \quad b \prod_{i\in M} f_i^{s_i} = \delta \bullet \prod_{i\in M} f_i^{s_i+1},$$

which is the functional equation of $\mathcal{B}_{1,p}$. In $q \in V_2$, we have $\mathcal{B}_{1,q} = \tilde{\mathcal{B}}_{1,q} = \langle 1\rangle$, so $\tilde{\mathcal{B}}_1 = \mathcal{B}_1$.

Analogously it follows that $\tilde{\mathcal{B}}_2 = \mathcal{B}_2$, which shows the claim. $\square$

Now we apply this result to an example.

Example 3.20. Consider the pair of two cuspidal curves given by

$$f = (x^2 - y^3, (y-1)^3 - x^2) \in \mathbb{C}[x,y]^2.$$

As $\mathbb{V}(f_1)$ and $\mathbb{V}(f_2)$ are connected by a linear transformation, they share the same Bernstein-Sato polynomial $b_{f_1}(s) = b_{f_2}(s) = \frac{1}{36}(s+1)(6s+5)(6s+7)$. By Proposition 3.19, the local Bernstein-Sato ideal of f for p not from the four intersection points (in particular for real p) is given by $\mathcal{B}_p = \langle b_{f_1,p}(s_1) \cdot b_{f_2,p}(s_2)\rangle$, which is a principal ideal in particular.

In order to treat more interesting examples in which the irreducible components of $\mathbb{V}(F)$ intersect, we need to consider tangent spaces in points of intersection.

Definition 3.21. Let $f \in \mathbb{C}[\underline{x}]$ and $p \in \mathbb{V}(f)$. The *tangent space* of $\mathbb{V}(f)$ at p is defined as

$$T_p(\mathbb{V}(f)) := \ker(J_f)(p) \subseteq \mathbb{C}^n,$$

where J_f denotes the Jacobian matrix $J_f = (\partial_1 \bullet f, \ldots, \partial_n \bullet f) \in \mathbb{C}[\underline{x}]^{1\times n}$.

For both Bernstein-Sato polynomials and Bernstein-Sato ideals, the singular locus of f plays an important role.

Definition 3.22. For $f \in \mathbb{C}[\underline{x}]$, the *singular locus* is defined as

$$\mathrm{Sing}(f) := \mathbb{V}(\langle f, \partial_1 \bullet f, \ldots, \partial_n \bullet f\rangle) = \{p \in \mathbb{V}(f) \mid T_p(\mathbb{V}(f)) = \mathbb{C}^n\}.$$

With these concepts, we can give a proof of the following, classical result about Bernstein-Sato polynomials.

Lemma 3.23. Let $F \in \mathbb{C}[\underline{x}]$. For $p \in \mathbb{V}(F) \setminus \mathrm{Sing}(F)$, the Bernstein-Sato polynomial of F in p is given by $b_{F,p}(s) = s+1$.

Proof. Let p be as described and $v \in \mathbb{C}^n$ such that $v \notin T_p(\mathbb{V}(f)) = \ker_{\mathbb{C}}((J_F)(p))$. We set $\delta(x,s) := \sum_{i=1}^n v_i \partial_i$, a differential operator that is homogeneous of order 1 in the ∂_i (i.e. a derivation) and has constant coefficients. Applying δ to F^{s+1} yields

$$\delta \bullet F^{s+1} = (s+1)F^s(\delta \bullet f) = (s+1)F^s \underbrace{((J_f)(x)v)}_{\text{unit in } \mathbb{C}[\underline{x}]_p},$$

so $\frac{1}{((J_F)(x)v)}\delta$ is a Bernstein-Sato operator that shows $b_{F,p} \mid (s+1)$. By Lemma 3.7, we know that $(s+1) \mid b_{F_p}$, which shows the claim and additionally that the Bernstein-Sato operator can be chosen to be a derivation in $\bigoplus_{i=1}^n \mathbb{C}[\underline{x}]_p \partial_i$. $\square$

Corollary 3.24. Combining the previous result with Lemma 3.10, we obtain that for $1 \le i \le r$ with $p \in \mathbb{V}(f_i) \setminus \left(\bigcup_{j\neq i} \mathbb{V}(f_j) \cup \mathrm{Sing}(\mathbb{V}(f_i))\right)$ it holds that $\mathcal{B}_p = \langle s_i + 1\rangle$. In this situation, we also have $\mathcal{B}_{(i),p} = \langle s_i + 1\rangle$.

Remark 3.25. This statement does not hold for $\mathcal{B}_{\Sigma,p}$. We consider the example $f = (x, 1-x)$. Here, $1 \in {}_{\mathbb{C}[x]}\langle f_1, f_2\rangle$, so we naturally obtain $\mathcal{B}_{\Sigma,p} = \langle 1\rangle$ for all $p \in \mathbb{C}$, i.e. in particular for $p \in \mathbb{V}(F)$.

The following definition allows us to consider a type of intersection which has more convenient properties.

Definition 3.26 ([EH10])**.** Two vanishing sets $\mathbb{V}(f), \mathbb{V}(g)$ for $f, g \in \mathbb{C}[\underline{x}]$ irreducible *intersect transversally* at $p \in \mathbb{V}(f) \cap \mathbb{V}(g)$ if

$$T_p(\mathbb{V}(f)) \oplus T_p(\mathbb{V}(g)) = \mathbb{C}^n.$$

We extend the definition to reducible f, g by allowing also such f, g with only one irreducible factor vanishing at p, i.e. $f = \hat{f}\tilde{f}$ with $\hat{f}$ irreducible, $\hat{f}(p) = 0$ and $\tilde{f}(p) \neq 0$ and analogous g.

The following lemma shows that common factors do not directly contribute to a transversal intersection.

Lemma 3.27. Assume that $f \in \mathbb{C}[\underline{x}]$ and $g \in \mathbb{C}[\underline{x}]$ have a common factor $h \in \mathbb{C}[\underline{x}]$ and intersect transversally at $p \in \mathbb{C}^n$. Then $h(p) \neq 0$.

Proof. First, we define $\hat{f} := \frac{f}{h}$ and $\hat{g} := \frac{g}{h}$. We calculate the Jacobian matrices by the Leibniz rule as

$$J_f = J_h \cdot \hat{f} + J_{\hat{f}} \cdot h \text{ and } J_g = J_h \cdot \hat{g} + J_{\hat{g}} \cdot h.$$

Assume that $p \in \mathbb{V}(h)$ and $\hat{f}(p) \neq 0$, $\hat{g}(p) \neq 0$. Then the Jacobians become

$$J_f(p) = J_h(p) \cdot \underbrace{\hat{f}(p)}_{\in\mathbb{C}} \text{ and } J_g(p) = J_h(p) \cdot \underbrace{\hat{g}(p)}_{\in\mathbb{C}},$$

but these have the same kernel because the constant factors do not vanish, so the intersection cannot be transversal. □

Remark 3.28. For $n = 2$ and $p \in \mathbb{V}(f_1) \cap \mathbb{V}(f_2)$ non-singular on both $\mathbb{V}(f_1)$ and $\mathbb{V}(f_2)$, the transversality condition $T_p\mathbb{V}(f_1) + T_p\mathbb{V}(f_2) = \mathbb{C}^n$ specializes to $T_p\mathbb{V}(f_1) \neq T_p\mathbb{V}(f_2)$. In this case, the construction of the proof of Lemma 3.23 for both f and g allows us to obtain obtain a shared basis $B = \{j_1, j_2\} \subseteq \mathbb{C}^2$ such that ${}_{\mathbb{C}}\langle j_1\rangle = \ker(J_{f_1}(p))$, ${}_{\mathbb{C}}\langle j_2\rangle = \ker(J_{f_2}(p))$ and corresponding operators $\delta_1(s_1) \in \mathbb{C}[\underline{x}]_p[s_1]\langle j_2^T(\partial_1, \partial_2)^T\rangle$, $\delta_2(s_2) \in \mathbb{C}[\underline{x}, s_1]\langle j_1^T(\partial_1, \partial_2)^T\rangle$. Then

$$\begin{aligned}
\delta_1(s_1)\delta_2(s_2) \bullet f_1^{s_1+1}f_2^{s_2+1} &= \delta_1(s_1) \bullet (\delta_2(s_2) \bullet f_1^{s_1+1}f_2^{s_2+1}) \\
&= \delta_1(s_1) \bullet ((\delta_2(s_2) \bullet f_1^{s_1+1}) \cdot f_2^{s_2+1} + f_1^{s_1+1} \cdot (\delta_2(s_2) \bullet f_2^{s_2+1})) \\
&= \delta_1(s_1) \bullet (f_1^{s_1+1} \cdot (\delta_2(s_2) \bullet f_2^{s_2+1})) \\
&= \delta_1(s_1) \bullet (f_1^{s_1+1} \cdot (b_{f_2}(s)f_2^{s_2})) \\
&= b_{f_2}(s_2)\delta_1(s_1) \bullet (f_1^{s_1+1}f_2^{s_2}) \\
&= b_{f_2}(s_2)((\delta_1(s_1) \bullet f_1^{s_1+1}) \cdot f_2^{s_2} + (\delta_1(s_1) \bullet f_2^{s_2}) \cdot f_1^{s_1+1}) \\
&= b_{f_2}(s_2)(\delta_1(s_1) \bullet f_1^{s_1+1}) \cdot f_2^{s_2} = b_{f_1}(s_1)b_{f_2}(s_2)f_1^{s_1}f_2^{s_2} \\
&= (s_1+1)(s_2+1)f_1^{s_1}f_2^{s_2}.
\end{aligned}$$

We can generalize this in the following lemma.

Lemma 3.29. Let $p \in \mathbb{V}(f_1) \cap \mathbb{V}(f_2)$ such that $T_p\mathbb{V}(f_1) \oplus T_p\mathbb{V}(f_2) = \mathbb{C}^n$ and consider $f = (f_1, f_2)$. Then $\mathcal{B}_{f,p} = \langle b_{f_1}(s_1) b_{f_2}(s_2) \rangle$.

Proof. The claim follows analogously as the previous remark. $\square$

Remark 3.30. In the more general case with $p \in \bigcap_i \mathbb{V}(f_i)$, $T_p\mathbb{V}(f_j) \oplus \left(\sum_{i \neq j} T_p\mathbb{V}(f_i)\right) = \mathbb{C}^n$ and $T_p\mathbb{V}(f_{i_1}) = T_p\mathbb{V}(f_{i_2})$ for all $i_1 \neq j \neq i_2$, the result still holds as

$$\mathcal{B}_p = \langle b_{f_j}(s_j) \rangle \cdot \tilde{\mathcal{B}}_p,$$

where $\tilde{\mathcal{B}}_p$ is the Bernstein-Sato ideal of $(f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_n)$ in the variable set $(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_n)$.

Remark 3.31. For $\mathcal{B}_{(j)}$ with $1 \leq j \leq r$, the statement of Remark 3.30 becomes simpler, since then, $\tilde{\mathcal{B}}_p = \langle 1 \rangle$ by Remark 3.6 and

$$\mathcal{B}_p = \langle b_{f_j}(s_j) \rangle.$$

Example 3.32. With the instruments previously developed, we can treat the example $f = (x, y, 1 - x - y) \in \mathbb{C}[x, y]$ with the points of intersection $(0,0)$ of f_1, f_2, $(1,0)$ of f_2, f_3 and $(0,1)$ of f_1, f_3. The tangent spaces are $T_p\mathbb{V}(f_1) = \mathbb{C}(1,0)^T$, $T_p\mathbb{V}(f_2) = \mathbb{C}(0,1)^T$ and $T_p\mathbb{V}(f_3) = \mathbb{C}(1,1)^T$ for all p on the varieties, so in the intersection points the two relevant tangent spaces form a basis. As the f_i are smooth, we get

$$\mathcal{B} = \bigcap_{p \in \mathbb{V}(f_1 f_2 f_3)} \mathcal{B}_p \overset{\text{smoothness}}{=} \bigcap_{p \in \{(0,0),(1,0),(0,1)\}} \mathcal{B}_p \overset{\text{Lemma 3.29}}{=} (s_1 + 1)(s_2 + 1)(s_3 + 1).$$

For intersections with each pair of components intersecting transversally, the situation becomes more complex, which we can see in the following example.

Example 3.33. Consider $f = (x, y, x + y) \in \mathbb{C}[x, y]^3$. Then

$$\mathcal{B} = \mathcal{B}_{(0,0)} = \langle (s_1 + 1)(s_2 + 1)(s_3 + 1)(s_1 + s_2 + s_3 + 2)(s_1 + s_2 + s_3 + 3)(s_1 + s_2 + s_3 + 4) \rangle.$$

Observations in examples like this one lead us to the following conjecture for a special case.

Conjecture 3.34. Let $n = 2$, $r = 3$, $\ker(J_{f_i}(p)) \cap \ker(J_{f_j}(p)) = \{0\}$ for all $i \neq j$ and p a smooth point of $\mathbb{V}(f_i)$ for all i. Then

$$\mathcal{B}_p = \langle (s_1 + 1)(s_2 + 1)(s_3 + 1)(s_1 + s_2 + s_3 + 2)(s_1 + s_2 + s_3 + 3)(s_1 + s_2 + s_3 + 4) \rangle.$$

We can show the following lemma which is a far weaker version.

