The Briaçon-Maisonobe Algorithm & Some Theory Behind

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Notation

- K arbirary field of characteristic zero (there should however be an embedding $K \hookrightarrow \mathbb{C}$)
- A_n (resp. $A_n(K)$) denotes the Weyl-algebra over K
- $f_1, \ldots, f_p$ are arbitrary (fixed) polynomials in $K[\underline{x}]$
- $F := f_1 \cdots f_p$
- $\underline{x} := (x_1, \ldots, x_\mathbf{n})$, $\underline{s} := (s_1, \ldots, s_\mathbf{p})$ $\underline{t} := (t_1, \ldots, t_\mathbf{p})$

The $A_n(K)[\underline{s}]$-Module $K[\underline{x}, \underline{s}, \frac{1}{F}]\underline{f}^{\underline{s}}$

Definition

$$M := K[\underline{x}, \underline{s}, \frac{1}{F}] f_1^{s_1} \cdots f_p^{s_p}$$

is an $A_n(K)[\underline{s}]$-module where the action of ∂_i on $\underline{f}^{\underline{s}}$ is given by

$$\partial_i \underline{f}^{\underline{s}} := \sum_{k=1}^{n} \frac{\frac{\partial f_j}{\partial x_k} s_k}{f_k} \underline{f}^{\underline{s}}$$

and is extended to arbitrary elements of M using

$$\partial_i (g \cdot \underline{f}^{\underline{s}}) := \frac{\partial g}{\partial x_i} \underline{f}^{\underline{s}} + g \cdot \partial_i \underline{f}^{\underline{s}}$$

The Bernstein Ideal

Definition

The Bernstein ideal of $f_1, \dots, f_p$ is defined as

$$I_{K,f_1,\dots,f_p} := \{b \in K[\underline{s}] \mid b \cdot \underline{f}^{\underline{s}} \in A_n(K)[\underline{s}] f_1^{s_1+1} \cdots f_p^{s_p+1}\}$$

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- $I_{K,f_1,\ldots,f_p}$ is an ideal in $K[\underline{s}]$
- For $p = 1$, $K[\underline{s}] = K[s_1]$ is a PID, hence $I_{K,F}$ is principal. Its generator is then called the *Bernstein polynomial* of F.

Properties

Lemma

Let (L/K) be a field extension. Then

$$I_{L,f_1,\ldots,f_p} = L \otimes_K I_{K,f_1,\ldots,f_p}$$

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Lemma (Action Of The Shift Algebra)

Consider the $A_n(K)[\underline{s}]$ module $M = K[\underline{s}, \frac{1}{F}]\underline{f}^{\underline{s}}$. The $A_n(K)[\underline{s}]$-action can be extended to an

$$\mathbf{A_n(K)\langle \underline{s}, \underline{t} | [t_i, s_i] = -t_i \rangle} \quad \textit{shift-algebra in } s_i, t_i$$

action via

$$\mathbf{t_i b(\underline{x}, \underline{s})\underline{f}^{\underline{s}} := -s_i \cdot b(\underline{x}, s_1, .., s_i - 1, .., s_p) f_1^{s_1} \cdot \cdot f_i^{s_i - 1} \cdot \cdot f_p^{s_p}}$$

The Bernstein Ideal is Nonzero

Theorem

$I_{K,f_1,\ldots,f_p} \neq \{0\}$.

What we need for the proof:

- the action of the shift algebra $\mathbf{A_n(K)}\langle \underline{\mathbf{s}}, \underline{\mathbf{t}} | [\mathbf{t_i}, \mathbf{s_i}] = -\mathbf{t_i} \rangle$: $t_i b(\underline{x}, \underline{s}) \underline{f}^{\underline{s}} := -s_i \cdot b(\underline{x}, s_1, .., s_i - 1, .., s_p) f_1^{s_1} \cdot\cdot f_i^{s_i - 1} \cdot\cdot f_p^{s_p}$
- $K(\underline{s})[\underline{x}, \frac{1}{F}]\underline{f}^{\underline{s}}$ is a holonomic $A_n(K(\underline{s}))$-module (*without proof*).
- Holonomic modules are artinian.

The Briaçon-Maisonobe Algorithm (Part I)

Setup

- $M := K[\underline{x}, \underline{s}, \frac{1}{F}]\underline{f}^{\underline{s}}$, $R := A_n(K)[\underline{s}]$,
  $\tilde{R} := A_n(K)\langle \underline{s}, \underline{t} | [t_i, s_i] = -t_i \rangle$

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Part I

$$\mathsf{Ann}_{\tilde{R}}(\underline{f}^{\underline{s}}) = \underbrace{\langle \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} t_j, s_k + f_k t_k | i \in [1, n], k \in [1, p] \rangle}_{=:J}$$

Hence $\mathsf{Ann}_R(\underline{f}^{\underline{s}}) = \mathsf{Ann}_{\tilde{R}}(\underline{f}^{\underline{s}}) \cap R$.

