Annihilator of a Power of a Polynomial

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Problem formulation

Given a ring $R = \mathbb{C}[x_1, \ldots, x_n]$, a polynomial $f \in R$ and a number $\alpha \in \mathbb{C}$. Compute the left ideal $\text{Ann}(f^\alpha) \in D(R)$, where $D(R)$ is the Weyl algebra in $2n$ variables $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ subject to usual relations.

Preliminaries

We utilize a D-module structure of a left module in $R[f^s] := \mathbb{C}[x_1, \ldots, x_n, \frac{1}{f}] \cdot f^s$.
The algorithm AnnFs computes a D–module structure on $R[f^s]$, that is a left ideal $I \subset D$, such that $R[f^s] \cong D/I$.

Algebraic Analysis

Indeed, for A a G-algebra of Lie type, GrA is commutative and we have $\operatorname{GK.dim}_A(M) = \operatorname{GK.dim}_{GrA} L(M) = \operatorname{Kr.dim}_{GrA} L(M)$.

Theorem (Weak FTAA, SST)

A proper left ideal in $D(R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \ldots \rangle$ has GK–dimension $\geq n$.

Let $I \subset D(R)$. Compute the left Gröbner basis of I with respect to the elimination ordering for ∂_i (a weight vector $(0, \ldots, 0, 1, \ldots, 1)$). Then the **characteristic ideal** of I is the ideal in the commutative ring $GrD(R) = \mathbb{K}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$, generated by the leading terms of I. The zero set of this ideal is called the **characteristic variety**.

Theorem (Strong FTAA, SST)

Let I be a proper left ideal in $D(R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \ldots \rangle$. Then every minimal prime of a char. ideal of I has dimension $\geq n$.

Multiplicatively Closed Ore Subsets

Let S be a multiplicatively closed (**m.c.**) subset of some algebra A, that is $1 \in S$ and $a, b \in S \Rightarrow ab \in S$. S is called an **Ore set** in A, if

$$\forall s \in S, a \in A, \ \exists r \in S, b \in A \text{ such that } ar = sb \ (s^{-1}a = br^{-1}).$$

Ore condition

$\forall s \in S, a \in A, \quad sA \cap aS \neq \emptyset.$

For an associative $\mathbb{K}$–algebra A and a m.c. subset S, we consider $S \times A$ and introduce the following equivalence relation $\simeq$ on it: $(s, a) \simeq (r, b)$, if for some $x, y \in A$, $ax = by, sx = ry$.

Ore localization

Then, $S \times A / \simeq$ is called an **Ore localization** of A w.r.t. S.
It is often denoted by $A_S = \{s^{-1}a | s \in S, a \in A\}$.

Ann F^s Method: From Kashiwara to Malgrange

Recall, that for $s \in \mathbb{K}$, $\mathrm{Ann}_{D(R)} f^s = \{a \in D(R) \mid a \bullet f^s = 0\}$.

Theorem (Kashiwara 1981)

$D(R)/\mathrm{Ann}_{D(R)} f^s$ is a (regular) holonomic $D(R)$–module for any $s \in \mathbb{K}$.

Malgrange's construction for $f = f_1 \cdot \dots \cdot f_p$: consider the left ideal

$$I_f := \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i \} \rangle, 1 \leq j \leq p, 1 \leq i \leq n,$$

$$I_f \subset \mathbb{K}\langle \{t_j, \partial t_j\} \mid [\partial t_j, t_j] = 1 \rangle \otimes_{\mathbb{K}} \mathbb{K}\langle \{x_i, \partial_i\} \mid [\partial_i, x_i] = 1 \rangle$$

Theorem (1.)

The ideal of operators in $D[s] := D(R) \otimes_{\mathbb{K}} \mathbb{K}[s]$, annihilating f^s equals to the image of the $I_f \cap D[t \cdot \partial t]$ under the substitution $t \cdot \partial t \mapsto -s - 1$.

Proof: next slides.

Recall: Generalized Product Criterion

Let A be an associative $\mathbb{K}$–algebra. We use the following notations: $[a, b] := ab - ba$, a *commutator* or a *Lie bracket* of $a, b \in A$.

$\forall a, b, c \in A$ the following bracket identities hold

- $[a, b] = -[b, a]$, in particular $[a, a] = 0$
- $[ab, c] = a[b, c] + [a, c]b$

Recall Levandovskyy and Schönemann, ISSAC 2003.

