The Noro Algorithm for Computing the Global b-function

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Let K be an field and $D = K\langle x, \partial\rangle = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ the n-dimensional Weyl algebra over K.

Definition

Let $0 \neq w \in \mathbb{R}^n$ be a weight vector. For $p = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \in D$ put $m = \max_{\alpha,\beta}\{-w\alpha + w\beta \mid c_{\alpha\beta} \neq 0\}$.
We call

$$\mathrm{in}_{(-w,w)}(p) := \sum_{\substack{\alpha,\beta \\ -w\alpha+w\beta=m}} c_{\alpha\beta} x^\alpha \partial^\beta \in D$$

the *initial form* of p w.r.t. to w.
For a D-ideal I, $\mathrm{in}_{(-w,w)}(I) = K \cdot \{\mathrm{in}_{(-w,w)}(p) \mid p \in I\}$ is a D-ideal.

Definition

Let I be a holonomic D-ideal and $0 \neq w \in \mathbb{R}^n$.
Set $s = \sum_{i=1}^{n} w_i\theta_i$ for $\theta_i = x_i\partial_i$. Then $\mathrm{in}_{(-w,w)}(I) \cap K[s]$ is a non zero principal ideal in $K[s]$.
We call its monic generator $b(s)$ the *global b-function of I* w.r.t. w.

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Recall: For a polynomial f, the Malgrange ideal
$I_f = \langle t - f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \ldots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t \rangle \subseteq D\langle t, \partial_t \rangle$ is holonomic.

Definition

Let $w = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ be a weight vector such that the weight of ∂_t is 1, and $B(s)$ the b-function of I_f w.r.t. w.
We call $b(s) = B(-s-1)$ the *global b-function of f*.

Masterplan

Computing the b-function of a holomonic D-ideal I w.r.t. a weight w in two steps:

1. Compute $J = \mathrm{in}_{(-w,w)}(I)$
2. Compute the minimal polynomial of s in D/J

Masterplan

Computing the b-function of a holonomic D-ideal I w.r.t. a weight w in two steps:

1. Compute $J = \text{in}_{(-w,w)}(I)$
2. Compute the minimal polynomial of s in D/J

Step 1: weighted homogenization to improve the effeciency for $I = I_f$

Step 2: method of indeterminate coefficient

Computation of $\mathrm{in}_{(-w,w)}(I)$

1. Define a (non-term) monomial order $<_{(-w,w)}$ with a term order $<$:

$$\begin{array}{rl} & x^\alpha \partial^\beta <_{(-w,w)} x^\gamma \partial^\delta \\ \Leftrightarrow & -w\alpha + w\beta < -w\gamma + w\delta \\ \text{or} & -w\alpha + w\beta = -w\gamma + w\delta \text{ and } x^\alpha \partial^\beta < x^\gamma \partial^\delta \end{array}$$

   Compute a Gröbner basis G of I w.r.t. $<_{(-w,w)}$.

2. $G_{(-w,w)} = \{\mathrm{in}_{(-w,w)}(g) \mid g \in G\}$ is a Gröbner basis of $\mathrm{in}_{(-w,w)}(I)$ w.r.t. $<$.

Weighted homogenization

Let $u, v \in \mathbb{R}^n_{>0}$. Consider $D^{(h)}_{(u,v)} = K\langle x, \partial, h\rangle$ with non commutative relations $\partial_i x_i = x_i \partial_i + h^{u_i+v_i}, 1 \leq i \leq n$.

For $p = \sum c_{\lambda\alpha\beta} h^\lambda x^\alpha \partial^\beta$ define the *weighted total degree* of p:

$$\deg_{(u,v)}(p) = \max\{\lambda + u\alpha + v\beta \mid c_{\lambda\alpha\beta} \neq 0\}$$

For $p = \sum c_{\alpha\beta} x^\alpha \partial^\beta$ define the *weighted homogenization* of p:

$$H_{(u,v)}(p) = \sum c_{\alpha\beta} h^{\deg_{(u,v)}(p)-(u\alpha+v\beta)} x^\alpha \partial^\beta$$

For a monomial order $<$ in D define a term order $<^h$ in $D^{(h)}_{(u,v)}$:

$$
\begin{array}{ll}
 & p <^h q \\
\Leftrightarrow & \deg_{(u,v)}(p) < \deg_{(u,v)}(q) \\
\text{or} & \deg_{(u,v)}(p) = \deg_{(u,v)}(q) \text{ and } p\,|_{h=1} < q\,|_{h=1}
\end{array}
$$

Theorem

Let F be a finite subset of D.
If G^h is a Gröbner basis of $\langle H_{(u,v)}(F)\rangle$ w.r.t. $<^h$, then $G^h\,|_{h=1}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $<$.