The Briançon-Maisonobe Algorithm (Part II)

Part II

Consider

$$N := R \cdot f_1^{s_1} \cdots f_p^{s_p} / R \cdot f_1^{s_1+1} \cdots f_p^{s_p+1}$$

Then

$$\mathsf{Ann}_R(\underbrace{\overline{f_1^{s_1} \cdots f_p^{s_p}}}_{\in N}) = \langle \underbrace{\mathsf{Ann}_R(f_1^{s_1} \cdots f_p^{s_p})}_{= J \cap R}, F \rangle$$

and hence

$$\mathbf{I_{K,f_1,\ldots,f_p} = \langle J \cap R, F \rangle_R \cap K[\underline{s}]}$$

Briaçon-Maisonobe in SINGULAR

- The Briaçon-Maisonobe algorithm is implemented in SINGULAR as part of the library "`dmod.lib`".
- The function is called `annfsBMI`, and it takes a list of polynomials (i.e. "ideal" in SINGULAR) as argument. It returns a ring which contains $\mathsf{Ann}_{A_n(K)[\underline{s}]}\, \underline{f}^{\underline{s}}$ (in `LD`) and $I_{K,f_1,\dots,f_p}$ (in `BS`).

A Filtration of $A_n(K)[\underline{s}]$

The filtration of the ring

Put $(A_n(K)[\underline{s}])_j := \{\sum_{|\underline{\alpha}|+|\underline{\beta}|\leq j} c_{\alpha,\beta}\underline{\partial}^{\underline{\alpha}}\underline{s}^{\underline{\beta}} | c_{\alpha,\beta} \in K[\underline{x}]\}$ *This is not the Bernstein-filtration!* Then

$$\mathsf{Gr}\, A_n(K)[\underline{s}] \cong K[\underline{x}, \underline{\xi}, \underline{s}]$$

where the x_i have degree 0 and the ξ_i, s_i degree 1. We will always take deg in this ring w.r.t. those degrees.

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Good filtration

Let M be an $A_n(K)[\underline{s}]$-module. We call a filtration $\Gamma = (\Gamma_k)_k$, $M = \bigcup_k \Gamma_k$ *good* if

- Each Γ_k is finitely generated $K[\underline{x}]$-module.
- $A_n(K)[\underline{s}]_j \Gamma_k = \Gamma_{j+k}$ for all $j \geq 0$ and all $k \gg 0$

Characteristic Varieties

Definition

Let M be an $A_n(K)[\underline{s}]$-module with a good filtration Γ. We call

$$V(\operatorname{Ann}_{\operatorname{Gr} A_n(K)[\underline{s}]} \operatorname{Gr} M) \subset K^n \times K^n \times K^p$$

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Theorem

The characteristic variety of $A_n(K)[\underline{s}]\underline{f}^{\underline{s}}$ *is*

$$V\left(\left\{P \in K[\underline{x}, \underline{\xi}, \underline{s}] \middle| P(\underline{x}, \sum_{j=1}^{p} \frac{\nabla f_j}{f_j} s_j, \underline{s}) = 0\right\}\right)$$

Finiteness of $A_n(K)[\underline{s}]\underline{f}^{\underline{s}}$ over $A_n(K)$

Lemma

Without proof Let $F \in K[\underline{x}]$. Denote by $J(F)$ the Jacobian ideal $J(F) := \langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \rangle$. Then $F \in \mathrm{rad}\, J(F)$ if and only if $\exists r \in \mathbb{N}, a_i \in J(F)^i$ such that $F^r + a_1 F^{r-1} + \ldots + a_r = 0$.

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Theorem (Case $p = 1$, i.e. $F = f_1$, $s := s_1$)

The following are equivalent:

1. $F \in \operatorname{rad} J(F)$
2. $\exists P = s^r + A_1 s^{r-1} + \ldots + A_r \in A_n(K)[s]$ such that $A_i \in A_n(K)$ is of degree (in $\underline{\partial}$) $\leq i$ and $PF^s = 0$.
3. $A_n(K)[s]F^s$ is finitely generated as $A_n(K)$-module.

What if $p > 1$?

Remark

In the case $p > 1$, $F \in \operatorname{rad} J(F)$ is no longer a sufficient condition for $A_n(K)\underline{s}]\underline{f}^{\underline{s}}$ being finitely generated over $A_n(K)$. It remains however necessary.

The End