Generalized Product Criterion

Let A be a G–algebra of Lie type (that is, all relations are of the type $x_j x_i = x_i x_j + d_{ij}, \forall 1 \leq i < j \leq n$.
Let $f, g \in A$. Suppose that $\text{lm}(f)$ and $\text{lm}(g)$ have no common factors, then $\text{spoly}(f, g) \to_{\{f,g\}} [f, g]$.

Proving the Theorem, part I

Let $f = f_1 \cdot \ldots \cdot f_p$ and let $g_i = \partial_i + \sum_j \frac{\partial f_j}{\partial x_i} \partial t_j$. A_{p+n} below stays for $D(R) \otimes_{\mathbb{K}} \mathbb{K}\langle \{t_i, \partial t_i \mid 1 \leq i \leq p\} \mid t_i \partial t_i = \partial t_i \cdot t_i + 1 \rangle$.

Lemma

$I_f = \langle \{t_j - f_j, \{g_i\}\}, 1 \leq j \leq p, 1 \leq i \leq n \rangle \subset A_{p+n}$ is a maximal ideal, hence A_{p+n}/I_f is a simple module.

Proof.

Choose an ordering with $\{t_i, \partial_i\} \gg \{x_i, \partial t_j\}$. Running Buchberger's algorithm, we see that

spoly$(g_i, g_k) \to [g_i, g_k] = \partial t_j \sum_j [\partial_i, \frac{\partial f_j}{\partial x_k}] + \partial t_j \sum_j [\frac{\partial f_j}{\partial x_i}, \partial_k]$. Since $[\partial_i, \frac{\partial f_j}{\partial x_k}] = \frac{\partial^2 f_j}{\partial x_i x_k}$, the spoly$(g_i, g_k)$ reduces to zero.

spoly$(t_k - f_k, g_i) \to [t_k - f_k, g_i] = \sum_j \frac{\partial f_j}{\partial x_i} [t_k, \partial t_j] - [f_k, \partial_i] = 0$.

Hence, I_f is given in a Gröbner basis and its leading monomials are $\{t_j, \partial_i\}$. Thus, the GK.dim$(A/I_f) = 2(p+n) - (p+n) = p+n$, hence I_f is holonomic. □

Proving the Theorem, part II

Consider the shift algebra $\mathbb{K}\langle s, E_s \mid E_s s = sE_s + E_s = (s+1)E_s\rangle$.
The Mellin transform is an injective $\mathbb{K}$–algebra homomorphism

$$\mathbb{K}\langle s, E_s = sE_s + E_s\rangle \to \mathbb{K}\langle t, \partial t \mid t \cdot \partial t = \partial t \cdot t + 1\rangle, \quad s \mapsto -t\partial t - 1,\ E_s \mapsto t.$$

Its image is the subalgebra $\mathbb{K}\langle t\partial t, t \mid \ldots\rangle$.

Lemma

I_f is the annihilator of f^s in $D(R) \otimes_{\mathbb{K}} \mathbb{K}\langle \{t_j, \partial t_j\} \mid t_j \partial t_j = \partial t_j t_j + 1\rangle$.

Proof.

The Mellin transform allows to supply $\mathbb{K}[\mathbf{s}, \mathbf{x}, f^s]$ with the following structure of $D(R) \otimes_{\mathbb{K}} \mathbb{K}\langle t, \partial t\rangle$–module ($p = 1$ for simplicity):

$$x_i \bullet g(s,x) f^s = x_i g(s,x) f^s,\ \partial_i \bullet g(s,x) f^s = \frac{\partial g}{\partial x_i} f^s + s g(s,x) \frac{\partial f}{\partial x_i} f^{s-1},$$

$$t \bullet g(s,x) f^s = g(s+1,x) f^{s+1},\ \partial t \bullet g(s,x) f^s = -s g(s-1,x) f^{s-1}.$$

□

Proving the Theorem, part III

Proof.

In particular, for $g = 1$ and $p = 1$ we have

$$x_i \bullet f^s = x_i f^s, \ \partial_i \bullet f^s = s \frac{\partial f}{\partial x_i} f^{s-1},$$

$$t \bullet f^s = f^{s+1}, \ \partial t \bullet f^s = -s f^{s-1}.$$

$$\text{Then } (t_j - f_j) \bullet f^s = 0, \ (\sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i) \bullet f^s = 0.$$

Since by Lemma before I_f is a maximal ideal, it is the Ann f^s. □

Ann F^s Algorithm in D–module Theory

Let $f = f_1 \cdot \ldots \cdot f_p$.