Lemma 3.35. Let $n = 2$, $r = 3$, $\ker(J_{f_i}(p)) \cap \ker(J_{f_j}(p)) = \{0\}$ for all $i \neq j$ and p a smooth point of $\mathbb{V}(f_i)$ for all i. Then

$$(s_1+1)(s_2+1)(s_3+1) \mid \mathcal{B}_p$$

and for $k \in \{2,3,4\}$, $b(s) \in \mathcal{B}_p$ it holds that

$$\Big(\big(s_2 \mid b(s) \ \vee \ (-s_1-s_3-k) \mid b(s)\big) \ \wedge \ \big(s_3 \mid b(s) \ \vee \ (-s_1-s_2-k) \mid b(s)\big) \Big) \\ \vee \ (s_1+s_2+s_3+k) \mid b(s).$$

Proof. The claim about the s_i+1 follows from Lemma 3.7. We proceed similarly to the constructions in the proof of Lemma 3.12. For some $2 \le k \le 4$ we set $s_1 := -s_2-s_3-k$ and restrict the other s_i to values from $\mathbb{N}$. The defining equation of the Bernstein-Sato ideal becomes

$$b(-s_2-s_3-k,\ldots)f_1^{-s_2-s_3-k}f_2^{s_2}f_3^{s_3} = \delta(-s_2-s_3-k,\ldots) \bullet f_1^{-s_2-s_3-k+1}f_2^{s_2+1}f_3^{s_3+1}, \quad (6)$$

where δ is without poles at p.

Let the tangent spaces of the $\mathbb{V}(f_i)$ in the smooth point p be given by $\ker(J_{f_i}(p)) = \langle j_i \rangle \subseteq \mathbb{C}^2$.

It holds that $g_i(t) := f_i(p+tj_i) \in \mathbb{C}[t]$ has a root of order at least 2 in $t=0$, since $g_i(0) = 0$ and

$$\frac{\partial}{\partial t} g_i(t) = J_{f_i}(p+tj_i)j_i$$

has a root in 0 as well. On the other hand, $g_j(t) := f_j(p+tj_i) \in \mathbb{C}[t]$ for $j \neq i$ has – by assumption – a root of order 1 in 0. We apply this by considering $g_1(t) := f_1(p+tj_2)$, $g_2(t) := f_2(p+tj_2)$ and $g_3(t) := f_1(p+tj_2)$. In (6) with $x := p_1+tj_{2,1}$ and $y := p_2+tj_{2,2}$, there exists an $\ell \in \mathbb{N}$, $\ell \ge 2$ such that

$$\begin{aligned} & b(-s_2-s_3-k,s_2,s_3) \quad \underbrace{g_1^{-s_2-s_3-k}}_{\text{pole of order } s_2+s_3+k,} \quad \underbrace{g_2^{s_2}}_{\text{root of order } \ell s_2,} \quad \underbrace{g_3^{s_3}}_{\text{root of order } s_3} \\ & = \underbrace{\delta(-s_2-s_3-k,s_2,s_3)}_{\text{no pole in } t=0} \bullet \quad \underbrace{g_1^{-s_2-s_3-k+1}}_{\text{pole of order } s_2+s_3+k-1,} \quad \underbrace{g_2^{s_2+1}}_{\text{root of order } \ell(s_2+1),} \quad \underbrace{g_3^{s_3+1}}_{\text{root of order } s_3+1} . \end{aligned}$$

We will now show that there is a $\tilde{s}_2$ for which the left hand side has a pole and the right hand side has none, implying $b(-\tilde{s}_2-s_3-k,\tilde{s}_2,s_3) = 0$ as a polynomial. The exponents have to fulfill

$$\begin{aligned} -(s_2+s_3+k)+\ell s_2+s_3 < 0 \quad &\wedge \quad -(s_2+s_3+k-1)+\ell(s_2+1)+s_3+1 \ge 0 \\ \Leftrightarrow \quad s_2(\ell-1) < k \quad &\wedge \quad s_2(\ell-1) \ge k-\ell-2 \\ \Leftrightarrow \quad & k-\ell-2 \le s_2(\ell-1) < k \\ \Leftarrow \quad & 4-\ell-2 \le s_2(\ell-1) < 2 \\ \Leftrightarrow \quad & 2-\ell \le s_2(\ell-1) < 2 \end{aligned}$$

which is fulfilled for $\tilde{s}_2 = 0$. In this case, we obtain $b(-\tilde{s}_2 - s_3 - k, \ldots) = 0$.

Analogously, we get $b(-s_2 - \tilde{s}_3 - k, s_2, \tilde{s}_3) = 0$ for $\tilde{s}_3 = 0$, so

$$s_2 s_3 \mid b(-s_2 - s_3 - k, s_2, s_3),$$

which implies $s_2 \mid b(s)$ or, with the substitution of s_1, $(-s_1 - s_3 - k) \mid b(s)$, or $b(-s_2 - s_3 - k, s_2, s_3)$ which implies $(s_1 + s_2 + s_3 + k) \mid b(s)$, so combined

$$s_2 \mid b(s) \quad \vee \quad (-s_1 - s_3 - k) \mid b(s) \quad \vee \quad (s_1 + s_2 + s_3 + k) \mid b(s)$$

and the analogue for s_3. $\square$

Remark 3.36. If we use substitutions $s_2 := -s_1 - s_3 - k$ and $s_3 := -s_1 - s_2 - k$ instead of $s_1 := -s_2 - s_3 - k$ in the proof of Lemma 3.35, for $b(s) \in \mathcal{B}_p$ we obtain additional results about factors which combined are equivalent to

$$\Big(\big(s_1 \mid b(s) \vee (-s_2 - s_3 - k) \mid b(s)\big) \wedge \big(s_2 \mid b(s) \vee (-s_1 - s_3 - k) \mid b(s)\big) \wedge \big(s_3 \mid b(s) \vee (-s_1 - s_2 - k) \mid b(s)\big) \Big) \vee (s_1 + s_2 + s_3 + k) \mid b(s).$$

In an attempt to eliminate the undesired factors s_1, s_2, s_3 as options we mention a conjecture by Budur about the form of the elements of the Bernstein-Sato ideal.

Conjecture 3.37 ([Bud12])**.** Let $f \in \mathbb{C}[\underline{x}], p \in \mathbb{C}^n$. There exists a generating system of $\mathcal{B}$ such that all generators b have the form

$$b = \prod_i (a_{i,1}s_1 + \ldots + a_{i,r}s_r + b_i)$$

with $a_{i,j} \in \mathbb{N}_0$ and $b_i \in \mathbb{Q}_{>0}$ for all i and $1 \leq j \leq r$.

This would imply that s_1 and s_2 are non-viable factors of $\mathcal{B}_p$. What is known so far is the following theorem, which guarantees at least one element of this form.

Theorem 3.38 ([Gyo93])**.** Let $f \in \mathbb{C}[\underline{x}], p \in \mathbb{C}^n$. There exists $b \in \mathcal{B}_p$ of the form

$$b = \prod_i (a_{i,1}s_1 + \ldots + a_{i,r}s_r + b_i)$$

with $a_{i,j} \in \mathbb{N}_0$ and $b_i \in \mathbb{Q}_{>0}$ for all i and $1 \leq j \leq r$.

For the case that we are interested in, we can conclude that $s_1, s_2 \nmid \mathcal{B}_p$, but this does not help with the proof of Conjecture 3.34, since s_1 and s_2 may still be factors of elements of a generating system of $\mathcal{B}_p$.

Now, we treat a different kind of intersection in the following proposition which generalizes Remark 3.28.

Proposition 3.39. Let $f = (f_1, \ldots, f_r) \in \mathbb{C}[\underline{x}]^r$ such that $f_1, \ldots, f_r$ intersect at $p \in \mathbb{C}^n$, p is a smooth point of $\mathbb{V}(f_i)$ for all i and the normal vectors of $T_p(\mathbb{V}(f_1)), \ldots, T_p(\mathbb{V}(f_r))$ are linearly independent. Then $\mathcal{B}_{f,p} = \prod_{i=1}^r (s_i + 1)$.

Proof. We remark that necessarily $r \leq n$ since otherwise the intersection could not have the desired form.

We proceed similarly as in Remark 3.28 by constructing a suitable basis of $\mathbb{C}^n$ and an associated generating system of D_n as $\mathbb{C}[\underline{x}]$-algebra. For this, we need vectors $j_i \in \mathbb{C}^n$ with $j_i \in T_p(\mathbb{V}(f_k))$ for all $k \neq i$ and $j_i \notin T_p(\mathbb{V}(f_i))$ for $1 \leq i \leq r$ and $j_i \in \bigcap_{k=1}^r T_p(\mathbb{V}(f_k))$ for $r+1 \leq i \leq n$.

Since the tangent spaces are hyperplanes in $\mathbb{C}^n$ and the tangent spaces of the $\mathbb{V}(f_i)$ at p are pairwise unequal, we conclude from linear algebra that $r-1$ of the tangent spaces intersect in a variety of dimension $n-r+1$. By this, we can construct vectors $j_1, \dots, j_r$ with $j_i \in \bigcap_{j \neq i} T_p(\mathbb{V}(f_j)) \setminus T_p(\mathbb{V}(f_i))$. We extend them to a basis of $\mathbb{C}^n$ through vectors $j_{r+1}, \dots, j_n \in \bigcap_i T_p(\mathbb{V}(f_i))$.

From $\{j_1, \dots, j_n\}$ we construct a basis $\{d_1, \dots, d_n\}$ of the $\mathbb{C}[\underline{x}]_p$-module

$$\{\delta \in D_{n,p} \mid \delta \text{ homogeneous of order } 1\}$$

via $d_i := \sum_{k=1}^n (j_i)_k \partial_k$.

From Lemma 3.23 we conclude that we can choose Bernstein-Sato operators $\delta_i \in {}_{S_p^{-1}\mathbb{C}[\underline{x},\underline{s}]}\langle d_i \rangle$ with

$$(s+1) f_i^s = \delta_i \bullet f_i^{s+1}$$

for $1 \leq i \leq n$, because

$$d_c \bullet f_i = \sum_{k=1}^n (j_c)_k \partial_k f_i = J_{f_i}(p) j_c \overset{j_c \in T_p(\mathbb{V}(f_i))}{=} 0$$

for all $c \neq i$.

With the Leibniz rule we conclude that

$$\begin{aligned}
&\delta_1 \cdot \ldots \cdot \delta_k \bullet f_1 \cdot \ldots \cdot f_k f^s \\
&= \delta_1 \cdot \ldots \cdot \delta_{k-1} \bullet \left(\sum_{i=1}^{k-1} \underbrace{(\delta_k \bullet f_i^{s_i+1})}_{=0} \frac{1}{f_i^{s_i+1}} + \sum_{i=k+1}^{r} \underbrace{(\delta_k \bullet f_i^{s_i})}_{=0} \frac{1}{f_i^{s_i}} + \underbrace{(\delta_k \bullet f_k^{s_k+1})}_{=(s_k+1) f_k^{s_k}} \frac{1}{f_k^{s_k+1}} \right) \\
&\qquad \cdot f_1 \cdot \ldots \cdot f_k f^s \\
&= (s_k+1) \delta_1 \cdot \ldots \cdot \delta_{k-1} \bullet f_1 \cdot \ldots \cdot f_{k-1} f^s
\end{aligned}$$

and thus inductively

$$\delta_1 \cdot \ldots \cdot \delta_r \bullet f^{s+1} = (s_1+1) \cdot \ldots \cdot (s_r+1) f^s.$$

□

Remark 3.40. With Theorem 3.5 we conclude that the analogue of the previous proposition for $(f_1, \dots, f_r)$ with $\mathbb{V}(f_1), \dots, \mathbb{V}(f_k)$ intersecting at $p \in \mathbb{C}^n$ for a smooth p on $\mathbb{V}(f_i)$ for all $1 \leq i \leq k$ such that the normal vectors of $T_p(\mathbb{V}(f_1)), \dots, T_p(\mathbb{V}(f_r))$ are linearly independent holds as well as

$$b(s) = (s_1+1) \cdot \ldots \cdot (s_k+1).$$

Example 3.41. We can apply these results to the example

$$f = (x - y^7, x^3 - y, x^2 - y^3 + z) \in \mathbb{C}[x, y, z]^3.$$

We remark that $\mathbb{V}(f_1), \mathbb{V}(f_2)$ and $\mathbb{V}(f_3)$ are smooth and their points of intersection fulfill the requirements of the previous remark. Thus, all local Bernstein-Sato ideals are principal and their generators are products of $s_1 + 1, s_2 + 1$ and $s_3 + 1$, making the global Bernstein-Sato ideal principal as well with generator $(s_1 + 1)(s_2 + 1)(s_3 + 1)$.

