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On Preimages of Ideals in Certain Non–commutative Algebras

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Abstract. In this paper we present new algorithms for non–commutative Gröbner ready algebras, which enable one to perform advanced operations with ideals and modules. In spite of the big interest in algorithmic treatment of related problems, preimage of ideal and central character decomposition were not discussed before.
An important algorithm for computation of the kernel of a homomorphism of left modules is described in the form, optimized for performance. We present these algorithms together with their implementation in computer algebra system SINGULAR:PLURAL and detailed applications.

Keywords. PBW algebras, Gröbner bases, intersection with subalgebra, central character decomposition, kernel of homomorphism

To Prof. Dr. Yuriy Drozd on his 60th birthday

Introduction

Non–commutative algebras with PBW basis admitting a Gröbner bases theory, quite similar to the one in the commutative case, appeared as a class in the 1980's and have been studied until now under different names: G–algebras ([2,12]), algebras of solvable type ([10,14]), Poincaré–Birkhoff–Witt (or, shortly, PBW) algebras ([3,4]). Teo Mora treated them in [15,16] without giving a special name.

After many important works and several implementations, the interest grew — reflected by the appearance of two recent books, namely by H. Li ([14], 2002) and by J. Bueso et.al. ([3], 2003) on the subject. Both books feature many interesting applications of Gröbner bases, related in particular to the ring theory and to the representation theory of algebras, but such an important question as the algorithmic treatment of morphisms between G–algebras was not discussed at all.

Especially great role is played by two commutative subalgebras - a center and a Gel'fand–Zetlin subalgebra. In the representation theory there are many constructions involving them and there is a big need for, in particular, intersection of modules with such subalgebras.

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We are going to present corresponding algorithms and applications together with the efficient implementation in the computer algebra system SINGULAR:PLURAL ([13]). Note, that at present no other computer algebra system features such non–commutative functionality as we provide with the SINGULAR:PLURAL and only a few systems can be compared with our rich collection of commutative procedures.

1. G–algebras and Morphisms of Algebras

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$ be a commutative polynomial ring in n variables. Suppose there is a well–ordering $<$ on R and two sets of data: $C = \{c_{ij}\} \subset \mathbb{K}^*$ and $D = \{d_{ij}\} \subset R$ (here $1 \leq i < j \leq n$).

If $\forall\ i < j$, $\mathrm{lm}(d_{ij}) < x_i x_j$ (by $\mathrm{lm}(f)$ we denote the leading monomial of f with respect to the given ordering), we can associate to the data $(R, <, C, D)$ a non–commutative algebra

$$A = \mathbb{K}\langle x_1, \ldots, x_n \mid \forall\ i < j\ x_j x_i = c_{ij} x_i x_j + d_{ij}\rangle.$$

We say that the algebra A has a PBW basis, if the $\mathbb{K}$–basis of A is $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$. A construction above does not guarantee us this property in general.

For $1 \leq i < j < k \leq n$ we define the **non–degeneracy condition** for (i, j, k) to be the polynomial

$$\mathcal{NDC}_{ijk} = c_{ik}c_{jk} \cdot d_{ij}x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik}x_j + d_{jk}x_i - c_{ij}c_{ik} \cdot x_i d_{jk}.$$

Theorem 1. *([12]). Let A be as before. Then algebra A has a PBW basis if and only if $1 \leq i < j < k \leq n$, $\mathcal{NDC}_{ijk} = 0$.*

We say, that algebra A is a G–**algebra**, if it satisfies the condition of the previous theorem.

Theorem 2. *([12]). Let A be a G–algebra. Then*

1) A is left and right noetherian,
2) A is an integral domain,
3) A has left and right quotient rings.

Let T be a proper two–sided ideal in the G–algebra A. Then the factor algebra $B = A/T$ is well–defined; we call such algebras GR–**algebras**. Note, that tensor products over a field and taking an opposite algebra operations are invariant with respect to GR–algebras.

The framework of GR–algebras provides a common roof for many interesting algebras. For example, universal enveloping algebras of finite dimensional Lie algebras, many quantum groups, some iterated Ore extensions ([10]) and many important algebras, associated to operators ([5]) are GR–algebras.