For $f = \sum c_\alpha x^\alpha \in K[x_1, \dots x_n]$ and $u \in \mathbb{R}^n_{>0}$ let $\deg_u(f) = \max_\alpha\{u\alpha\}$ denote the weighted total degree of f w.r.t. u.
Choose a weight vector $(\hat{u}, \hat{v})$ in $K\langle t, x, \partial_t, \partial\rangle$ defined by
$\hat{u} = (\deg_u(f), u_1, \dots, u_n)$,
$\hat{v} = (1, \deg_u(f) - u_1 + 1, \dots, \deg_u(f) - u_n + 1)$.
Let $F = \{t - f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t\}$ be the generator set of I_f and $\hat{f} = H_{(\hat{u},\hat{v})}(f)$, then

$$H_{(\hat{u},\hat{v})}(F) = \{t - \hat{f}, \partial_1 + \frac{\partial \hat{f}}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial \hat{f}}{\partial x_n}\partial_t\}.$$

For $f = \sum c_\alpha x^\alpha \in K[x_1, \dots x_n]$ and $u \in \mathbb{R}^n_{>0}$ let $\deg_u(f) = \max_\alpha\{u\alpha\}$ denote the weighted total degree of f w.r.t. u.
Choose a weight vector $(\hat{u}, \hat{v})$ in $K\langle t, x, \partial_t, \partial\rangle$ defined by
$\hat{u} = (\deg_u(f), u_1, \dots, u_n)$,
$\hat{v} = (1, \deg_u(f) - u_1 + 1, \dots, \deg_u(f) - u_n + 1)$.
Let $F = \{t - f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t\}$ be the generator set of I_f and $\hat{f} = H_{(\hat{u},\hat{v})}(f)$, then

$$H_{(\hat{u},\hat{v})}(F) = \{t - \hat{f}, \partial_1 + \frac{\partial \hat{f}}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial \hat{f}}{\partial x_n}\partial_t\}.$$

Lemma

For any term order $<_0$ such that the leading monomials of $H_{(\hat{u},\hat{v})}(F)$ are $t, \partial_1, \dots, \partial_n$, $H_{(\hat{u},\hat{v})}(F)$ is a Gröbner basis of $\langle H_{(\hat{u},\hat{v})}(F)\rangle$ w.r.t. $<_0$.

Let $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \notin p\mathbb{Z}\}$ and $\phi_p : \mathbb{Z}_{(p)} \to \mathbb{Z}_p$ be the canonical projection.

Algorithm: ModularChangeOfOrdering($G_0, <_0, <$)

Input: a Gröbner basis $G_0 \subset D$ w.r.t. $<_0$ with each element having the monic head term
a term order $<$
Output: the reduced Gröbner basis G of G_0 w.r.t. $<$
do
  $p \leftarrow$ a new prime such that $G_0 \subset \mathbb{Z}_{(p)}[x, \partial]$
  $G_p \leftarrow$ the reduced Gröbner basis w.r.t. $<$
  If there exists $G \subset \langle G_0 \rangle$ such that $\phi_p(G) = G_p$
  then return G
end do

Computation of the minimal polynomial in D/J

Saito, Sturmfels, Takayama:

1. Compute $J' = J \cap K[\theta_1, \ldots, \theta_n]$
2. Compute $J' \cap K[s]$

But we can make use of the Gröbner basis we have already computed.

Algorithm: `MinimalPolynomial`$(G, <, P)$

```
Input:   a Gröbner basis G of a D-ideal J w.r.t. <,
         P ∈ D such that J ∩ K[P] ≠ {0}
Output:  b(s) ∈ K[s] such that J ∩ K[P] = ⟨b(P)⟩
i ← 1
do  {
      If there exist a_{i-1}, ..., a_0 ∈ K such that
         NF(P^i, G) + Σ_{j=0}^{i-1} a_j NF(P^j, G) = 0
      then return s^i + Σ_{j=0}^{i-1} a_j s^j
      else i ← i + 1
}
```

Input: a Gröbner basis G of a D-ideal J w.r.t. $<$, $P \in D$ such that $J \cap K[P] \neq \{0\}$

Output: $b(s) \in K[s]$ such that $J \cap K[P] = \langle b(P) \rangle$

$i \leftarrow 1$

do {

If there exist $a_{i-1}, \ldots, a_0 \in K$ such that $\mathrm{NF}(P^i, G) + \sum_{j=0}^{i-1} a_j \, \mathrm{NF}(P^j, G) = 0$

then return $s^i + \sum_{j=0}^{i-1} a_j s^j$

else $i \leftarrow i + 1$

}

Assumptions

- $K = \mathbb{Q}$
- J a D-ideal, $P \in D$ such that $J \cap \mathbb{Q}[P] = \langle b(P) \rangle$ for $b(s) \in \mathbb{Z}[s]$
- G a Gröbner basis of J w.r.t. a term order $<$ such that each element in G is monic w.r.t. $<$.
- P has integral coefficients
- b is primitive over $\mathbb{Z}$

Lemma

For a prime p such that $G \subset \mathbb{Z}_{(p)}\langle x, \partial \rangle$, $\phi_p(G)$ *is a Gröbner basis of* $\langle \phi_p(G) \rangle$ *w.r.t.* $<$ *and* $\phi_p(b(P)) \in \langle \phi_p(G) \rangle$.