3.3. Ucha-Enríquez's conjecture

In [And14, 4.4.1], the following conjecture by Ucha-Enríquez was shown for the case that $\operatorname{ann}(f^s) = \operatorname{ann}^1(f^s)$, where

$$\operatorname{ann}^1(f^s) := \left\{ \delta \in \operatorname{ann}_{D_n[\underline{s}]}(f^s) \;\middle|\; \delta = \sum_{i=1}^{n} c_i \partial_i \text{ for some } c_i \in \mathbb{C}[\underline{x}, \underline{s}] \right\}.$$

Conjecture 3.42 (Ucha-Enríquez's conjecture, see [And14, 4.47], global case)**.** Let $f = (f_1, \ldots, f_r) \in \mathbb{C}[\underline{x}]^r$ and $F = \prod_{i=1}^r f_i$. Denote by φ the ring homomorphism

$$D_n \left[\underline{s}, \frac{1}{F} \right] \to D_n \left[s, \frac{1}{F} \right], \quad s_j \mapsto s, x_i \mapsto x_i, \partial_i \mapsto \partial_i \text{ for } 1 \leq j \leq r, 1 \leq i \leq n.$$

Then

$$\varphi(\operatorname{ann}_{D_n[\underline{s}]}(\prod_{i=1}^{r} f_i^{s_i})) = \operatorname{ann}_{D_n[s]}(F^s).$$

We will now show that, even if the conjecture holds in general, it does not imply

$$\varphi(\mathcal{B}_f) = \langle b_F(s) \rangle$$

by considering a class of examples which further restricts the limitations from Proposition 3.19.

Lemma 3.43. If the vanishing sets of the f_i are pairwise disjoint and $r > 1$, it holds that

$$(\mathcal{B}_F)|_{s_i = s} \subsetneq \langle b_{f_1 \cdot \ldots \cdot f_r}(s) \rangle \tag{7}$$

and

$$\sqrt{(\mathcal{B}_F)|_{s_i = s}} = \sqrt{\langle b_{f_1 \cdot \ldots \cdot f_r}(s) \rangle}.$$

Proof. In (7), '$\subseteq$' obviously always holds, as the functional equation on the left hand side imposes more restrictions than the one on the right hand side.

We will now show that this is a proper inclusion and the radicals are equal. By Proposition 3.19, we have

$$\mathcal{B}_F = \langle b_{f_1}(s_1) \cdot \ldots \cdot b_{f_r}(s_r) \rangle$$

and with Lemma 3.7 we know that $b_{f_j}(s_j) = (s_j+1)^{\mu_j} \cdot \tilde{b}_j$ for some $\mu_j \in \mathbb{N}$ and some $\tilde{b}_j \in \mathbb{C}[s_j]$ with $(s_j+1) \nmid \tilde{b}_j$.

On the other hand, if we choose $p \in \mathbb{V}(f_i) = \mathbb{V}(f_i) \setminus \bigcup_{j \neq i} \mathbb{V}(f_j)$, in the univariate case we get a functional equation of the form

$$b_{f_1 \cdot \ldots \cdot f_r, p}(s) f^s = \delta(s) \bullet f^{s+1}.$$

Applying $\phi_{\frac{f}{f_i}, s}$ from Proposition 3.2 and multiplying δ by $\frac{f_i}{f}$ from the right yields

$$b_{f_1 \cdot \ldots \cdot f_r, p}(s) f_i^s = \tilde{\delta}(s) f_i^{s+1}.$$

In the multivariate case, we get a functional equation of the form

$$b_{f,p}(\underline{s}) f^{\underline{s}} = \delta(\underline{s}) f^{\underline{s}+1}$$

and iteratively apply ϕ_{f_j, s_j} for $j \neq i$ and multiply δ by $\frac{f_i}{f}$, obtaining the same functional equation

$$b_{f,p}(\underline{s}) f_i^{s_i} = \tilde{\delta}(\underline{s}) f_i^{s_i+1}.$$

Thus, $\mathcal{B}_p = \langle b_{f,p}(s_i) \rangle$. Now we have

$$\begin{aligned} \mathcal{B} = \bigcap_{p \in \mathbb{V}(F)} \mathcal{B}_p \overset{\mathcal{B}_p \subseteq \mathbb{C}[s_i]}{\underset{\text{for } p \in \mathbb{V}(f_i)}{=}} \prod_{p \in \mathbb{V}(F)} \mathcal{B}_p = \langle \prod_{i=1}^{r} b_{f_i}(s_i) \rangle, \\ \langle b_F \rangle = \bigcap_{p \in \mathbb{V}(F)} \langle b_p(s) \rangle = \bigcap_{p \in \mathbb{V}(F)} \varphi\left(\mathcal{B}_p\right) = \mathrm{lcm}_{i=1,\ldots,r}(b_{f_i}(s)), \end{aligned}$$

where we obtain equality of the radicals, whereas the ideals themselves form a strict inclusion, which we can see by considering the factor $(s+1)^{\mu_1+\ldots+\mu_r} \neq (s+1)^{\max_i(\mu_i)}$, since $\mu_i > 0$ for all i. $\square$

Example 3.44. If we consider $F = (x, x+1) \in \mathbb{C}[x]$, we obtain $\mathcal{B}_F = \langle (s_1+1)(s_2+1) \rangle$ but $b_{f_1 \cdot f_2}(s) = s+1$.

Remark 3.45. It is obvious that the problems mentioned above already arise when we can partition $\{1, \ldots, r\} = I \cup J$ such that $\left(\bigcup_{i \in I} \mathbb{V}(f_i) \right) \cap \left(\bigcup_{j \in J} \mathbb{V}(f_j) \right) = \varnothing$ and the Bernstein-Sato ideals for $\prod_{i \in I} f_i$ and $\prod_{j \in J} f_j$ have common factors after the substitution $s_i \mapsto s$.

Example 3.46. An example for this observation is $f = (y, y - x^2 - 1, y + 2)$ with

$$\mathcal{B} = \langle (s_1+1)(s_2+1)(s_3+1)\rangle, \quad \mathcal{B}_{(f_1,f_2)} = \langle (s_1+1)(s_2+1)\rangle, \quad b_F = (s+1)^2.$$

A more complex example that shall hint us at further steps is $F = (x, x+1, y, y+1) \in \mathbb{C}[x,y]^4$ with

$$\mathcal{B} = \langle (s_1+1)(s_2+1)(s_3+1)(s_4+1)\rangle, \quad b_F = (s+1)^2,$$

where at each point of intersection of the irreducible components of $\mathbb{V}(F)$ only two of the f_i vanish.

The following lemma is included here although it does not contribute to the goal of showing Ucha-Enríquez's conjecture because it arose during the attempt of doing so. It deals with the order and total degree of the application of differential operators and can be seen as a step towards a general formula for the application of differential operators (for such a formula, compare e.g. [And14, 4.59]).

For $\delta = \sum_{\alpha,\beta} p_{\alpha,\beta} s^\alpha \partial^\beta$ with $p_{\alpha,\beta} \in \mathbb{C}[\underline{x}]$ we use the notations

$$\operatorname{ord}(\delta) = \max\{|\beta| \mid p_{\alpha,\beta} \neq 0 \text{ for some } \alpha\}$$

and

$$\operatorname{tdeg}_s(\delta) = \max\{|\alpha| \mid p_{\alpha,\beta} \neq 0 \text{ for some } \beta\}.$$

Lemma 3.47. If $b(s) \in \mathbb{C}[\underline{s}]$ and $\delta(s) \in D_n[\underline{s}]$ such that $b(s)f^s = \delta(s) \bullet f^{s+1}$, then $\operatorname{tdeg}_s(b) \leq \operatorname{ord}(\delta) + \operatorname{tdeg}_s(\delta)$.

Proof. We show the claim

$$\operatorname{tdeg}_s(\tilde{b}) \leq \operatorname{ord}(\delta) + \operatorname{tdeg}_s(\delta) + \operatorname{tdeg}_s(\hat{f})$$

for the more general case of $\tilde{b}(s) = \hat{b}f^s \in \mathbb{C}[\underline{x},\underline{s}]f^s$ with $\hat{b} \in \mathbb{C}[\underline{x},\underline{s}]$, $\tilde{f} = \hat{f}f^s \in \mathbb{C}[\underline{x},\underline{s},\frac{1}{F}]f^s$ with $\hat{f} \in \mathbb{C}[\underline{x},\underline{s},\frac{1}{F}]$, $\delta(s) \in D_n[\underline{s}]$ such that $\tilde{b}(s) = \delta(s) \bullet \tilde{f}$ by induction on $\operatorname{ord}(\delta) =: o$.

For $\operatorname{ord}(\delta) = 1$, we have

$$\partial_k \bullet \tilde{f} = f^s(\partial_k \bullet \hat{f}) + \hat{f}(\partial_k \bullet f^s) = f^s(\partial_k \bullet \hat{f}) + \hat{f}\sum_{i=1}^{r}\left(\prod_{j\neq i} f_j^{s_j}\right) s_i f_i^{s_i-1}(\partial_k \bullet f_i)$$

with maximal total degree $1 + \operatorname{tdeg}_s(\hat{f})$ in the s_i. Linear combinations over $\mathbb{C}[\underline{x},\underline{s}]$ of such terms with at most n summands increase the total degree in the s_i only by the total degree of the coefficients in the s_i, which shows the claim for $o = 1$.

Now let the claim be shown for $\operatorname{ord}(\delta) < o$ and consider $\partial^\alpha \tilde{f}$ for $|\alpha| = o$ for which w.l.o.g. $\alpha_1 > 0$. Then we have

$$\partial^\alpha \bullet \tilde{f} = \partial_1 \partial^{\alpha - e_1} \bullet \tilde{f} \overset{\text{IH}}{=} \partial_1 \bullet \underbrace{\hat{b}}_{\operatorname{tdeg}_s(\cdot) \leq |\alpha|-1+\operatorname{tdeg}_s(\hat{f})} f^s \overset{IB}{=} \hat{\hat{b}} f^s$$

with total degree at most $|\alpha| + \operatorname{tdeg}_s(\hat{f}) = o + \operatorname{tdeg}_s(\hat{f})$ in the s_i. Again, linear combinations contribute only with $\operatorname{tdeg}_s(\delta)$, if at all. $\square$

4. Budur's upper and lower bounds

In [Bud12], Budur introduced the notion of a generalized Bernstein-Sato ideal.

Definition 4.1 ([Bud12])**.** For $M \in \mathbb{N}_0^{u \times r}$ for some $u \in \mathbb{N}$ and $f \in \mathbb{C}[\underline{x}]^r$, we define

$$\mathcal{B}_f^M := (\mathrm{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle f^{M_{1,-}}, \ldots, f^{M_{u,-}} \rangle) \cap \mathbb{C}[\underline{s}].$$

Remark 4.2. We can again reformulate this definition by using functional equations:

$$b \in \mathcal{B}_f^M \quad \Leftrightarrow \quad b(s) f^s = \left(\sum_{i=1}^{u} \delta_i f^{M_{i,-}} \right) \bullet f^s \text{ for some } \delta_i \in D_n[\underline{s}].$$

The generalized Bernstein-Sato ideal indeed generalizes all types of Bernstein-Sato ideals that we defined so far. For $M = I_r$, the $r \times r$ identity matrix, the resulting ideal is $\mathcal{B}_f^{I_r} = \mathcal{B}_\Sigma$. For $M = (1, \ldots, 1)^T \in \mathbb{N}_0^r$, the construction results in $\mathcal{B}_f^M = \mathcal{B}$. For $M = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{N}_0^r$ the ith standard basis vector, we get $\mathcal{B}_f^M = \mathcal{B}_{(i)}$.

From Theorem 4.8 and the fact that $\mathcal{B}_{(i)} \neq 0$ for all $1 \leq i \leq r$ we conclude that $\mathcal{B}_f^M \neq \{0\}$ for all M.

Remark 4.3. The computation of $\mathcal{B}_f^M$ can be conducted analogously as the computation of $\mathcal{B}, \mathcal{B}_{(i)}$ and $\mathcal{B}_\Sigma$. After determining a Gröbner basis of $\mathrm{ann}_{D_n[\underline{s}]}(f^s)$, we append the additional generators $f^{M_{1,-}}, \ldots, f^{M_{p,-}}$ and compute the intersection with $\mathbb{C}[\underline{s}]$ by means of Gröbner bases with an appropriate elimination ordering.

We introduce shift maps t_i that shift the s_i in order to formulate upper and lower bounds for $\mathcal{B}_f^m$ for $m \in \mathbb{N}_0^r$.

Definition 4.4 ([Bud12])**.** For $i \in \{1, \ldots, r\}$, we define

$$t_i : \mathbb{C}[\underline{s}] \to \mathbb{C}[\underline{s}], s_i \mapsto s_i + 1, s_j \mapsto s_j \text{ for } j \neq i.$$

We will denote the action of t_i as right multiplication, i.e. $t_i p = t_i(p)$ for $p \in \mathbb{C}[\underline{s}]$ and use multi-index notation, i.e. $t^\alpha p = t_1^{\alpha_1} \circ t_2^{\alpha_2} \circ \ldots \circ t_r^{\alpha_r}(p)$ for $\alpha \in \mathbb{N}_0^r, p \in \mathbb{C}[\underline{s}]$. With this notation, we have $t^\alpha p(s) = p(s + \alpha)$.