A finite Gröbner bases theory exists in GR–algebras and it is well investigated, although it is not as complete as the contemporary books on commutative Gröbner bases show ([8]). We will not even sketch the theory of Gröbner bases in this article, directing the reader to [3,13,14]. However, we use several notations from the Gröbner bases theory: $\mathrm{NF}(X \mid G)$ denotes a normal form of an object (polynomial or a module) X with respect to the set G, and $\mathrm{Syz}(M)$ denotes a module of syzygies of a module M.

Let A and B be associative $\mathbb{K}$–algebras. Recall, that a map $\psi : A \to B$ is defined by its values on generators $\{x_i\}$ of A, that is $\psi : x_i \mapsto p_i$, $\{p_1, \dots, p_n\} \subset B$. ψ is called **a (homo)morphism of $\mathbb{K}$–algebras**, if $\forall x, y \in A$

- $\psi(1) = 1, \quad \psi(x + y) = \psi(x) + \psi(y),$
- $\psi(xy) = \psi(x)\psi(y).$

Let $\mathcal{GR}$ denote the category of GR–algebras and $\mathcal{G}$ be its subcategory of G–algebras. We denote by $\mathrm{Mor}(A, B)$ (respectively $\mathrm{Mor}(\mathcal{A}, \mathcal{B})$) the set of morphisms between $A, B \in \mathcal{G}$ (respectively $\mathcal{A}, \mathcal{B} \in \mathcal{GR}$).

Let $A, B \in \mathcal{G}$. Suppose there are proper two–sided ideals $T_A \subset A$, $T_B \subset B$, already given as two–sided Gröbner bases and there are GR–algebras $\mathcal{A} = A/T_A$ and $\mathcal{B} = B/T_B$.

Starting with the map $\psi : A \to B$, we define the induced map $\Psi : \mathcal{A} \to \mathcal{B}$ by setting $\Psi(\bar{a}) := \overline{\psi(a)}$, where we can choose $a = \mathrm{NF}(\bar{a} \mid T_A)$ as a representative for $\bar{a} \in \mathcal{A}$.

Remark 3. *On the contrary to the commutative case, not every map of GR–algebras is a morphism.*

Define the **obstruction polynomials** $o_{ij} := \psi(x_j x_i) - \psi(x_j)\psi(x_i)$ *and the* **ideal of obstructions of** ψ *to be* $O_\psi := \langle \{o_{ij} | 1 \le i < j \le n\} \rangle \subseteq B$. *Respectively, the ideal of obstructions of* Ψ *is* $\mathcal{O}_\Psi = BO_\psi/T_B \subseteq \mathcal{B}$. *Following the definition, we see that*

- $\psi \in \mathrm{Mor}(A, B) \Leftrightarrow O_\psi = \langle 0 \rangle \subset B,$
- $\Psi \in \mathrm{Mor}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{O}_\psi = \langle 0 \rangle \subset \mathcal{B} \Leftrightarrow \mathrm{NF}(O_\psi \mid T_B) = 0.$

For each G–algebra A, there are several natural commutative subalgebras.

- $Z(A) := \{z \in A \mid za = az \ \forall \ a \in A\}$ is the center of A ([6]);
- if there exists a Cartan subalgebra $H(A)$ ([6]), it is commutative;
- from two previous subalgebras, we can construct a bigger subalgebra $CZ(A) := H(A) \otimes_{\mathbb{K}} Z(A)$;
- Gel'fand–Zetlin subalgebra $GZ(A)$ ([7]), if it exists.

Note, that if both $CZ(A)$ and $GZ(A)$ exist, then $GZ(A) \supseteq CZ(A) \supset Z(A)$ holds. Ovsienko ([18]) proved, that if $GZ(A)$ exists, it is the biggest commutative subalgebra of A. Note, that the construction of Gel'fand–Zetlin subalgebra has not been yet completely algorithmized.

On the contrary, there is a general algorithm for computing the center of a GR–algebra up to a given degree. Recently it has been implemented in Singular:Plural ([17]); we used it for computations of examples below.