The Ann F^s Algorithm, step I

Compute the preimage of the left ideal

$$L = \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i \} \rangle, 1 \leq j \leq p, 1 \leq i \leq n$$

in the subalgebra $\mathbb{K} [\{t_j \cdot \partial t_j\}] \langle \{x_i, \partial_i \mid [\partial_i, x_i] = 1\} \rangle$ of

$$\mathbb{K}\langle \{t_j, \partial t_j\} \mid [\partial t_j, t_j] = 1 \rangle \otimes_{\mathbb{K}} \mathbb{K}\langle \{x_i, \partial_i\} \mid [\partial_i, x_i] = 1 \rangle$$

Moreover, in the preimage, $t_j \cdot \partial t_j$ will be replaced by $-s_j - 1$ (algebraic Mellin transform), where s_j are new variables, commuting with $\{x_k, \partial_k\}$.

Ann F^s Algorithm in D–module Theory

Remember, $f = f_1 \cdot \ldots \cdot f_p$.

The Ann F^s Algorithm, step II

Denote the result of step I by $L' \in \mathbb{K}[\{s_j\}]\langle\{x_i, \partial x_i \mid [\partial x_i, x_i] = 1\}\rangle$.
Compute the preimage of the left ideal $\langle L', f\rangle$ in the commutative subalgebra $\mathbb{K}[\{s_j\}]$.

If $p = 1$, e.g. $f = f_1$, the output is a principal ideal. Its monic generator is called a global Bernstein–Sato polynomial $b(s)$.
There exists an operator $B \in D(R)$, such that $B \bullet f^{s+1} = b(s) \cdot f^s$.

Theorem (Kashiwara)

All roots of $b(s)$ are rational numbers.

Note, that $s + 1$ always divides $b(s)$.

OT method for Step I

Oaku–Takayama method, 1999

$\{u_j, v_j, s_j\}$ commute with everything, $\{[\partial_i, x_i] = 1, [\partial t_j, t_j] = 1\}$.

$$\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i, u_j, v_j \mid \ldots \rangle \supset \langle \{t_j - u_j f_j, \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} u_k \partial t_j + \partial_i, u_j v_j - 1\}\rangle$$

1. Intersect the ideal with the subalgebra $\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i \mid \ldots \rangle$ i.e. eliminate $\{u_j, v_j\}$.
2. Intersect the result of p.1. with $\mathbb{K}[-t_j \partial t_j - 1] \otimes_{\mathbb{K}} \mathbb{K}\langle x_i, \partial_i \mid \ldots \rangle$, replace $-t_j \partial t_j - 1$ by s_j.

$p = 1$

$$\langle t - uf, uv - 1, \{\frac{\partial f}{\partial x_i} u \partial t + \partial_i\}\rangle$$

The total result lives in $\mathbb{K}\langle x_i, \partial_i \mid \ldots \rangle \otimes_{\mathbb{K}} \mathbb{K}[\{s\}]$

Given $\alpha \in \mathbb{C}$. Then we have the following:

Theorem (2.)

Let $f \in R = \mathbb{C}[x_1, \ldots, x_n]$ and α_0 is the minimal integer root of the global b-function $b(s)$ of f. If $\alpha \notin \alpha_0 + 1 + \mathbb{N}_0$, then $\mathrm{Ann}_{D(R)} f^{\alpha} = \mathrm{Ann}_{D(R)[s]} f^{s} |_{s=\alpha}$.

For the case, when $\alpha \in \alpha_0 + 1 + \mathbb{N}_0$, we apply the **Algorithm 3**:

1. Compute $\mathrm{Ann}\, f^s = \{g_1(s), \ldots, g_r(s)\} \subset D(R)[s]$.
2. Compute $b(s)$ of f; let $\alpha_0 :=$ the minimal integer root of $b(s)$.
3. Let $d = \alpha - \alpha_0$. If $d \leq 0$ output $\{g_i(\alpha)\}$ and stop.
4. For $d \in \mathbb{Z}_+$, compute the generators $\{s^{(k)}\}$ of the module
$$\mathrm{syz}(\ \{f^d, g_1(\alpha_0), \ldots, g_r(\alpha_0)\}\ ) \subset D(R)^{r+1}$$
5. Output $\{g_i(\alpha)\} \cup \{s_1^{(k)}\}$, where $s_1^{(k)}$ is the 1st component of $s^{(k)}$.