With this preparation, we can show the following lemma, which iteratively leads towards upper and lower bounds for $\mathcal{B}_f^m$ with $m \in \mathbb{N}_0^r$.

Lemma 4.5 ([Bud12])**.** Let $m, n \in \mathbb{N}_0^r$. For the corresponding Bernstein-Sato ideals, the following holds:

$$\mathcal{B}_f^m (t^m \mathcal{B}_f^n) \subseteq \mathcal{B}_f^{m+n} \subseteq \mathcal{B}_f^m \cap (t^m \mathcal{B}_f^n).$$

Proof. First, let $b_1(s)b_2(s) \in \mathcal{B}_f^m(t^m\mathcal{B}_f^n)$ with $b_1(s) \in \mathcal{B}_f^m, b_2(s) \in t^m\mathcal{B}_f^n$. For $b_1(s)$ it holds that $b_1(s)f^s = \delta_1(s)f^{s+m}$ for some $\delta_1(s) \in D_n[\underline{s}]$. The functional equation of $b_2(s)$ reads

$$b_2(s-m)f^s = \delta_2(s) \bullet f^{s+n}$$

for some $\delta_2(s) \in D_n[\underline{s}]$ or, after applying t^m,

$$b_2(s)f^{s+m} = \delta_2(s+m) \bullet f^{s+m+n}.$$

With this

$$b_1(s)b_2(s)f^s = b_2(s)\delta_1(s) \bullet f^{s+m} = \delta_1(s)\delta_2(s+m) \bullet f^{s+m+n},$$

which shows the first inclusion.

Now, let $b(s) \in \mathcal{B}_f^{m+n}$, e.g. $b(s)f^s = \delta(s) \bullet f^{s+m+n}$. Then,

$$b(s)f^s = \underbrace{\delta(s)f^n}_{\in D_n[\underline{s}]} \bullet f^{s+m},$$

which shows $b(s) \in \mathcal{B}_f^m$. On the other hand

$$b(s-m)f^{s-m} = \delta(s-m) \bullet f^{s+n},$$

which, after left multiplication with f^m results in

$$b(s-m)f^s = f^m\delta(s-m) \bullet f^{s+n},$$

which shows $b(s) \in t^m\mathcal{B}_f^n$, implying the second inclusion. $\square$

Remark 4.6. As the roles of m and n do not differ, we can restrict the upper bound even more to

$$\mathcal{B}_f^{m+n} \subseteq \mathcal{B}_f^m \cap (t^m\mathcal{B}_f^n) \cap \mathcal{B}_f^n \cap (t^n\mathcal{B}_f^m),$$

but in all of the examples checked the inclusion towards the upper bound from Lemma 4.5 is an equality, which leads to the conjecture that equality always holds (cf. [Bud12]).

Lemma 4.7. Let $m, n \in \mathbb{N}_0^r$ be such that $pf^m + qf^n = 1$ for some $p, q \in \mathbb{C}[\underline{x}]$. Then

$$\mathcal{B}_f^{m+n} = (t^m\mathcal{B}_f^n) \cap (t^n\mathcal{B}_f^m).$$

Proof. It remains to be shown that '$\supseteq$' holds. For this, let $b(s)f^{s+m} = \delta_1(s) \bullet f^{s+m+n}$ and $b(s)f^{s+n} = \delta_2(s) \bullet f^{s+m+n}$. Then

$$(p\delta_1(s) + q\delta_2(s)) \bullet f^{s+m+n} = pb(s)f^{s+m} + qb(s)f^{s+n} = b(s)f^s(pf^m + qf^n) = b(s)f^s,$$

as desired. $\square$

We apply Lemma 4.5 iteratively to obtain the following result.

Theorem 4.8 ([Bud12])**.** Denote by $e_i \in \mathbb{N}_0^r$ the ith standard basis vector. Then

$$\prod_{\substack{j=1\\m_j\neq 0}}^{r}\prod_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k\mathcal{B}_f^{e_j} \subseteq \mathcal{B}_f^m \subseteq \bigcap_{\substack{j=1\\m_j\neq 0}}^{r}\bigcap_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k\mathcal{B}_f^{e_j}.$$

Proof. First, we consider $m = m_i \cdot e_i$ with $m_i \in \mathbb{N}_0$. By induction over $m_i \in \mathbb{N}_0$, we show that

$$\prod_{k=0}^{m_i-1} t_i^k\mathcal{B}_f^{e_i} \subseteq \mathcal{B}_f^{m_ie_i} \subseteq \bigcap_{k=0}^{m_i-1} t_i^k\mathcal{B}_f^{e_i} \tag{8}$$

For $m_i \in \{0,1\}$ the claim is obvious. Let the claim be shown for all $m_i < m_{i_0}$. We apply Lemma 4.5 and get

$$\begin{aligned}
\prod_{k=0}^{m_{i_0}-1} t_i^k\mathcal{B}_f^{e_i} &= \left(\prod_{k=0}^{m_{i_0}-2} t_i^k\mathcal{B}_f^{e_i}\right) t_i^{m_{i_0}-1}\mathcal{B}_f^{e_i} \overset{\text{IH}}{\subseteq} \mathcal{B}_f^{(m_{i_0}-1)e_i}(t^{(m_{i_0}-1)e_i}\mathcal{B}_f^{e_i})\\
&\overset{4.5}{\subseteq}\mathcal{B}_f^{(m_{i_0}-1)e_i+e_i} \overset{4.5}{\subseteq} \mathcal{B}_f^{(m_{i_0}-1)e_i} \cap (t^{(m_{i_0}-1)e_i}\mathcal{B}_f^{e_i})\\
&\overset{\text{IH}}{\subseteq}\left(\bigcap_{k=0}^{m_{i_0}-2} t_i^k\mathcal{B}_f^{e_i}\right) \cap t_i^{m_{i_0}-1}\mathcal{B}_f^{e_i} = \bigcap_{k=0}^{m_{i_0}-1} t_i^k\mathcal{B}_f^{e_i}.
\end{aligned}$$

We now show the main claim by induction on $r \in \mathbb{N}$, again using Lemma 4.5. For $r = 1$, the claim follows from (8). Let the claim be shown for all $r < r_0$. With Lemma 4.5, (8) and the induction hypothesis, for $m \in \mathbb{N}_0^{r_0}$ we obtain

$$\prod_{\substack{j=1\\m_j\neq 0}}^{r_0}\prod_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k\mathcal{B}_f^{e_j} \overset{\text{IH}}{\subseteq} \mathcal{B}_f^{m-m_{r_0}e_{r_0}}\cdot(t^{m-m_{r_0}e_{r_0}}\mathcal{B}_f^{m_{r_0}e_{r_0}}) \subseteq \mathcal{B}_f^m$$

$$= \mathcal{B}_f^{(m-m_{r_0}e_{r_0})+m_{r_0}e_{r_0}} \subseteq \mathcal{B}_f^{m-m_{r_0}e_{r_0}} \cap (t^{m-m_{r_0}e_{r_0}}\mathcal{B}_f^{m_{r_0}e_{r_0}}) \overset{\text{IH}}{\subseteq} \bigcap_{\substack{j=1\\m_j\neq 0}}^{r_0}\bigcap_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k\mathcal{B}_f^{e_j}.$$

□

Remark 4.9 ([Bud12])**.** Theorem 4.8 can be used to compute upper bounds and lower bounds of $\mathcal{B}_f^m$ (see Code A.2). When taking the radicals of the inclusions, we get equalities (because $\sqrt{I\cap J} = \sqrt{I\cdot J}$ for ideals $I, J \subseteq \mathbb{C}[\underline{s}]$), so we can obtain factors of the Bernstein-Sato ideals from lower or upper bounds and the vanishing sets of the bounds in $\mathbb{C}^r$ do not differ, which is especially useful for the application given in Chapter 5.

There is no known example in which the upper bound is really a strict upper bound, so the upper inclusion may even be an equality (see [Bud12]).

The method is especially useful to compute $\mathcal{B}_f = \mathcal{B}_f^{(1,\dots,1)}$, because in practice it is often much easier to determine $\mathcal{B}_f^{e_i}$ than to determine $\mathcal{B}_f$ directly. This can be reasoned by the fact that the total degrees of the generators of $\operatorname{ann}(f^s) + \langle f_i \rangle$ are in general significantly smaller than the ones of the generators of $\operatorname{ann}(f^s)+\langle F\rangle$, which may simplify the Gröbner basis computations needed to eliminate variables.

We can regard the lower bound obtained as a generalization of Lemma 3.29, where we gathered information about a product where shifts did not play any role in the special case of transversal intersections.

Remark 4.10 ([Bud12])**.** The analogous statement as in Theorem 4.8 can be shown for permutations of $\{1,\dots,r\}$ as follows. In Theorem 4.8 we have only considered the shifts $t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k$ but we might as well change the order of the j through a permutation $\pi:\{1,\dots,r\}\to\{1,\dots,r\}$ to obtain

$$\prod_{\substack{j=1\\ m_{\pi(j)}\neq 0}}^{r}\prod_{k=0}^{m_{\pi(j)}-1} t_{\pi(1)}^{m_{\pi(1)}}\dots t_{\pi(j-1)}^{m_{\pi(j-1)}}t_{\pi(j)}^{k}\mathcal{B}_f^{e_{\pi(j)}} \subseteq \mathcal{B}_f^m \subseteq \bigcap_{\substack{j=1\\ m_{\pi(j)}\neq 0}}^{r}\bigcap_{k=0}^{m_{\pi(j)}-1} t_{\pi(1)}^{m_{\pi(1)}}\dots t_{\pi(j-1)}^{m_{\pi(j-1)}}t_{\pi(j)}^{k}\mathcal{B}_f^{e_{\pi(j)}}.$$

Remark 4.11. Although in practice the upper inclusion is an equality for all known examples, we can only show the following bound.

The inductive proof of Theorem 4.8 suggests a very rough upper bound for the difference of powers of the upper and lower bounds. For this let $b(s)\in\mathcal{B}_f^m\cap(t^m\mathcal{B}_f^n)$, e.g. $b(s)f^s=\delta_1(s)\bullet f^{s+m}$ and $b(s)f^{s+m}=\delta_2(s)\bullet f^{s+m+n}$. Then

$$b(s)^2f^s=b(s)\delta_1(s)\bullet f^{s+m}=\delta_1(s)\bullet b(s)f^{s+m}=\delta_1(s)\delta_2(s)\bullet f^{s+m+n},$$

so for each application of Lemma 4.5, we have to take the Bernstein-Sato ideal to the power of two, which through the two inductions yields

$$\left(\bigcap_{\substack{j=1\\ m_j\neq 0}}^{r}\bigcap_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^{k}\mathcal{B}_f^{e_j}\right)^{2^{|m|-1}} \subseteq \prod_{\substack{j=1\\ m_j\neq 0}}^{r}\prod_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^{k}\mathcal{B}_f^{e_j},$$

so the gap between upper and lower bound is at most a power $2^{|m|-1}$.

