The LOT Algorithm

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Problem formulation

Given a ring $R = \mathbb{C}[x_1, \ldots, x_n]$, a polynomial $f \in R$ and a number $\alpha \in \mathbb{C}$. Compute the left ideal $\mathrm{Ann}(f^\alpha) \in D(R)$.

Preliminaries

We utilize a D-module structure of a left module in $R[f^s] := \mathbb{C}[x_1, \ldots, x_n, \frac{1}{f}] \cdot f^s$.
The algorithm ANNFS computes a D–module structure on $R[f^s]$, that is a left ideal $I \subset D$, such that $R[f^s] \cong D/I$.

LOT = Levandovskyy's modification on Oaku-Takayama algorithm.

Ann F^s Algorithm in D–module Theory

Let $f = f_1 \cdot \ldots \cdot f_p$.

The Ann F^s Algorithm, step I

Compute the preimage of the left ideal

$$L = \langle \; \{ \; t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i \; \} \; \rangle, 1 \leq j \leq p, 1 \leq i \leq n$$

in the subalgebra $\mathbb{K} \, [\{t_j \cdot \partial t_j\}] \, \langle \{x_i, \partial_i \mid [\partial_i, x_i] = 1\} \rangle$ of

$$\mathbb{K}\langle \{t_j, \partial t_j\} \mid [\partial t_j, t_j] = 1 \rangle \otimes_{\mathbb{K}} \mathbb{K}\langle \{x_i, \partial_i\} \mid [\partial_i, x_i] = 1 \; \rangle$$

Moreover, in the preimage, $t_j \cdot \partial t_j$ will be replaced by $-s_j - 1$ (algebraic Mellin transform), where s_j are new variables, commuting with $\{x_k, \partial_k\}$.

Recall

Let $f = f_1 \cdot \ldots \cdot f_p$ and let $g_i = \partial_i + \sum_j \frac{\partial f_j}{\partial x_i} \partial t_j$.

Lemma

$L = \langle \{t_j - f_j, \{g_i\}\}, 1 \le j \le p, 1 \le i \le n \rangle \subset A_{p+n}$ is a maximal ideal, hence A_{p+n}/L is a simple module.

Oaku–Takayama method, 1999

$\{u_j, v_j, s_j\}$ commute with everything, $\{[\partial_i, x_i] = 1, [\partial t_j, t_j] = 1\}$.

$$\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i, u_j, v_j \mid \ldots \rangle \supset \langle \{t_j - u_j f_j, \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} u_k \partial t_j + \partial_i, u_j v_j - 1\} \rangle$$

1. Intersect the ideal with the subalgebra $\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i \mid \ldots \rangle$ i.e. eliminate $\{u_j, v_j\}$.
2. Intersect the result of p.1. with $\mathbb{K}[-t_j \partial t_j - 1] \otimes_{\mathbb{K}} \mathbb{K}\langle x_i, \partial_i \mid \ldots \rangle$, replace $-t_j \partial t_j - 1$ by s_j.

Anomalies With Elimination

Contrast to Commutative Case

In terminology, we rather use ”intersection with subalgebras” instead of ”elimination of variables”, since the latter may have no sense.

Let $A = \mathbb{K}\langle x_1, \dots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij}\}_{1 \le i < j \le n}\rangle$ be a G–algebra.
Consider a subalgebra A_r, generated by $\{x_{r+1}, \dots, x_n\}$.
We say that such A_r is an *admissible subalgebra*, if d_{ij} are polynomials in $x_{r+1}, \dots, x_n$ for $r+1 \le i < j \le n$ and $A_r \subsetneq A$ is a G–algebra.

Definition (Elimination ordering)

Let A and A_r be as before and $B := \mathbb{K}\langle x_1, \dots, x_r \mid \dots\rangle \subset A$
An ordering $\prec$ on A is an **elimination ordering for** $x_1, \dots, x_r$
if for any $f \in A$, $\operatorname{lm}(f) \in B$ implies $f \in B$.

Constructive Elimination Lemma

"Elimination of variables $x_1, \ldots, x_r$ from an ideal I"

means the intersection $I \cap A_r$ with an admissible subalgebra A_r.
In contrast to the commutative case:

- not every subset of variables determines an admissible subalgebra
- there can be no admissible elimination ordering $\prec_{A_r}$ on A

Lemma

Let A be a G–algebra, generated by $\{x_1, \ldots, x_n\}$ and $I \subset A$ be an ideal. Suppose, that the following conditions are satisfied:

- $\{x_{r+1}, \ldots, x_n\}$ *generate an essential subalgebra B,*
- $\exists$ *an admissible elimination ordering $\prec_B$ for $x_1, \ldots, x_r$ on A.*

Then, if S is a left Gröbner basis of I with respect to $\prec_B$, we have $S \cap B$ is a left Gröbner basis of $I \cap B$.

Anomalies With Elimination: Example

Example

Consider the algebra $A = \mathbb{K}\langle a, b \mid ba = ab + b^2 \rangle$.
It is a G–algebra with respect to any well–ordering, such that $b^2 \prec ab$, that is $b \prec a$. Any elimination ordering for b must satisfy $b \succ a$, hence A is not a G–algebra w.r.t. any elimination ordering for b.