Examples with SINGULAR:PLURAL

1. (don't forget to) start SINGULAR
2. load the `dmod.lib` library by typing

```
> LIB "dmod.lib";
```

3. define a commutative ring R and a polynomial F, e.g.

```
> ring R = 0,(x,y,z),dp;  poly F = x3+y3+z3;
```

4. run the `annfsOT` routine. It returns a ring, call it, say, S

```
> def S = annfsOT(F);   setring S;
```

5. in the ring S (= Weyl algebra of R), there are the following computed objects:

   a) an ideal LD (the desired D-module structure)

   b) a list BS containing the roots (with mult's) of Bernstein poly.

6. If you wish to compute an s-parametric annihilator, run

```
> setring R;   def P = SannfsOT(F);   setring P;
```

7. in the output ring the ideal LD is the parametric D-module structure

Examples

In the same way as on the previous slide you can see how the algorithms of Brianson–Maisonobe and LOT work.

- OT: `annfsOT, SannfsOT`
- Brianson–Maisonobe, BM: `annfsBM, SannfsBM`
- LOT: `annfsLOT, SannfsLOT`
- Multivariate BM: `annfsBMI`
- All the relevant data at once: `operatorBM`

If you wish to see progress of each step of the algorithm, set before computation `printlevel = 1;`. If you wish to see additionally all intermediate data, set `printlevel = 2;`.

Example Session

x; x^4; $x^3 - y^2$; $y^5 + xy^4 + x^4$ (a Reiffen curve), $x^3 + y^3 + z^2 w$ (4 variables), $(x^3 + y^2) \cdot (x^2 + y^3)$.

Non-genericity with SINGULAR:PLURAL

Assume we have $F = x^3 + y^3 + z^3$ and we'd like to compute Ann F^n for $n \geq 1$. As we know from the example before, the minimal integer root is -2. So, any $n \geq -1$ leads to the exceptional situation (Algorithm 3 instead of Theorem 2 above). Let us compute the structure of the annihilator for $n = 3$.

1. define a commutative ring R and a polynomial F, e.g.

```
> ring R = 0,(x,y,z),dp;  poly F = x3+y3+z3;
```

2. run the `SannfsBM` routine. It returns a ring, call it, say, S

```
> def S = SannfsBM(F);  setring S;
```

3. in the ring $S = D(R)[s]$, there is an ideal LD (s-parametric D-module structure)

```
> int n = 3; poly F = imap(R,F);
> ideal I = annfspecial(LD,F,-2,n); I = groebner(I);
```

4. the ideal I is the desired D-module structure of F^3.

Thank you for your attention!

Please visit the SINGULAR homepage

- `http://www.singular.uni-kl.de/`
- there you find among others the online manual (with detailed documentation and examples for each command, procedure and library)

Integrals

For generic f, an integral $\int_C f(x,t)^\alpha t^\gamma dt$ satisfies a Gel'fand–Kapranov–Zelevinsky system [SST]. However, one needs to treat non–generic polynomials too.

Hypergeometric Integral

$$\text{Consider} \quad F(\alpha; x) = \int_C \prod_{i=1}^{p} f_i(x,t)^{\alpha_i} dt_1 \cdots dt_m,$$

where $\alpha_i \in \mathbb{K} \subseteq \mathbb{C}$ and C is an m–cycle. The function $F(\alpha; x)$ depends on the homology class of C. Let

$$D = \mathbb{K}\langle t_1, \ldots, t_m, \partial_{t_1}, \ldots, \partial_{t_m}, x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n} \mid$$

$$\partial_{t_j} t_i = t_i \partial_{t_j} + \delta_{ij}, \partial_{x_j} x_i = x_i \partial_{x_j} + \delta_{ij} \rangle$$

Theorem (SST, Th. 5.5.1)

Let $I \subset D$ be a left ideal, annihilating the function $f_\alpha(x,t) = \prod_{i=1}^{p} f_i(x,t)^{\alpha_i}$, $f_i \in \mathbb{K}[x_1,\ldots,x_n,t_1,\ldots,t_m]$. Then, the ideal

$$J = (I + \langle \partial_{t_1},\ldots,\partial_{t_m}\rangle_D) \cap \mathbb{K}\langle x_1,\ldots,x_n,\partial_{x_1},\ldots,\partial_{x_n} \mid \partial_{x_j}x_i = x_i\partial_{x_j} + \delta_{ij}\rangle$$

annihilates the function $F(\alpha;x)$.

The left ideal J is called the integral ideal of I with respect to t.

Note

In the Theorem, we have to intersect the sum of a left and a right ideals with a subalgebra. There are no general methods (only very specific, e.g. of Takayama) for treating this situation.