Example 4.12 (see [HKS05])**.** Consider the example $f=(x(1-y)^2+(1-x)(1-z)^2, xy(1-y)+(1-x)z(1-z), xy^2+(1-x)z^2)\in\mathbb{C}[x,y,z]^3$ with $f_1-f_2+f_3=1$ for which we are interested in $\mathcal{B}_f=\mathcal{B}_f^{(1,1,1)}$. We can easily determine $\mathcal{B}_\Sigma=\langle 1\rangle=\langle b_{\langle f\rangle}\rangle$. We can use Theorem 4.8 to compute upper and lower bounds for $\mathcal{B}_f$. The computation of the $\mathcal{B}_j^{e_j}$ yields

$$\begin{aligned}
\mathcal{B}_f^{e_1}&=\mathcal{B}_1=\langle(s_1+1)(2s_1+s_2+2)(2s_1+s_2+3)(2s_1+s_2+4)\rangle,\\
\mathcal{B}_f^{e_2}&=\mathcal{B}_2=\langle(s_2+1)(2s_1+s_2+2)(2s_1+s_2+3)(s_2+2s_3+2)(s_2+2s_3+3)\rangle,\\
\mathcal{B}_f^{e_3}&=\mathcal{B}_3=\langle(s_3+1)(s_2+2s_3+2)(s_2+2s_3+3)(s_2+2s_3+4)\rangle.
\end{aligned}$$

The inclusions

$$\prod_{\substack{j=1\\ m_j\neq 0}}^{r}\prod_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k\mathcal{B}_f^{e_j} \subseteq \mathcal{B}_f^m \subseteq \bigcap_{\substack{j=1\\ m_j\neq 0}}^{r}\bigcap_{k=0}^{m_j-1} t_1^{m_1}\dots t_{j-1}^{m_{j-1}}t_j^k\mathcal{B}_f^{e_j}$$

become

$$\mathcal{B}_f^{e_1}(t_1\mathcal{B}_f^{e_2})(t_1t_2\mathcal{B}_f^{e_3}) = \prod_{j=1}^{3} t_1\dots t_{j-1}\mathcal{B}_f^{e_j} \subseteq \mathcal{B}_f^m \subseteq \bigcap_{j=1}^{3} t_1\dots t_{j-1}\mathcal{B}_f^{e_j} = \mathcal{B}_f^{e_1}\cap(t_1\mathcal{B}_f^{e_2})\cap(t_1t_2\mathcal{B}_f^{e_3})$$

in our case. For the further steps we determine

$$\begin{aligned} t_1\mathcal{B}_f^{e_2} &= \langle(s_2+1)(2s_1+s_2+4)(2s_1+s_2+5)(s_2+2s_3+2)(s_2+2s_3+3)\rangle,\\ t_1t_2\mathcal{B}_f^{e_3} &= \langle(s_3+1)(s_2+2s_3+3)(s_2+2s_3+4)(s_2+2s_3+5)\rangle. \end{aligned}$$

We notice that the pairs $\mathcal{B}_f^{e_1}, t_1\mathcal{B}_f^{e_2}$ and $t_1\mathcal{B}_f^{e_2}, t_1t_2\mathcal{B}_f^{e_3}$ each share a common factor, so the upper and lower bound differ in our example. The upper bound is given by

$$\begin{aligned}&\langle(s_1+1)(s_2+1)(s_3+1)(2s_1+s_2+2)(2s_1+s_2+3)(2s_1+s_2+4)\\ &\qquad(2s_1+s_2+5)(s_2+2s_3+2)(s_2+2s_3+3)(s_2+2s_3+4)(s_2+2s_3+5)\rangle\end{aligned}$$

and the lower bound is given by

$$\begin{aligned}&\langle(s_1+1)(s_2+1)(s_3+1)(2s_1+s_2+2)(2s_1+s_2+3)(2s_1+s_2+4)^2\\ &\qquad(2s_1+s_2+5)(s_2+2s_3+2)(s_2+2s_3+3)^2(s_2+2s_3+4)(s_2+2s_3+5)\rangle,\end{aligned}$$

so up to the multiplicity of two factors we know the Bernstein-Sato polynomial.

We may as well use other orders of the factors as described in Remark 4.10, but these yield the same upper bound.

With Algorithm 2.18 and our computational means, we were unable to determine $\mathcal{B}_f$ exactly.

5. The annihilator of f^α

An application of Bernstein-Sato ideals is the computation of the annihilator $\operatorname{ann}_{D_n}(f^\alpha)$, where now $\alpha \in \mathbb{C}^r$. In this chapter, we follow the approach of [SST00] and [OT99]. We fix α for this chapter. We have already dealt with the computation of $\operatorname{ann}_{D_n[\underline{s}]}(f^s)$ for symbolic f^s in Algorithm 2.18.

Remark 5.1. In general it holds that $\operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha} \subseteq \operatorname{ann}_{D_n}(f^\alpha)$, since the 'polynomial' equalities for f^s hold as well after evaluating s. An example in which the proper inclusion holds is given in the following.

Example 5.2. Consider $f \in \mathbb{C}[\underline{x}]^r$ and $\alpha = (0, \dots, 0) \in \mathbb{C}^r$. Then

$$\operatorname{ann}_{D_n}(f^\alpha) = \operatorname{ann}_{D_n}(1) = {}_{D_n}\langle \partial_1, \dots, \partial_n \rangle$$

but in general $\operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=0} \neq {}_{D_n}\langle \partial_1, \dots, \partial_n \rangle$ which we can already see in the example $f = x \in \mathbb{C}[x]$, since here $\operatorname{ann}_{D_n[s]}(f^s) = {}_{D_n[s]}\langle x\partial_x - s \rangle$, in particular $\partial_x \notin \operatorname{ann}_{D_n[s]}(f^s)|_{s=0}$.

It is surprising that for most $\alpha \in \mathbb{C}^n$ the equality $\operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha} = \operatorname{ann}_{D_n}(f^\alpha)$ holds, which we can see in the following theorem.

Theorem 5.3 ([OT99])**.** Let $\alpha \notin \{\alpha_0 \in \mathbb{C}^r \mid b(\alpha_0) = 0 \text{ for all } b \in \mathcal{B}\} + \mathbb{N} \cdot (1, \dots, 1)$. Then

$$\operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha} = \operatorname{ann}_{D_n}(f^\alpha).$$

Proof. First, let $\delta \in \operatorname{ann}_{D_n[\underline{s}]}(f^s)$, i.e. $\delta(s) \bullet f^s = 0$. Substituting $s_i \mapsto \alpha_i$ yields $\delta(\alpha) \bullet f^\alpha = 0$, so $\delta(\alpha) \in \operatorname{ann}_{D_n}(f^\alpha)$.

For the other inclusion, let $\delta \in \operatorname{ann}_{D_n}(f^\alpha)$. We need to find $\tilde{\delta} \in \operatorname{ann}_{D_n[\underline{s}]}(f^s)$ such that $\tilde{\delta}|_{s=\alpha} = \delta$. Let δ be of the form

$$\delta = \sum_{\gamma \leq \gamma_0 \text{ component-wise}} \underbrace{\delta_\gamma}_{\in \mathbb{C}[\underline{x}, \underline{s}]} \partial^\gamma$$

for $\gamma_0 \in \mathbb{N}_0^n$.

We claim that $\delta \bullet f^s \in \sum_{j=1}^n \mathbb{C}[\underline{x}, \underline{s}](s_j - \alpha_j) f^{s - |\gamma_0|}$.

In order to show this, we prove the auxiliary statement that

$$\partial^\gamma \bullet f^{s-\alpha} g \in \sum_{j=1}^n \mathbb{C}[\underline{x}, \underline{s}](s_j - \alpha_j) f^{s-|\gamma_0|} + f^{s-\alpha} \partial^\gamma \bullet g$$

for all $\gamma \in \mathbb{N}_0^r$ with $\gamma \leq_{\text{cw.}} \gamma_0$, $g \in \mathbb{C}[\underline{x}]$. This follows from an iterated application of the Leibniz rule, since for γ with w.l.o.g. $\gamma_1 \neq 0$ we have

$$\partial^\gamma \bullet f^{s-\alpha} g = \partial^{\gamma - e_1} \left(\partial_1 \bullet f^{s-\alpha} g\right) = \partial^{\gamma - e_1} \left(\underbrace{\left(\partial_1 \bullet \prod_{i=1}^r f_i^{s_i - \alpha_i}\right) f^\alpha}_{\in \sum_{j=1}^n \mathbb{C}[\underline{x},\underline{s}](s_j - \alpha_j) f^{s-|\gamma_0|}} + f^{s-\alpha} \left(\partial_1 \bullet f^\alpha\right) \right)$$

and by iterated application to the second summand of the right hand side

$$\partial^\gamma \bullet f^{s-\alpha} g \in \sum_{j=1}^n \mathbb{C}[\underline{x}, \underline{s}](s_j - \alpha_j) f^{s-|\gamma_0|} + f^{s-\alpha} \partial^\gamma \bullet g.$$

Application of this result to $g = f^\alpha$ yields

$$\begin{aligned}
\delta \bullet f^s = \delta \bullet f^{s-\alpha} f^\alpha &= \sum_{\gamma \leq \gamma_0 \text{ cw.}} \underbrace{\delta_\gamma}_{\in \mathbb{C}[\underline{x},\underline{s}]} \partial^\gamma \bullet f^{s-\alpha} f^\alpha \\
&\overset{\text{aux. statement}}{=} f^{s-\alpha} \underbrace{(\delta \bullet f^\alpha)}_{=0} + \sum_{j=1}^r f^\alpha (s_j - \alpha_j) \underbrace{g_j}_{\in \mathbb{C}[\underline{x},\underline{s}]} f^{s-|\gamma_0|} \\
&\in \sum_{j=1}^n \mathbb{C}[\underline{x}, \underline{s}](s_j - \alpha_j) f^{s-|\gamma_0|}.
\end{aligned}$$

We choose $b_1, \ldots, b_{|\gamma_0|} \in \mathcal{B}$ such that $b_i(\alpha_1 - i, \ldots, \alpha_r - i) \neq 0$ for all $1 \leq i \leq |\gamma_0|$. These b_i exist by assumption. Through successive application of the functional equation of the Bernstein-Sato ideal we obtain

$$\underbrace{b_{|\gamma_0|}(s_1 - |\gamma_0|, \ldots, s_r - |\gamma_0|) \cdot \ldots \cdot b_1(s_1 - 1, \ldots, s_r - 1)}_{=:\tilde{b}} f^{s-|\gamma_0|} = \hat{\delta} \bullet f^s$$

for some $\hat{\delta} \in D_n[\underline{s}]$.

Now it follows that

$$\begin{aligned}
\underbrace{\left(\tilde{b}\delta - \sum_{j=1}^r f^\alpha (s_j - \alpha_j) g_j \hat{\delta}\right)}_{=:\tilde{\delta}} \bullet f^s &= \tilde{b}\delta \bullet f^s - \sum_{j=1}^r f^\alpha (s_j - \alpha_j) g_j \hat{\delta} \bullet f^s \\
&= \tilde{b} \sum_{j=1}^r f^\alpha (s_j - \alpha_j) g_j f^{s-|\gamma_0|} - \tilde{b} \sum_{j=1}^r f^\alpha (s_j - \alpha_j) g_j f^{s-|\gamma_0|} = 0,
\end{aligned}$$

so $\frac{\tilde{\delta}}{\tilde{b}(\alpha)} \in \operatorname{ann}_{D_n[\underline{s}]}(f^s)$. On the other hand, $\left(\frac{\tilde{\delta}}{\tilde{b}(\alpha)}\right)|_{s=\alpha} = \delta$, because in $\tilde{\delta}$ the term $\sum_{j=1}^r f^\alpha (s_j - \alpha_j) g_j \hat{\delta}$ vanishes in α. This implies the claim. $\square$

Remark 5.4. This gives another explanation for the factor $s+1$ of Bernstein-Sato polynomials or at least for the necessity of a factor of the form $s+n$ for some $n \in \mathbb{N}$, because in general $\operatorname{ann}_{D_n}(f^0) = \operatorname{ann}_{D_n}(1) = \langle \partial_1, \ldots, \partial_n \rangle \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=0}$, so there needs to be a factor of that form.

Speaking in a vague two-dimensional chess metaphor, in the previous theorem we have used only a bishop's moves, whereas the moves of the king were available through the $\mathcal{B}_{(i)}$ combined with $\mathcal{B}$, which we use in the following generalization of the result of [OT99].

Lemma 5.5. Let $\alpha \in \mathbb{C}^r$ such that there exists a sequence $(\beta_i)_{i\in\mathbb{N}_0}$ with values in $\mathbb{N}_0^r$ such that $\beta_0 = 0$, $\lim_{i\to\infty}(\beta_i)_j = \infty$ for all $1 \le j \le r$ and for all $i \in \mathbb{N}$ one of the following properties holds:

- $\beta_i - \beta_{i-1} = e_j$ and $b_i(\alpha - \beta_i) \neq 0$ for some $1 \le j \le r$ and some $b_i \in \mathcal{B}_{(j)} \supseteq \mathcal{B}$,
- $\beta_i - \beta_{i-1} = (1, \ldots, 1)$ and $b_i(\alpha - \beta_i) \neq 0$ for some $b_i \in \mathcal{B}$.

Then it holds that

$$\operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha} = \operatorname{ann}_{D_n}(f^\alpha).$$

Proof. We proceed similarly as in the proof of Theorem 5.3. Again, we choose $\delta \in \operatorname{ann}_{D_n}(f^\alpha)$ and apply it to f^s to obtain

$$\delta \bullet f^s = \sum_{j=1}^{r} f^\alpha (s_j - \alpha_j) \underbrace{g_j}_{\in\mathbb{C}[\underline{x},\underline{s}]} f^{s-|\gamma_0|}.$$

Now we choose $i_0 \in \mathbb{N}_0$ such that $\beta_{i_0} \ge_{\text{cw.}} |\gamma_0| \cdot (1, \ldots, 1)$. We iteratively apply the functional equations of $\mathcal{B}$ and the one of the $\mathcal{B}_{(i)}$ and obtain

$$\begin{aligned} &b_1(s) f^{s-\beta_1} = \delta_1 \bullet f^s, b_2(s) b_1(s) f^{s-\beta_2} = \delta_2\delta_1 \bullet f^s, \ldots \\ &b_{i_0}(s-\beta_{i_0}) \cdot \ldots \cdot b_1(s-\beta_1) f^{s-|\gamma_0|\cdot(1,\ldots,1)} f^{|\gamma_0|\cdot(1,\ldots,1)-\beta_{i_0}} = \hat{\delta} \bullet f^s \end{aligned}$$

for some $\hat{\delta} \in D_n[\underline{s}]$. The remaining steps are the same ones as in the proof of Theorem 5.3. $\square$

Remark 5.6. Since it holds that $\mathcal{B} \subseteq \mathcal{B}_{(i)}$, which implies $\mathbb{V}_{\mathbb{C}^r}(\mathcal{B}) \supseteq \mathbb{V}_{\mathbb{C}^r}(\mathcal{B}_{(i)})$, we can replace the first condition from Lemma 5.5 with the sufficient condition that

$$\beta_i - \beta_{i-1} = e_j \text{ and } b_i(\alpha - \beta_i) \neq 0 \text{ for some } 1 \le j \le r \text{ and } b_i \in \mathcal{B},$$

which spares us the computation of the $\mathcal{B}_{(i)}$.