2. Morphisms from Commutative Algebras to GR–algebras

Let $A = \mathbb{K}[y_1, \dots, y_m]$, $T_A \subset A$ be an ideal and $\mathcal{A} = A/T_A$ be a commutative GR–algebra. Let $B = \mathbb{K}\langle x_1, \dots, x_n \mid x_j x_i = c_{ij} x_i x_j + d_{ij}, \forall j > i \rangle$ be a G–algebra, $T_B \subset B$ be a two–sided ideal and $\mathcal{B} = B/T_B$ be a GR–algebra.

For polynomials a, b we use the notation $[a, b] = ab - ba$.

Let $F = \{f_1, \dots, f_m\} \subset \mathcal{B}$ be the set of pairwise commuting polynomials. Consider a map of $\mathbb{K}$–algebras $\mathcal{A} \xrightarrow{\phi} \mathcal{B}$, $\phi : y_i \mapsto f_i \in \mathcal{B}$. Then, according to the Remark 3, such ϕ is always a morphism.

Suppose there is an ideal $\mathcal{J} \subset \mathcal{B}$. In this section we present an algorithm for computation of the preimage of an ideal under such map.

2.1. Algorithm for Computing a Preimage

Let us describe the structure of $\mathcal{E} = \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$. Let $E = A \otimes_{\mathbb{K}} B$ be the algebra in variables $\{x_i \otimes 1 \mid 1 \le i \le n\}$ and $\{1 \otimes y_j \mid 1 \le j \le m\}$, which we identify with $\{x_i\}$ and $\{y_j\}$ respectively. Then E is a G–algebra

$$E = \mathbb{K}\langle y_1, \dots, y_m, x_1, \dots, x_n \mid [y_k, y_\ell] = [y_k, x_i] = 0, x_j x_i = c_{ij} x_i x_j + d_{ij} \rangle,$$

with indices $\forall\, 1 \le k, \ell \le m, \forall\, 1 \le i < j \le n$.

If T_A and T_B were given as two–sided Gröbner bases, their images in E under canonical inclusions keep this property. Hence, the ideal $T_E = T_A + T_B$ is a two–sided ideal, given in its two–sided Gröbner basis. Then $\mathcal{E} \cong E/T_E$ is a GR–algebra. We denote such construction as $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B})$ in the sequel and identify $\mathcal{A}$ and $\mathcal{B}$ with corresponding admissible subalgebras of $\mathcal{E}$.

Theorem 4. *Let $\mathcal{A} = \mathbb{K}[y_1, \dots, y_m]/T_A$, $\mathcal{B} \in \mathcal{GR}$, $\Phi \in \mathrm{Mor}(\mathcal{A}, \mathcal{B})$ and $\mathcal{J} \subset \mathcal{B}$ be a left ideal. Let I_Φ be a left ideal $\langle \{y_i - \phi(y_i) \mid 1 \le i \le m\} \rangle \subset \mathcal{E}(\mathcal{A}, \mathcal{B})$. Then*

$$\Phi^{-1}(\mathcal{J}) = (I_\phi + \mathcal{J}) \cap \mathcal{A}.$$

Proof. 1. Consider some polynomial $p = \sum_{\alpha \in \mathbb{N}} c_\alpha y^\alpha \in \mathcal{A}$ with all but finite number of c_α are zero. For $0 \le k \le n$ we define polynomials

$$q_k = \sum_{\alpha \in \mathbb{N}} c_\alpha \left(\prod_{i=1}^{k} y_i^{\alpha_i} \right) \left(\prod_{i=k+1}^{n} \phi(y_i)^{\alpha_i} \right).$$

One has $q_0 = \Phi(p)$, $q_n = p$ and $q_k - q_{k+1} \in I_\Phi$ for $0 \le k \le n-1$. Then

$$p = q_n + \sum_{k=0}^{n-1} (q_k - q_{k+1}) \in I_\Phi$$

and hence $\forall p \in \mathcal{A}$, $p - \Phi(p) \in I_\Phi$.

2. Since f_i commute pairwise, we have $I_\Phi \cap \mathcal{J} \subseteq I_\Phi \cap \mathcal{B} = 0$. Hence, the sum of ideals is a direct sum and $(I_\Phi + \mathcal{J}) \cap \mathcal{B} = \mathcal{J}$.