Theorem

Let $b_p(s)$ *be the minimal polynomial of* $\phi_p(P)$ *in* $\phi_p(D)/\langle \phi_p(G) \rangle$. *If there exists* $f \in \mathbb{Z}[s]$ *such that* $\deg(f(s)) = \deg(b_p(s))$ *and* $f(P) \in \langle G \rangle$, *then* $f(s) = b(s)$.

Algorithm: ModularMinimalPolynomial$(G, <, P)$

```
Input:   a Gröbner basis G of a D-ideal J w.r.t. <,
         each element in G is monic w.r.t. <,
         P ∈ D such that P is integral and J ∩ Q[P] ≠ {0}
Output:  b(s) ∈ Q[s] such that J ∩ Q[P] = ⟨b(P)⟩
start:
p ← a new prime such that G ∈ Z_(p)⟨x, ∂⟩
i ← 1
do    {
      If there exist a_{i-1}, ..., a_0 ∈ Z_p such that
(L_p) φ_p(NF(P^i, G)) + Σ_{j=0}^{i-1} a_j φ_p(NF(P^j, G)) = 0
      then  {
            If there exist a_{i-1}, ..., a_0 ∈ Q such that
(L)         NF(P^i, G) + Σ_{j=0}^{i-1} a_j NF(P^j, G) = 0
              return s^i + Σ_{j=0}^{i-1} a_j s^j
            else goto start
      } else i ← i + 1
}
```

Input: a Gröbner basis G of a D-ideal J w.r.t. $<$, each element in G is monic w.r.t. $<$, $P \in D$ such that P is integral and $J \cap \mathbb{Q}[P] \neq \{0\}$

Output: $b(s) \in \mathbb{Q}[s]$ such that $J \cap \mathbb{Q}[P] = \langle b(P) \rangle$

start:

$p \leftarrow$ a new prime such that $G \in \mathbb{Z}_{(p)}\langle x, \partial \rangle$

$i \leftarrow 1$

do {

If there exist $a_{i-1}, \ldots, a_0 \in \mathbb{Z}_p$ such that

(L_p) $\phi_p(\mathrm{NF}(P^i, G)) + \sum_{j=0}^{i-1} a_j \phi_p(\mathrm{NF}(P^j, G)) = 0$

then {

If there exist $a_{i-1}, \ldots, a_0 \in \mathbb{Q}$ such that

(L) $\mathrm{NF}(P^i, G) + \sum_{j=0}^{i-1} a_j \mathrm{NF}(P^j, G) = 0$

return $s^i + \sum_{j=0}^{i-1} a_j s^j$

else goto start

} else $i \leftarrow i + 1$

}

Algorithm

```
 Input:    an integral polynomial f(x_1, ..., x_n),
           an optional weight vector u
 Output:   the global b-function of f
```

If u is not given $u \leftarrow (1, \ldots, 1)$

$d \leftarrow \deg_u(f)$

$\hat{u} \leftarrow (d, u_1, \ldots, u_n)$

$\hat{v} \leftarrow (1, d + a - u_1, \ldots, d + 1 - u_n)$

$\hat{f} \leftarrow H_{(\hat{u},\hat{v})}(f)$

$\hat{G}_f \leftarrow \{t - \hat{f}, \partial_1 + \frac{\partial \hat{f}}{\partial x_1}\partial_t, \ldots, \partial_n + \frac{\partial \hat{f}}{\partial x_n}\partial_t\}$

$<_0 \leftarrow$ a term order s.t. $\hat{G}_f$ is a Gröbner basis w.r.t. $<_0$

$< \leftarrow$ the wgrlex order w.r.t. $(\hat{u}, \hat{v})$

$<^h_{(-w,w)} \leftarrow$ the homogenized term order with a tie breaker $<$

$G^h \leftarrow$ ModularChangeOfOrdering$(\hat{G}_f, <_0, <^h_{(-w,w)})$

$G \leftarrow \mathrm{in}_{(-w,w)}(G^h \mid_{h=1})$

$B(s) \leftarrow$ ModularMinimalPolynomial$(G, <, t\partial_t)$

return $B(-s-1)$