Remark 5.7. Naturally, we can generalize Theorem 5.3 and Lemma 5.5 to the computation of $\operatorname{ann}_{D_{n,p}}(f^\alpha)$ since the approach was only dependent on roots of $b(s)$.

In examples like the following, we notice that the additional possibilities of Lemma 5.5 in fact do not contribute to the computation of $\operatorname{ann}(f^\alpha)$.

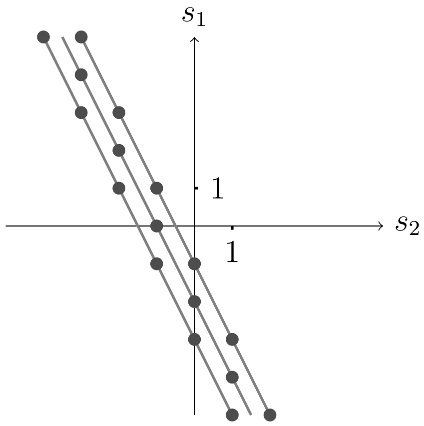

Figure 5.1.: The real vanishing set of $\mathcal{B}$ from Example 5.8 and $\mathbb{V}(\mathcal{B}) \cap \mathbb{Z}^2$.

Example 5.8. Consider $f = (x, x^2) \in \mathbb{C}[x]^2$. The Bernstein-Sato ideal is given by $\mathcal{B} = \langle (s_1 + 2s_2 + 1)(s_1 + 2s_2 + 2)(s_1 + 2s_2 + 3) \rangle \subseteq \mathbb{C}[s_1, s_2]$. The real vanishing set of $\mathcal{B}$ is shown in Figure 5.1.

It holds that for any $\alpha \in \mathbb{V}_{\mathbb{C}^2}(\mathcal{B}) + \mathbb{N} \cdot (1, \ldots, 1)$ there is no sequence of the form from Lemma 5.5, which we can see in Figure 5.1 as follows: W.l.o.g. we may assume that $\alpha \in \mathbb{N}^2$ (otherwise we can shift it along in the direction of the components of $\mathbb{V}(\mathcal{B})$). But the intersection of $\mathbb{V}(\mathcal{B})$ with $\mathbb{N}^2$ is such that through the 'king's moves' one necessarily needs to pass a point of $\mathbb{V}(\mathcal{B})$ when starting from α right of the vanishing set.

We see this example as a hint towards a general property which we can show under the following condition.

Conjecture 5.9. We conjecture that for any $\alpha \in \mathbb{V}(\mathcal{B})$ with $(\alpha - \mathbb{N} \cdot (1, \ldots, 1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$ it holds that $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$.

This conjecture can be seen as part of the converse direction of Theorem 5.3 with interchanged assumption and conclusion. With it, we can prove the following lemma geometrically, which is part of a statement shown in [Gyo93] (compare also [Bud12]) with a different proof.

Lemma 5.10. Let $r > 1$. Under the assumption of Conjecture 5.9 with $f \in \mathbb{R}[\underline{x}]^r$, all common irreducible factors of the generators of $\mathcal{B}$ have a representation of the form

$$b_0 = \sum_{i=1}^{r} c_i s_i + c$$

for $c_i \in \mathbb{Q}_{\geq 0}$ for all $1 \leq i \leq r$ and $c \in \mathbb{R}$.

Proof. First we remark that $\mathcal{B}$ is generated by polynomials with real coefficients in $\mathbb{R}[\underline{s}]$, because in the functional equation $b(s) f^s = \delta(s) \bullet f^{s+1}$ we can obtain imaginary valued coefficients of b only from the coefficients of f and δ, which can be chosen real for δ, and the exponent $s+1$ which is contained in $\mathbb{R}[\underline{s}]$.

In the following, we will work with proofs by contradiction with a relatively arbitrary $\alpha \in \mathbb{V}(b_0)$ for some irreducible b_0, from which we make 'king's steps' and 'bishop's steps' of the form

$$\alpha + 1 \xrightarrow{\text{bishop's step}} \alpha \xrightarrow{\text{bishop's step}} \alpha - 1 \text{ and} \tag{9}$$

$$\begin{aligned} \alpha + 1 &\xrightarrow{\text{king's step}} \alpha + 1 - e_1 \xrightarrow{\text{bishop's step}} \alpha - e_1 \xrightarrow{\text{king's step}} \alpha - e_1 - e_2 \\ &\xrightarrow{\text{king's step}} \dots \xrightarrow{\text{king's step}} \alpha - 1. \end{aligned} \tag{10}$$

Since we are interested in the intersection of those values with the vanishing set $\mathbb{V}(\mathcal{B})$ for the application of Theorem 5.3 and Lemma 5.5, we have to consider only the finite set

$$\mathbb{V}(\mathcal{B}) \cap \{\alpha + 1 - e_1, \alpha - e_1, \alpha - e_1 - e_2, \dots, \alpha - 1\}.$$

This allows us to assume w.l.o.g. that $\mathbb{V}(\mathcal{B})$ consists only of shifted copies of the form $\mathbb{V}(b_0) + \kappa$ with $\kappa \in \mathbb{C}^r$ because otherwise we can move α along the hypersurface $\mathbb{V}(b_0)$.

More precisely, when considering a Euclidean neighbourhood U of α and $U \cap \mathbb{V}(b_0)$ we obtain uncountably many $\tilde{\alpha} \in U \cap \mathbb{V}(b_0)$. Assume that for all of these $\tilde{\alpha}$ there exists some irreducible $b_{\tilde{\alpha}}$ with $b_{\tilde{\alpha}} \mid \mathcal{B}$ and $\mathbb{V}(b_{\tilde{\alpha}}) \neq \mathbb{V}(b_0) + \kappa$ for all $\kappa \in \mathbb{C}^r$ such that

$$\mathbb{V}(b_{\tilde{\alpha}}) \cap \{\tilde{\alpha} + 1 - e_1, \tilde{\alpha} - e_1, \alpha - e_1 - e_2, \dots, \tilde{\alpha} - 1\} \neq \varnothing.$$

Since the Bernstein-Sato ideal has only finitely many common factors and thus $\mathbb{V}(\mathcal{B})$ has only finitely many irreducible components, it follows that uncountably many of the $\tilde{\alpha}$ share one common $b_{\tilde{\alpha}}$, e.g. b_1. This implies that b_1 is such that $\mathbb{V}(b_1) = \mathbb{V}(b_0) + \kappa$ for some $\kappa \in \mathbb{C}^r$, so we can choose $\tilde{\alpha}$ with $b_{\tilde{\alpha}} = b_1$ instead of α.

In the following, we will use the following argument: For $\alpha \in \mathbb{V}(b_0)$ such that $(\alpha - \mathbb{N} \cdot (1, \dots, 1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$, we have $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$ by Conjecture 5.9.

Furthermore, we know that the maps

$$D_n f^{\alpha-1} \xrightarrow{\cdot f_n} D_n f^{\alpha-1+e_n} \xrightarrow{\cdot f_{n-1}} \dots \xrightarrow{\cdot f_2} D_n f^{\alpha-e_1} \xrightarrow{\cdot F} D_n f^{\alpha+1-e_1} \xrightarrow{\cdot f_1} D_n f^{\alpha+1}$$

and

$$D_n f^{\alpha-1} \xrightarrow{\cdot F^2} D_n f^{\alpha+1}$$

commute.

We will use this idea together with Lemma 5.5, because by contraposition we conclude with this lemma from $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$ that there is no sequence of the form $\beta = (0, e_1, (1, \dots, 1) + e_1, (1, \dots, 1) + e_1 + e_2, \dots, (1, \dots, 1) + e_1 + \dots + e_{n-1}, 2 \cdot (1, \dots, 1))$ with $((\alpha + 1) - \beta)_i \notin \mathbb{V}(\mathcal{B})$ for all i. This results in contradictions for the cases considered.

Next, we will show that there are no irreducible factors of total degree greater or equal to two. Assume towards a contradiction that there is a b_0 of this form. We choose b_0 'minimal' of this form in the sense that there is an $\alpha \in \mathbb{V}(b_0)$ with $(\alpha - \mathbb{N}(1,\ldots,1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$.

We consider the set

$$A := \mathbb{V}(b_0) + (1,\ldots,1).$$

By assumption of Conjecture 5.9 and the 'minimality' of b_0, we know that for uncountably many $\alpha \in A$ it holds that $\operatorname{ann}_{D_n}(f^\alpha) \neq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha}$. For these, the sequence $(\alpha-2, \alpha-3, \ldots)$ has an empty intersection with $\mathbb{V}(\mathcal{B})$. We combine this sequence with the sequence (10) and obtain the sequence

$$\beta = (0, e_1, 1+e_1, 1+e_1+e_2, \ldots, 2, 3, \ldots).$$

If it fulfilled the conditions of Lemma 5.5, this would imply the equality $\operatorname{ann}_{D_n}(f^\alpha) = \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha}$, a contradiction. It follows that

$$\mathbb{V}(\mathcal{B}) \cap (\alpha - \{e_1, 1+e_1, 1+e_1+e_2, \ldots, 2-e_n\}) \neq \varnothing.$$

Since there are uncountably many such α, we conclude that a shifted copy of $\mathbb{V}(b_0)$ of the form $\mathbb{V}(b_0) + \kappa$ with $\kappa \in \{1-e_1, -e_1, -e_1-e_2, \ldots, -1+e_n\}$ is contained in $\mathbb{V}(\mathcal{B})$. In particular, this copy is not $\mathbb{V}(b_0)$ itself, since it is not linear. Applying the same argument to the newly found copy, inductively we conclude that $\mathbb{V}(\mathcal{B})$ contains infinitely many shifted copies of $\mathbb{V}(b_0)$, which contradicts $\mathcal{B} \neq \{0\}$. In conclusion, all b_0 are linear.

Next, we will show that for all c_i of a 'minimal' factor $b_0 = \sum_{i=1}^r c_i s_i + c$ we can choose $c_i \geq 0$ for all $1 \leq i \leq r$. Here, 'minimal' means that no copies of $\mathbb{V}(b_0)$ shifted by $-\mathbb{N}(1,\ldots,1)$ are contained in $\mathbb{V}(\mathcal{B})$. We already showed that $c_i \in \mathbb{R}$. Assume towards a contradiction that $c_i < 0$ and $c_j > 0$.

First, we consider $\frac{c_i}{c_j} = -1$. We choose $\alpha \in \mathbb{V}(b_0)$. For $\alpha+1$ we know by Conjecture 5.9 that $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$. We construct the sequence

$$(\beta_i)_{i\in\mathbb{N}_0} := (0, e_i, \ldots, k\cdot e_i, k\cdot e_i + (1,\ldots,1), (k+1)\cdot e_i + (1,\ldots,1), \ldots, 2k\cdot e_i + (1,\ldots,1), \ldots),$$

such that $\alpha + 1 - (\beta_i)_{i\in\mathbb{N}_0}$ has empty intersection with $\mathbb{V}(\mathcal{B})$ by construction for k sufficiently large. With Lemma 5.5 we conclude that $\operatorname{ann}_{D_n}(f^{\alpha+1}) = \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$, a contradiction.

Next, let $\frac{c_i}{c_j} \neq -1$ and w.l.o.g. $c_{j_2} > 0$ for all $j_2 \neq i$. Again, we choose b_0 'minimal' and $\alpha \in \mathbb{V}(b_0)$ that shows the 'minimality' of b_0). We use the sequence

$$\beta = (0, e_i, 1+e_i, 1+e_i+e_2, 1+e_i+e_2+e_3, \ldots, 2)$$

for which $\alpha + 1 - \beta$ by construction has an empty intersection with $\mathbb{V}(b_0)$. By contraposition of Lemma 5.5 and the fact that α could be chosen such that $(\alpha - \mathbb{N}\cdot(1,\ldots,1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$ we know that $\alpha+1-\beta$ contains an element of $\mathbb{V}(\mathcal{B})$, because $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$. Thus, a shifted copy of $\mathbb{V}(b_0)$ is contained in $\mathbb{V}(\mathcal{B})$ (see Figure 5.2).