3. For any $q \in (I_\Phi + \mathcal{J}) \cap \mathcal{A}$ we can present $\Phi(q)$ as a sum $q + \Phi(q) - q$. Hence, $\Phi(q) \in (I_\Phi + \mathcal{J}) \cap \mathcal{B} = \mathcal{J}$ and inclusion $\Phi^{-1}(\mathcal{J}) \supset (I_\Phi + \mathcal{J}) \cap \mathcal{A}$ follows.

Let $p \in \Phi^{-1}(\mathcal{J})$. Again one has $p = p - \Phi(p) + \Phi(p) \in (I_\Phi + \mathcal{J}) \cap \mathcal{A}$. This completes the proof. □

The computational part of the theorem is formulated in the following algorithm. We need two subalgorithms, described in details in the article [13]:
TwoSidedGröbnerBasis(ideal I): computes a two–sided Gröbner basis of a given set of generators;
Eliminate(module M, subalgebra S): computes the intersection of a module M with the subalgebra S, generated by a subset of the set of variables. This is done by computing a Gröbner basis with the special "elimination" ordering (cf. [8]). Note, that this operation is quite complicated in general, requiring most of computing time in the algorithm which follows.

We may take $J \subset B$ as input instead of its reduced form $\mathcal{J} = \mathrm{NF}(J + T_B \mid T_B)$, since not the summands separately but the sum $J + T_B$ is used within the algorithm.

From now on, for an ideal I and a two–sided ideal T_A, we denote $\mathrm{NF}(I + T_A \mid T_A)$ simply by "$I \mod T_A$".

Algorithm 1 PreimageInCommutativeAlgebra($\mathcal{A}, \mathcal{B}, J, \Phi$);

Input 1: $A = \mathbb{K}[y_1, \dots, y_m]$, $T_A \subset A$ an ideal; ▷ $\mathcal{A}$
Input 2: B (G–algebra), $T_B \subset B$ (two–sided ideal); ▷ $\mathcal{B}$
Input 3: $J \subset B$ (left ideal); ▷ $\mathcal{J}$
Input 4: $\{\Phi(y_i)\} \subset B$ (pairwise commuting polynomials); ▷ Φ
Output: $\Phi^{-1}(\mathcal{J})$.

T_B = TwoSidedGröbnerBasis(T_B);
$E = A \otimes_{\mathbb{K}} B$; $T_E = T_A + T_B$; $\mathcal{E} = E/T_E$; ▷ $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B})$
$I_\Phi = \{y_i - \Phi(y_i) \mid 1 \le i \le m\}$;
$P = T_B + I_\Phi + J$; ▷ $P \subset E$
P = Eliminate(P, B); ▷ $P = P \cap A$
$P = \mathrm{NF}(T_A + P \mid T_A)$;
return P; ▷ $\Phi^{-1}(\mathcal{J}) = (T_A + (T_B + I_\Phi + J) \cap A) \mod T_A$;

2.2. Kernel of a map

Since $\ker(\Phi) = \Phi^{-1}(\langle 0 \rangle)$, with this theorem one can compute the kernel of a map between commutative and non–commutative G–algebras using the formula

$$\ker(\Phi) = (T_A + (T_B + I_\Phi) \cap A) \mod T_A.$$

For the rest of this section, let A be a G–algebra with the set of pairwise commuting polynomials $f_1, \dots, f_k \in A$.

2.3. Algebraic Dependency of Elements

Speaking on the algebraic dependency of non–commuting polynomials, one usually think on polynomials in the free algebra. However, if $\{f_i\}$ pairwise commute, the dependency could be expressed by a polynomial from the commutative ring. We will say that $\{f_1, \ldots, f_k\}$ are *algebraically dependent*, if they are pairwise commutative and there exists a non–zero polynomial $g \in \mathbb{K}[y_1, \ldots, y_k]$ such that $g(f_1, \ldots, f_k) = 0$.

Define a morphism $\varphi : \mathbb{K}[y_1, \ldots, y_k] \to A$, $\varphi(y_i) = f_i$.