The Gröbner basis of a two–sided ideal, generated by $b^2 - ba + ab$ in $\mathbb{K}\langle a, b \rangle$ w.r.t. an ordering $b \succ a$ is infinite and equals to

$$\{ba^{n-1}b - \frac{1}{n}(ba^n - a^n b) \mid n \geq 1\}.$$

Finding an admissible elimination ordering can be done by solving a linear programming problem.

Elimination Orderings in G–algebras

1. lexicographical orderings: mainly for (q–) Weyl algebras

$$\begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

2. block orderings $M_A \otimes M_B$: a universal tool

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

3. extra weight orderings $(a(w_1, \dots, w_k), ord)$: the champion!

$$\begin{pmatrix} w_1 & \dots & w_k & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}$$

Symmetric Deformation: Motivation

Let $\phi : A \to B$ be a map of K–algebras. There are the natural actions of A on B, induced by ϕ:

$$a \circ_L b := \phi(a)b \text{ and } b \cdot a := b \circ_R a := b\phi(a).$$

Observation

These actions provide a well–defined left and right A–module structures on B if and only if ϕ is a morphism.

Hence, B is an (A, A)–bimodule. We extend both actions to A by $a_1 \circ_L a_2 := a_1 \cdot a_2$ and thus turn $A \otimes_{\mathbb{K}} B$ into an (A, A)–bimodule.

Lemma

Consider the set $G = \{g - \phi(g) \mid g \in A\} \subset A \otimes_{\mathbb{K}} B$. Then

$$G = {}_A\langle\{x_i - \phi(x_i) \mid 1 \leq i \leq n\}\rangle_A \subset A \otimes_{\mathbb{K}} B.$$

Symmetric Deformation: Theorem

Theorem (Preimage of a Left Ideal)

Let A, B be G–algebras of Lie type and $\phi \in \mathrm{Mor}(A, B)$.
Let I_ϕ be the (A, A)–bimodule ${}_A\langle\{x_i - \phi(x_i) \mid 1 \le i \le n\}\rangle_A \subset A \otimes_{\mathbb{K}} B$ and $f_i := \phi(x_i)$. Suppose there exists an elimination ordering for B on $A \otimes_{\mathbb{K}} B$, such that

$$1 \le i \le n, 1 \le j \le m, \qquad \mathrm{lm}(y_j f_i - f_i y_j) \prec x_i y_j.$$

Then
1) *$A \otimes_{\mathbb{K}}^{\phi} B$, obtained from $A \otimes_{\mathbb{K}} B$ by introducing additional relations $\{y_j x_i = x_i y_j + y_j f_i - f_i y_j\}$, is a G–algebra.*
2) *Let $J \subset B$ be a left ideal, then*

$$\phi^{-1}(J) = (I_\phi + J) \cap A.$$

Application of Preimage Algorithm to D–modules

Setup with the Symmetric Deformation

$$A := \mathbb{K}\langle s_j, X_i, D_i \mid D_i X_i = X_i D_i + 1 \rangle$$

$$B := \mathbb{K}\langle t_j, \partial t_j, x_i, d_i \mid d_i x_i = x_i d_i + 1, \partial t_j t_j = t_j \partial t_j + 1 \rangle$$

Consider the map $\phi : A \to B$, where $s_j \mapsto -t_j \partial t_j - 1$, $X_i \mapsto x_i, D_i \mapsto d_i$.

Hence, $I_\phi = \langle \{X_i - x_i, D_i - d_i, t_j \partial t_j + s_j + 1\} \rangle \subset A \otimes_{\mathbb{K}}^{\phi} B =: E$.

Due to the structure, we replace E with $E' = \mathbb{K}\langle t_j, \partial t_j, x_i, d_i, s_j \rangle$ subject to the relations

$$\{[d_i, x_i] = 1, [\partial t_j, t_j] = 1, s_j t_j = t_j s_j - t_j, s_j \partial t_j = \partial t_j s_j + \partial t_j\}.$$

Respectively, $I_\phi \subset E$ becomes $I'_\phi = \langle \{t_j \partial t_j + s_j + 1\} \rangle \subset E'$.
Any ordering $\prec$ satisfying $\{t_j, \partial t_j\} \gg \{x_i, d_i, s_j\}$ (which is very easy to find) satisfies the conditions of the Theorem.

The Computation

By the Theorem, for any $L \subset B$, $\phi^{-1}(L) = (I_\phi + L) \cap A$. Hence,

$$I_\phi + L = \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i, t_j \partial t_j + s_j + 1 \} \rangle =$$

$$= \langle \{ t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i, f_j \partial t_j + s_j \} \rangle$$

The last step is just a reduction of $t_j + s_j + 1$ with $t_j - f_j$. In most situations, the ordering prefers t_j over x_i and s_k.

Citing Gago–Vargas, Hartillo and Ucha JSC paper from 2005...