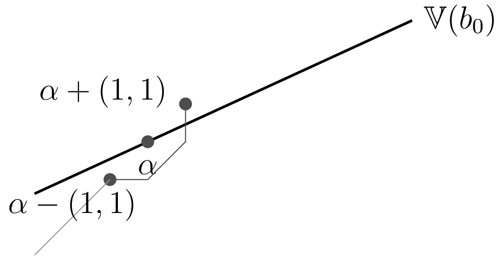

Figure 5.2.: Illustration of the construction for $c_i > 0, c_j < 0$ in the two-dimensional case.

Iterating this argument, using the form of b_0 we obtain either infinitely many copies of $\mathbb{V}(b_0)$ in $\mathbb{V}(\mathcal{B})$ or a copy that contradicts the minimality of $\mathbb{V}(b_0)$, both of which is a contradiction. In conclusion, there is no b_0 with $c_i < 0 < c_j$.

Next, we want to show that the coefficients of the linear terms of a common factor b_0 can be chosen from the rational numbers. We already showed that $b_0 = \sum_{i=1}^r c_i s_i + c$ for non-negative c_i. Assume towards a contradiction that the c_i cannot all be chosen rational, e.g. $\frac{c_i}{c_j} \notin \mathbb{Q}$. We choose b_0 'minimal' and $\alpha \in \mathbb{V}(b_0)$ such that α shows the 'minimality' of b_0, i.e. $(\alpha - \mathbb{N} \cdot (1, \ldots, 1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$. By Conjecture 5.9 we know that $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$. Again, we consider the sequence

$$\beta = (0, e_i, 1 + e_i, 1 + e_i + e_2, 1 + e_i + e_2 + e_3, \ldots, 2)$$

with $(\alpha + 1 - \beta) \cap \mathbb{V}(b_0) = \varnothing$, since the slope $\frac{c_i}{c_j}$ is irrational (see Figure 5.3). By

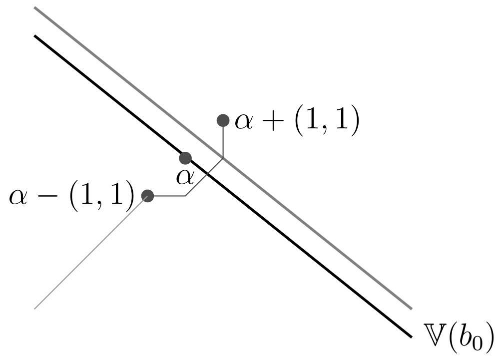

Figure 5.3.: Illustration of the construction for $\frac{c_i}{c_j}$ irrational in the case $r = 2$.

the contrapositive of Lemma 5.5 and the 'minimality' of b_0, which implies $(\alpha - 1 - \mathbb{N}_0(1, \ldots, 1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$, we know that $\alpha + 1 - \beta$ contains an element of $\mathbb{V}(\mathcal{B})$, because $\operatorname{ann}_{D_n}(f^{\alpha+1}) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha+1}$. We conclude that a shifted copy of $\mathbb{V}(b_0)$ shifted by some $\nu \in \mathbb{Z}_0^r \setminus \{0\}$ is contained in $\mathbb{V}(\mathcal{B})$. In particular, this copy is not $\mathbb{V}(b_0)$ itself. Iterating this argument by applying it to the said copy, we conclude that infinitely many shifted copies of $\mathbb{V}(b_0)$ are contained in $\mathbb{V}(\mathcal{B})$, a contradiction.

In conclusion, all common factors have non-negative rational coefficients in the linear terms. □

In order to give an algorithm for computing $\operatorname{ann}_{D_n}(f^\alpha)$ for any α in $\mathbb{C}^r$ we are still missing knowledge about the non-generic α which lie in $\mathbb{V}(\mathcal{B}) + \mathbb{N}\cdot(1,\ldots,1)$. For these, the following lemma offers a solution.

Lemma 5.11 (see [SST00, 5.3.15])**.** For $\alpha = \alpha_0 + k\cdot(1,\ldots,1)$ with $\alpha_0 \in \mathbb{V}(\mathcal{B})$ such that α_0 is minimal in the sense that $(\alpha_0 - \mathbb{N}\cdot(1,\ldots,1)) \cap \mathbb{V}(\mathcal{B}) = \varnothing$ holds, we have

$$\begin{aligned} &\operatorname{ann}_{D_n}(f^\alpha) = \\ &\quad {}_{D_n}\langle h \in D_n \mid hF^k + h_1 g_1(\alpha_0) + \ldots + h_\lambda g_\lambda(\alpha_0) = 0 \text{ for some } h_1,\ldots,h_\lambda \in D_n\rangle, \end{aligned} \tag{11}$$

where $\operatorname{ann}_{D_n[\underline{s}]}(f^s) = {}_{D_n[\underline{s}]}\langle g_1(s),\ldots,g_\lambda(s)\rangle$.

Proof. For $h \in D_n$ it holds that

$$h \in \operatorname{ann}_{D_n}(f^\alpha) \iff 0 = h \bullet f^\alpha = h \bullet F^k f^{\alpha_0} = hF^k \bullet f^{\alpha_0} \iff hF^k \in \operatorname{ann}_{D_n}(f^{\alpha_0}).$$

By the minimality of α_0 we furthermore know that $\operatorname{ann}_{D_n}(f^{\alpha_0}) = \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha_0}$. Now we have $h \in \operatorname{ann}_{D_n}(f^\alpha)$ if and only if $hF^k \in \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha_0}$, which is exactly the condition for the elements of the term on the right hand side of the equation. □

For the computation of the h with the property on the right hand side of (11), which are the first components of the syzygies

$$\operatorname{syz}_{D_n}(F^k, g_1(\alpha_0),\ldots,g_\lambda(\alpha_0)) := \left\{(h,h_1,\ldots,h_\lambda) \mid hF^k + h_1 g_1(\alpha_0) + \ldots + h_\lambda g_\lambda = 0\right\},$$

see [OT01, 9.10] for an algorithm based on Gröbner bases with respect to a specific ordering.

Now we can give an algorithm to compute $\operatorname{ann}_{D_n}(f^\alpha)$ for any $\alpha \in \mathbb{C}^r$ as a generalization of the algorithm given in [SST00].

Algorithm 5.12 (see also A.3)**.**
Input: $f \in \mathbb{C}[\underline{x}]^r$, $\alpha \in \mathbb{C}^r$.
Output: a generating system of $\operatorname{ann}_{D_n}(f^\alpha)$.

Compute $\langle g_1(s),\ldots,g_\lambda(s)\rangle := \operatorname{ann}_{D_n[\underline{s}]}(f^s)$ with the method from Algorithm 2.18.
Compute a generating set G of $\mathcal{B}_f$ with Algorithm 2.18.
Set $H := \{-s_j + s_1 + \alpha_j - \alpha_1 \mid j \in \{1,\ldots,r\}\}$.
$\quad\quad\triangleright\ \mathbb{V}(H) = \alpha + \mathbb{C}\cdot(1,\ldots,1)$
Compute a reduced Gröbner basis K of ${}_{\mathbb{C}[\underline{s}]}\langle G, H\rangle$.
$\quad\quad\triangleright\ \mathbb{V}(K) = \mathbb{V}(\mathcal{B}) \cap (\alpha + \mathbb{C}\cdot(1,\ldots,1))$
Set $k_0 := 0$.
for $\beta \in \mathbb{V}(K)$ **do**
    if $\alpha - \beta = k\cdot(1,\ldots,1)$ for some $k \in \mathbb{N}$ **then**
        Set $k_0 := \max(k_0, k)$.
    end if

end for $\triangleright\ \alpha_0 = \alpha - k_0 \cdot (1, \ldots, 1)$
if $k_0 = 0$ **then**
  return $\langle g_1(\alpha), \ldots, g_\lambda(\alpha) \rangle$.
else
  Set $\alpha_0 := \alpha - k_0 \cdot (1, \ldots, 1)$.
  return $\left\{ h \in D_n \mid hF^k + \sum_{i=1}^{\lambda} h_i g_i(\alpha - \alpha_0) \text{ for some } h_1(s), \ldots, h_\lambda(s) \in D_n[\underline{s}] \right\}$.
end if

Remark 5.13. The correctness of the algorithm follows from the sub-algorithms and Lemma 5.11. The termination of the algorithm follows from the fact that the set of intersection points of a hypersurface and a line not contained in the hypersurface $\mathbb{V}(K)$ is finite. An application example is given in Code A.3.

Conclusion

In Chapter 2, we dealt with two of the computer-algebraic aspects of Bernstein-Sato ideals: Their determination in Section 2.1 and stratifications with respect to them in Section 2.2, Section 2.4 and Section 2.7 which allow the computation of local Bernstein-Sato ideals. We were able to generalize the type of stratification used for Bernstein-Sato ideals (using primary decompositions) in Section 2.8. The steps contributed are Lemma 2.24 and the contents of Section 2.8. An instance of this generalized stratification is the stratification with respect to $b_{f,g}$ which allowed us to give the new Algorithm 2.66 presented in Section 2.7.

For these results we needed the basic definitions and properties of Bernstein-Sato ideals and their variants, Bernstein-Sato polynomials and Bernstein-Sato polynomials of varieties. We introduced local Bernstein-Sato ideals with respect to prime ideals or varieties and examined their properties.

The algorithm for the computation of $\mathcal{B}_f$ given in Section 2.1 is currently the most effective among known ones. It is unclear whether the approach of Chapter 4 with upper and lower bounds can be used to determine the Bernstein-Sato ideal exactly, which would in many cases lead to a speed-up of computations. We were only able to give an estimation of the powers through which upper and lower bound differ in Remark 4.11.

Another open problem is the adaption of more effective stratification algorithms for Bernstein-Sato polynomials such as the one from Section 2.4 to the case of Bernstein-Sato ideals. The foundations of these algorithms (see Remark 2.44) do not hold for Bernstein-Sato ideals so it remains to be shown whether they can be adapted at all.

In Chapter 3, we mainly dealt with factors of Bernstein-Sato ideals and their general form in certain geometric situations. For this, we refined a result about the irrelevance of units for Bernstein-Sato ideals (Theorem 3.5) in Section 3.1. This was especially useful for the determination of $\mathcal{B}_f$ for f with disjoint $\mathbb{V}(f_i)$. For common factors of the f_i and transversal intersections of the $\mathbb{V}(f_i)$ we used different approaches and arrived at some results which previously have not been studied to the best of the author's knowledge (Lemma 3.12, Proposition 3.14, Lemma 3.35). Many of the results here are rather unsatisfying and hint at new, bigger problems that are not yet solved, such as pairwise transversal intersections of components that are not transversal for all components combined. Another open problem is the intersection of vanishing sets in singular points for which the tangent cone seems to play an important role.

In Section 3.3, we examined the conjecture that $\operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s_i=s} = \operatorname{ann}_{D_n[s]}(F^s)$. In Lemma 3.43, we gave a systematic counterexample for the applicability to Bernstein-Sato ideals, i.e. $\mathcal{B}_f|_{s_i=s} \neq \langle b_F \rangle$ in general. It is unknown whether the conjecture holds in general and whether the equality of radicals holds for Bernstein-Sato ideals and polynomials.

In Chapter 5 we considered the computation of $\operatorname{ann}_{D_n}(f^\alpha)$. We modified the previously used approach in Lemma 5.5 and used this modification to give a different proof of some known facts about factors of the Bernstein-Sato ideal under certain conditions in Lemma 5.10. Obviously, the question arises whether Conjecture 5.9 holds.

The algorithms in Appendix A are implementations of the results presented throughout the thesis and were previously not implemented in Singular. Appendix B can be seen as an outlook on some of the most practical problems for the future. Many examples are still difficult to treat with computer algebra systems even in seemingly small instances, e.g. the exact determination of the Bernstein-Sato ideal from Code B.4. This shows the need for new, more effective algorithms.

Many of the results about factors of Bernstein-Sato ideals imply that these ideals can in parts be obtained from the Bernstein-Sato polynomials of components by a combinatorial process in which only intersections play a role. This process, the role that intersection multiplicities and tangent cones play, and whether the whole Bernstein-Sato ideal can be constructed in this way, are some of the big questions that remain unsolved. Answers to these questions could connect Bernstein-Sato polynomials and Bernstein-Sato ideals in a different, geometric way and could explain why and how their structure differs.

A. Procedures for Singular

Here, the function headers of algorithms presented throughout this thesis, but not yet implemented in SINGULAR/PLURAL ([DGPS15]/[GLMS15]), are given according to the author's implementations.