Then any $g \in \ker(\varphi) \setminus \{0\}$ defines an algebraic relation between the $f_1, \ldots, f_k$. In particular, $f_1, \ldots, f_k$ are algebraically independent if and only if $\ker(\varphi) = 0$. Hence, the check for dependency is computable, since $\ker(\varphi)$ could be computed with the formula of 2.2.

Example 5. *The Fairlie–Odesskii algebra $U'_q(\mathfrak{so}_3)$ ([1]) is an associative unital algebra with generating elements I_1, I_2, I_3 and defining relations*

$$q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \quad q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \quad q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2,$$

where $q \neq 0, \pm 1$, is a complex number, called deformation parameter. In the limit $q \to 1$, the algebra $U'_q(\mathfrak{so}_3)$ reduces to the enveloping algebra $U(\mathfrak{so}_3)$. Both algebras are, of course, G–algebras.

Recall, that the p–th Chebyshev polynomial of the first kind is defined to be

$$T_p(x) = \frac{p}{2}\sum_{k=0}^{[p/2]} \frac{(-1)^k(p-k-1)!}{k!(p-2k)!}(2x)^{p-2k},$$

where $[p/2]$ is an integral part of $p/2$. For example, $T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, T_4(x) = 8x^4 - 8x^2 + 1$.

Consider the algebra $U'_q(\mathfrak{so}_3)$. At arbitrary q, the algebra $U'_q(\mathfrak{so}_3)$ has central element $C = -q^{1/2}(q - q^{-1})I_1I_2I_3 + qI_1^2 + q^{-1}I_2^2 + qI_3^2$, which generates the center of $U'_q(\mathfrak{so}_3)$ when q is not a root of unity.

Let q be a p–th primitive root of unity ($p > 2$), that is $q^p = 1$, $q^{p'} \neq 1$, $1 \leq p' < p$. Then elements $C_k = 2\; T_p\Big(I_k(q - q^{-1})/2\Big)$, $k = 1, 2, 3$, where $T_p(x)$ is Chebyshev polynomial, are also central in $U'_q(\mathfrak{so}_3)$.

Using the algorithm from 2.3, we compute the polynomial, describing the algebraic dependency between C, C_1, C_2 and C_3. Let $f_n \in \mathbb{K}[C, C_1, C_2, C_3]$ be such, that $f_n(C, C_1, C_2, C_3) = 0$ for q be the n–th primitive root of unity. We use $Q = q^{1/2}$ below to simplify the presentation.

Then, $f_3 = (1-2Q)C^3 + (Q+1)C^2 - 243C_1C_2C_3 + 9(1-2Q)(C_1^2 + C_2^2 + C_3^2)$,

$f_4 = C^4 - C^2 - 8C^2(C_1 + C_2 + C_3) - 1024C_1C_2C_3 + 16(C1 - C2 - C3)^2$,

$f_5 = C^5 + Q(3Q^2 - 4Q + 3)C^4 + (3Q^3 - 8Q^2 + 8Q - 3)C^3 - (3Q^2 - 5Q + 3)C^2 - 625(3Q^3 + Q^2 + 2Q - 1)C_1C_2C_3 - 25(C_1^2 + C_2^2 + C_3^2)$ *and so on.*

We should note that despite the simplicity of the algorithm, revealing an algebraic dependency with the method above is one of the hardest computational problems we have ever encountered. In the example above it took us a lot of time and memory to obtain needed elements. We use examples like above further as a very good benchmark test for computer algebra systems.

We hope there could exist other methods for finding dependencies which have lower complexity than the Gröbner basis algorithm we use. We will report on further progress in this area.

2.4. Subalgebra Membership

Suppose we are given $f \in A$. How can we check whether it belongs to the subalgebra S, generated by pairwise commuting $f_1, \dots, f_k$?

If f does not commute with all f_i, it can not belong to S. Hence, our first task is to ensure that f commutes with every f_i.

Then, we have two following possibilities to perform further check and to compute the polynomial, describing the dependency of f on $\{f_1, \dots, f_k\}$.