"...As far as we know, the example $f = f_1 \cdot f_2 = (x^2 + y^3) \cdot (x^3 + y^2)$ is intractable for available computer algebra systems."
$\rightarrow$ Demonstration.

Comparison: OT vs. LOT

The relations on the variables: $\{u_j, v_j\}$ commute with everything,

$$\{[\partial_i, x_i] = 1, [\partial t_j, t_j] = 1, s_j t_j = t_j s_j - t_j, s_j \partial t_j = \partial t_j s_j + \partial t_j\}.$$

Oaku–Takayama method (1999)

$$\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i, u_j, v_j \mid \ldots \rangle \supset \langle \{t_j - u_j f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} u_j \partial t_j + \partial_i, u_j v_j - 1\} \rangle$$

to intersect with: $\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i \mid \ldots \rangle$, then with $\mathbb{K}[-t_j\partial t_j - 1] \otimes_{\mathbb{K}} \mathbb{K}\langle x_i, \partial_i \mid \ldots \rangle$, replace $-t_j\partial t_j - 1$ by s_j.

LOT method

$$\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i, s_j \mid \ldots \rangle \supset \langle \{t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j + \partial_i, f_j \partial t_j + s_j\} \rangle$$

to intersect with: $\mathbb{K}\langle x_i, \partial_i \mid \ldots \rangle \otimes_{\mathbb{K}} \mathbb{K}[\{s_j\}]$

Comparison: LOT vs. BM (Briançon–Maisonobe)

BM method (2002)

Non-comm relations: $\{[\partial_i, x_i] = 1, [\partial t_j, s_j] = -\partial t_j\}$.

$$\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i, s_j \mid \dots \rangle \supset \langle \{ s_j + f_j \partial t_j, \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k + \partial_i \} \rangle$$

Eliminate $\{\partial t_j\}$, i.e. intersect with $\mathbb{K}\langle \{x_i, \partial_i\} \mid \dots \rangle[\{s_j\}]$.

LOT method (2006)

NC relations: $\{[\partial_i, x_i] = 1, [\partial t_j, t_j] = 1, s_j t_j = t_j s_j - t_j, s_j \partial t_j = \partial t_j s_j + \partial t_j\}$.

$$\mathbb{K}\langle t_j, \partial t_j, x_i, \partial_i, s_j \mid \dots \rangle \supset \{ t_j - f_j, s_j + f_j \partial t_j, \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k + \partial_i \}$$

Eliminate $\{t_j, \partial t_j\}$, i.e. intersect with $\mathbb{K}\langle x_i, \partial_i \mid \dots \rangle \otimes_{\mathbb{K}} \mathbb{K}[\{s_j\}]$.

Comparison: LOT vs. BM contd

In LOT, we have to eliminate $\{t_j, \partial t_j\}$. We can eliminate $\{t_j\}$ first, that is intersect I_f above with the subalgebra

$$\mathbb{K}\langle\{\partial t_j, x_i, \partial_i, s_j\} \mid \{[\partial_i, x_i] = 1, s_j \partial t_j = \partial t_j s_j + \partial t_j\}\rangle.$$

This can be done by using any ordering giving precedence to t_j.
Let us fix an ordering $\prec_T$ with the property $\{t_j\} \gg \{\partial_i\} \succ \{\partial t_j, x_i, s_j\}$.
Clearly it is admissible. In the talk on Ann F^s we proved, that

$$S = \{t_j - f_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k\} \text{ is a left Gröbner basis w.r.t. } \prec_T$$

Lemma. $S' = S \cup \{s_j + f_j \partial t_j\}$ is a left Gröbner basis w.r.t. $\prec_T$.

Proof

Again, we apply generalized Product Criterion.

$\mathrm{spoly}(t_k - f_k, s_j + f_j\partial t_j) \to [t_k - f_k, s_j + f_j\partial t_j] =$

$$[t_k, s_j] + f_j[t_k, \partial t_j] - [f_k, s_j] - [f_k, f_j\partial t_j] = \delta_{jk}(t_k - f_k) \to 0$$

$$\mathrm{spoly}(s_j + f_j\partial t_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i}\partial t_k) \to [s_j + f_j\partial t_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i}\partial t_k] =$$

$$= [s_j, \partial_i] + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i}[s_j, \partial t_k] + \partial t_j[f_j, \partial_i] + [*, *] = \frac{\partial f_j}{\partial x_i}\partial t_j - [\partial_i, f_j]\partial t_j = 0$$

□

BM = LOT + elimination

Since S' is a left Gröbner basis with respect to $\prec_T$ (an ordering eliminating $\{t_j\}$), then from the Elimination Lemma it follows, that

$$\{\ \{s_j + f_j\partial t_j \mid 1 \le j \le p\}, \{\partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i}\partial t_k \mid 1 \le i \le n\}\ \}$$

is a left Gröbner basis of
$S' \cap \mathbb{K}\langle\{\partial t_j, x_i, \partial_i, s_j\} \mid \{[\partial_i, x_i] = 1, s_j\partial t_j = \partial t_j s_j + \partial t_j\}\rangle$.

And the latter is exactly the statement of BM! Hence the claim in the title.