Code A.1. First, we consider Algorithm 2.35 which computes compatible stratifications of

$$Q := (\operatorname{ann}_{D_n[\underline{s}]}(f^s) + {}_{D_n[\underline{s}]}\langle F\rangle) \cap \mathbb{C}[\underline{x}, \underline{s}], \quad Q \cap \mathbb{C}[\underline{x}] \quad \text{and} \quad Q \cap \mathbb{C}[\underline{s}],$$

which induce a stratification with respect to Bernstein-Sato ideals.

```
proc primDecStrat(ideal f, list #)
"USAGE:  primDecStrat(f [,outputFile]); f an ideal, outputFile a string
RETURN:  ring
PURPOSE: compute compatible primary decompositions for a stratification
@*  w.r.t. Bernstein-Sato ideals with the method of Bahloul/Oaku
ASSUME: basering is a commutative polynomial ring of characteristic 0
NOTE:  Activate the output ring with the @code{setring} command.
@*  It contains
@*    Lf=(ann(f^s)+<F>) intersected with K[x,s]
@*    B: list of primary components of Lf intersected with K[s]
@*    I: list of primary components of Lf intersected with K[x]
@*    Iprim: list of the radicals of the elements of I
@*  f: an ideal which contains the components of a vector of polynomials
@*  outputFile: if set, the results will be saved in outputFile
DISPLAY:  If printlevel=1, progress information will be printed.
@*  If printlevel>=2, progress and intermediate results will be printed.
"
```

An application example is given by

```
>   LIB "appendixA.lib"; //containing the procedures from Chapter A
>   ring R=0,(x,y),dp;
>   ideal f=x^2-y,y;
>   def A=primDecStrat(f);
>   setring A;
>   B; //primary components of B
[1]:
   _[1]=s(2)+1
[2]:
   _[1]=s(1)+1
[3]:
   _[1]=2*s(1)+2*s(2)+5
[4]:
   _[1]=2*s(1)+2*s(2)+3
```

```
>   I; //primary components of I
[1]:
   _[1]=y
[2]:
   _[1]=x^2-y
[3]:
   _[1]=y^2
   _[2]=x*y
   _[3]=x^3
[4]:
   _[1]=y
   _[2]=x
```

Code A.2. The result of Theorem 4.8 allows for a computation of upper and lower bounds of Bernstein-Sato ideals. The corresponding SINGULAR code is given in the following.

```
proc squeezer(ideal F,intvec m)
"USAGE:  squeezer(F,m); F an ideal, m an intvec
RETURN:  ring
ASSUME: basering is a commutative polynomial ring of characteristic 0
@*  m is a vector of non-negative integers
PURPOSE: determine upper and lower bounds of the Bernstein-Sato ideal associated to m
@*  (see [Bud13])
NOTE: returns ring with lists
@*  Bj, containing the Bernstein-Sato ideals associated to e_j,
@*  shiftedIdeals, containing the shifted ideals from [Bud13] 4.7,
@*  and ideals upperBound, lowerBound which give upper bounds
@*  and lower bounds for the Bernstein-Sato ideal associated to m.
"
```

An application example is given by

```
>   LIB "appendixA.lib"; //containing the procedures from Chapter A
>   ring R=0,(x,y),dp;
>   ideal f=x+y,y;
>   def A=squeezer(f, intvec(1,1));
>   setring A;
>   upperBound; //upper bound of the Bernstein-Sato polynomial
upperBound[1]=s(1)*s(2)+s(1)+s(2)+1
>   lowerBound; //lower bound of the Bernstein-Sato polynomial
lowerBound[1]=s(1)*s(2)+s(1)+s(2)+1
```

Code A.3. The following procedure computes $\operatorname{ann}_{D_n}(f^\alpha)$ for $f \in \mathbb{C}[\underline{x}]^r$ and $\alpha \in \mathbb{Q}^r$ following Algorithm 5.12.

```
proc annfalphaI(ideal f, vector alpha)
"USAGE:  annfalphaI(f,alpha); f an ideal, alpha a vector
RETURN:  ring
PURPOSE: determine annihilator of f^alpha in the n-th Weyl algebra
ASSUME: basering is a commutative polynomial ring in characteristic 0
EXAMPLE: example annfalphaI; shows example
NOTE: In the returned ring, annfalpha is the annihilator of f^alpha
@*  over the Weyl algebra
"
```

An application example is given by

```
>   LIB "appendixA.lib"; //containing the procedures from Chapter A
>   ring R = 0,(x,y,z),dp;
>   ideal f = x,y,z;
>   vector alpha = [1/4, 2/3, 1];
>   def A = annfalphaI(f,alpha);
>   setring A;
>   annfalpha;
annfalpha[1]=Dz^2
annfalpha[2]=z*Dz-1
annfalpha[3]=3*y*Dy*Dz-2*Dz
annfalpha[4]=3*y*Dy^2*Dz+Dy*Dz
annfalpha[5]=4*x*Dx*Dy^2*Dz-Dy^2*Dz
annfalpha[6]=4*x*Dx^2*Dy^2*Dz+3*Dx*Dy^2*Dz
annfalpha[7]=3*y*Dx^2*Dy^3*Dz+4*Dx^2*Dy^2*Dz
annfalpha[8]=4*x*Dx^3*Dy^2*Dz+7*Dx^2*Dy^2*Dz
```

Here, the generic annihilator is $\operatorname{ann}_{D_n[\underline{s}]}(f^s) = {}_{D_n[\underline{s}]}\langle x\partial_x - s_1, y\partial_y - s_2, z\partial_z - s_3\rangle$, so we see that $\operatorname{ann}_{D_n}(f^\alpha) \supsetneq \operatorname{ann}_{D_n[\underline{s}]}(f^s)|_{s=\alpha}$.

B. Applications of Singular

Here, some demonstrations of the application of the computer algebra system Singular/Plural ([DGPS15]/[GLMS15]) to examples are given. We will use both the procedures from Chapter A and procedures from the libraries `dmod.lib` ([LM15]) and `dmodvar.lib` ([ALM15]).

Code B.1 (Computation of $\mathcal{B}$ and $\mathrm{ann}_{D_n[\underline{s}]}(f^s)$).

```
> LIB "dmod.lib";
> ring R=0,(x,y),dp;
> ideal f=x^2-y^3,x-y+1;
> def A=annfsBMI(f);
> setring A;
> LD; //ann(f^s)
LD[1]=3*x^2*Dx-3*x*y*Dx+2*x*y*Dy-2*y^2*Dy+3*x*Dx+2*y*Dy-6*x*s(1)+6*y*s(1)
 -3*x*s(2)+2*y*s(2)-6*s(1)
LD[2]=3*y^3*Dx+3*y^3*Dy-3*x*y*Dx-3*x^2*Dy+2*x*y*Dy-2*y^2*Dy-9*y^2*s(1)
 +3*x*Dx+2*y*Dy+6*y*s(1)-3*x*s(2)+2*y*s(2)-6*s(1)
LD[3]=3*x*y^2*Dx+3*y^3*Dy-3*x*y*Dx+3*y^2*Dx-x^2*Dy-2*y^2*Dy-9*y^2*s(1)
 -3*y^2*s(2)+3*x*Dx+2*x*Dy+2*y*Dy+6*y*s(1)-x*s(2)+2*y*s(2)-6*s(1)
LD[4]=x*y^3*Dy-y^4*Dy-x^3*Dy+x^2*y*Dy+y^3*Dy-3*x*y^2*s(1)+3*y^3*s(1)
 +y^3*s(2)-x^2*Dy-3*y^2*s(1)-x^2*s(2)
> BS; //Bernstein-Sato ideal as factorization of its generator
[1]:
   _[1]=s(1)+1
   _[2]=6*s(1)+7
   _[3]=6*s(1)+5
   _[4]=s(2)+1
[2]:
   1,1,1,1
```

Code B.2 (primary ideals for a stratification with respect to $\mathcal{B}_p$).

```
> LIB "appendixA.lib"; //containing the procedures from Chapter A
> ring R=0,(x,y,z),dp;
> ideal f= x^2-y,y^3,x-1;
> primDecStrat(f);
//output is
Component Q_1: s(3)+1,x-1, dimension: 4
Component B_1: s(3)+1, dimension: 5
Component I_1: x-1, dimension: 5, radical: x-1
Component Q_2: 3*s(2)+2,y^2, dimension: 4
Component B_2: 3*s(2)+2, dimension: 5
Component I_2: y^2, dimension: 5, radical: y
Component Q_3: s(2)+1,y^3, dimension: 4
Component B_3: s(2)+1, dimension: 5
Component I_3: y^3, dimension: 5, radical: y
```

```
Component Q_4: 3*s(2)+1,y, dimension: 4
Component B_4: 3*s(2)+1, dimension: 5
Component I_4: y, dimension: 5, radical: y
Component Q_5: s(1)+1,x^2-y, dimension: 4
Component B_5: s(1)+1, dimension: 5
Component I_5: x^2-y, dimension: 5, radical: x^2-y
Component Q_6: 2*s(1)+6*s(2)+5,y^2,x*y,6*x^2*s(2)+2*x^2+y,x^3, dimension: 3
Component B_6: 2*s(1)+6*s(2)+5, dimension: 5
Component I_6: y^2,x*y,x^3, dimension: 4, radical: y,x
Component Q_7: 2*s(1)+6*s(2)+9,y^4,x*y^3,6*x^2*y^2*s(2)+6*x^2*y^2+y^3,6*x^3*y*s(2)
 +4*x^3*y+3*x*y^2,x^3*y^2,36*x^4*s(2)^2+36*x^4*s(2)+8*x^4+36*x^2*y*s(2)+24*x^2*y
 +3*y^2,6*x^5*s(2)+2*x^5+5*x^3*y,x^5*y,x^7, dimension: 3
Component B_7: 2*s(1)+6*s(2)+9, dimension: 5
Component I_7: y^4,x*y^3,x^3*y^2,x^5*y,x^7, dimension: 4, radical: y,x
Component Q_8: 2*s(1)+6*s(2)+3,y,x, dimension: 3
Component B_8: 2*s(1)+6*s(2)+3, dimension: 5
Component I_8: y,x, dimension: 4, radical: y,x
Component Q_9: 2*s(1)+6*s(2)+7,y^3,x*y^2,6*x^2*y*s(2)+4*x^2*y+y^2,6*x^3*s(2)
 +2*x^3+3*x*y,x^3*y,x^5, dimension: 3
Component B_9: 2*s(1)+6*s(2)+7, dimension: 5
Component I_9: y^3,x*y^2,x^3*y,x^5, dimension: 4, radical: y,x
```

Code B.3 (computation of $b_{\langle f \rangle}$).

```
> LIB "dmodvar.lib";
> ring R=0,(x,y),dp;
> ideal f=x^2-y^3,y^2;
> bfctVarAnn(f); //returns the roots of b_<f> and their multiplicities
[1]:
   _[1]=0
   _[2]=-1/2
   _[3]=-1
[2]:
   1,1,1
```

Code B.4 (application of Code A.2 to compute upper and lower bounds of $\mathcal{B}_f^m$).

```
> LIB "appendixA.lib"; //containing the procedures from Chapter A
> ring R=0,(x,y,z),dp;
> ideal f=x*y,(1-x)*y,x*(1-y),(1-x)*(1-y);
> intvec m=(0,2,3,1);
> def A=squeezer(f,m);
> setring A;
> Bj; //the ideals B_i
[1]:
   _[1]=s(1)^2+s(1)*s(2)+s(1)*s(3)+s(2)*s(3)+2*s(1)+s(2)+s(3)+1
[2]:
   _[1]=s(1)*s(2)+s(2)^2+s(1)*s(4)+s(2)*s(4)+s(1)+2*s(2)+s(4)+1
[3]:
   _[1]=s(1)*s(3)+s(3)^2+s(1)*s(4)+s(3)*s(4)+s(1)+2*s(3)+s(4)+1
[4]:
   _[1]=s(2)*s(3)+s(2)*s(4)+s(3)*s(4)+s(4)^2+s(2)+s(3)+2*s(4)+1
> size(upperBound); //the upper bound is principal
1
> size(lowerBound); //the lower bound is principal
```

```
1
> factorize(upperBound[1]);
[1]:
   _[1]=1
   _[2]=s(1)+s(2)+2
   _[3]=s(1)+s(2)+1
   _[4]=s(1)+s(3)+2
   _[5]=s(1)+s(3)+3
   _[6]=s(1)+s(3)+1
   _[7]=s(3)+s(4)+4
   _[8]=s(3)+s(4)+1
   _[9]=s(3)+s(4)+3
   _[10]=s(3)+s(4)+2
   _[11]=s(2)+s(4)+3
   _[12]=s(2)+s(4)+2
   _[13]=s(2)+s(4)+1
[2]:
   1,1,1,1,1,1,1,1,1,1,1,1,1
> factorize(lowerBound[1]);
[1]:
   _[1]=1
   _[2]=s(1)+s(2)+2
   _[3]=s(1)+s(2)+1
   _[4]=s(1)+s(3)+3
   _[5]=s(1)+s(3)+2
   _[6]=s(1)+s(3)+1
   _[7]=s(3)+s(4)+2
   _[8]=s(3)+s(4)+3
   _[9]=s(3)+s(4)+4
   _[10]=s(3)+s(4)+1
   _[11]=s(2)+s(4)+1
   _[12]=s(2)+s(4)+2
   _[13]=s(2)+s(4)+3
[2]:
   1,1,1,1,1,1,1,1,1,1,1,1,1
// upper and lower bound are the same, hence it is the Bernstein-Sato ideal
```

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Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Alle übernommenen Aussagen aus diesen Quellen habe ich als solche gekennzeichnet.

Aachen, September 2015, ____________________
(Robert Löw)