1. We define a map $\psi : \mathbb{K}[y_0, \dots, y_k] \to A$, $y_0 \mapsto f$, $y_i \mapsto f_i$ and compute $\ker(\psi)$ with the Algorithm 1. Then we take an ordering $<_0$ with y_0 greater than everything containing $y_1, \dots, y_k$ on $\mathbb{K}[y_0, \dots, y_k]$ and compute the Gröbner basis G of $\ker(\psi) \in \mathbb{K}[y_0, \dots, y_k]$ with respect to $<_0$. G contains an element g with the leading monomial $\mathrm{lm}(g) = y_0$ if and only if $f \in \mathbb{K}[f_1, \dots, f_k]$. The polynomial f, written in terms of $f_1, \dots, f_k$, is then $g - \mathrm{lc}(g)\,\mathrm{lm}(g)$.

2. We define a map $\phi : \mathbb{K}[y_1, \dots, y_k] \to A$, $y_i \mapsto f_i$ and a left ideal $I_\phi = \langle y_1 - f_1, \dots, y_k - f_k \rangle \subset \mathbb{K}[y_1, \dots, y_k] \otimes_{\mathbb{K}} A$ like in the algorithm. We compute a Gröbner basis G of I_ϕ with respect to the elimination ordering for $x_1, \dots, x_n$. Then we check whether the $\mathrm{NF}(f \mid G)$ does not involve any variable from A. This happens if and only if $f \in \mathbb{K}[f_1, \dots, f_k]$. The formula for f as a polynomial in $f_1, \dots, f_k$ is just the normal form polynomial.

Example 6. *Let us continue with the example 5. There arises a very natural question: since there is an algebraic dependency, could one of the known generators of the center C, C_1, C_2 and C_3 belong to the subalgebra, generated by the other three?*

We have checked it with the second method above, and obtained a negative answer. Note, that in comparison to finding the dependency explicitly, this procedure is much easier and requires less resources.

Our implementation of the algorithms above in SINGULAR:PLURAL *was useful for treating the general situation, exploring several conjectures, posed in [1]. In the work [9] Iorgov used the explicit form of dependency polynomials and finally showed, that there is a general formula for the dependency, which is moreover expressed in terms of Chebyshev polynomials.*

Klimyk and Iorgov posed a conjecture that $\{C, C_1, C_2, C_3\}$ is a minimal generating set of the center.

2.5. Intersection of Modules with Commutative Subalgebras

Suppose we have an ideal $I \subset A$. In order to compute the intersection of I with S, we set up the map $\mathbb{K}[y_1, \dots, y_k] \xrightarrow{\varphi} A$, $\varphi(y_i) = f_i$ and compute its kernel $K = \ker(\varphi)$ with the Algorithm 1. Then φ induces a monomorphism

$\mathbb{K}[y_1,\dots,y_k]/K \xrightarrow{\varphi} A$. Let $\mathbb{K}[y_1,\dots,y_k]/K \supset J = \varphi^{-1}(I)$ be the preimage of I. Since the algorithm guarantees that J is given in Gröbner basis $\{g_1,\dots,g_s\}$, we finish with the computation of the Gröbner basis of $I \cap S = \langle \varphi(g_1),\dots,\varphi(g_s)\rangle \subset A$.

Example 7. *(Weight vectors with respect to Gel'fand–Zetlin subalgebra)*

Consider $A = U(\mathfrak{sl}(3,\mathbb{K}))$ *for* $\operatorname{char}\mathbb{K} = 0$. *That is, A is the algebra over $\mathbb{K}$, generated by* $\{x_\alpha, x_\beta, x_\gamma, y_\alpha, y_\beta, y_\gamma, h_\alpha, h_\beta\}$ *subject to relations* $[x_\alpha,x_\beta] = x_\gamma, [x_\alpha,y_\alpha] = h_\alpha, [x_\alpha,y_\gamma] = -y_\beta, [x_\alpha,h_\alpha] = -2x_\alpha, [x_\alpha,h_\beta] = x_\alpha, [x_\beta,y_\beta] = h_\beta, [x_\beta,y_\gamma] = y_\alpha, [x_\beta,h_\alpha] = x_\beta, [x_\beta,h_\beta] = -2x_\beta, [x_\gamma,y_\alpha] = -x_\beta, [x_\gamma,y_\beta] = x_\alpha, [x_\gamma,y_\gamma] = h_\alpha + h_\beta, [x_\gamma,h_\alpha] = -x_\gamma, [x_\gamma,h_\beta] = -x_\gamma, [y_\alpha,y_\beta] = -y_\gamma, [y_\alpha,h_\alpha] = 2y_\alpha, [y_\alpha,h_\beta] = -y_\alpha, [y_\beta,h_\alpha] = -y_\beta, [y_\beta,h_\beta] = 2y_\beta,\ [y_\gamma,h_\alpha] = y_\gamma, [y_\gamma,h_\beta] = y_\gamma$.

With the help of SINGULAR:PLURAL *and its library* `center.lib` *we compute the central elements of $U(\mathfrak{sl}_3)$, which we denote by p_4 and p_5:*

$p_4 = 3x_\alpha y_\alpha + 3x_\beta y_\beta + 3x_\gamma y_\gamma + h_\alpha^2 + h_\alpha h_\beta + h_\beta^2 - 3h_\alpha - 3h_\beta$,

$p_5 = 27x_\gamma y_\alpha y_\beta + 27x_\alpha x_\beta y_\gamma + 9x_\alpha y_\alpha h_\alpha - 18x_\beta y_\beta h_\alpha + 9x_\gamma y_\gamma h_\alpha + 2h_\alpha^3 + 18x_\alpha y_\alpha h_\beta - 9x_\beta y_\beta h_\beta - 9x_\gamma y_\gamma h_\beta + 3h_\alpha^2 h_\beta - 3h_\alpha h_\beta^2 - 2h_\beta^3 - 36x_\alpha y_\alpha + 18x_\beta y_\beta - 9x_\gamma y_\gamma - 12h_\alpha^2 - 3h_\alpha h_\beta + 6h_\beta^2 + 18h_\alpha$.

Let $p_3 = h_\alpha^2 + 4x_\alpha y_\alpha - 2h_\alpha$ *be the central element of the subalgebra of A, generated by $x_\alpha, y_\alpha, h_\alpha$ (it is isomorphic to $U(\mathfrak{sl}_2)$). Let, moreover, $p_1 = h_\alpha$ and $p_2 = h_\beta$ be the generators of the Cartan subalgebra of A. Then $B_1 = Z(A)$ is generated by the $\{p_4,p_5\}$. Let B_2 be the Gel'fand-Zetlin subalgebra $GZ(A)$, generated by $\{p_1,p_2,p_3,p_4,p_5\}$.*

Consider the natural maps $\phi_i : B_i \to A$. We want to compute $\mathfrak{I}_i := \phi_i^{-1}(I)$ for certain left ideals I, what will give us the central ($i=1$) and the Gel'fand–Zetlin ($i=2$) characters of cyclic modules, for which I is the annihilator of a generator. In fact, one of the nice properties of Gel'fand–Zetlin subalgebra implies that it suffices to compute the Gel'fand–Zetlin character of a module, since the central character will be obtained from it.

1. First of all we perform the computations of kernels and obtain $\ker\phi_1 = \ker\phi_2 = 0$. *(It is no longer true if* $\operatorname{char}\mathbb{K} > 0$ *since then there appear additional generators in the center).*

2. Consider the parametric ideal $I = \langle x_\alpha, x_\beta, h_\alpha - a, h_\beta - b\rangle$. Then

$$\begin{aligned}\mathfrak{I}_2 = \big\langle\ & p_1 - a,\ \ p_2 - b,\ \ p_3 - a^2 - 2a,\\ & p_4 - a^2 - ab - b^2 - 3a - 3b,\\ & p_5 - 2a^3 - 3a^2b + 3ab^2 + 2b^3 - 6a^2 + 3ab + 12b^2 + 18b\ \big\rangle.\end{aligned}$$

Moreover, the fourth and the fifth polynomials of $\mathfrak{I}_2$ generate $\mathfrak{I}_1$. Note that both ideals $\mathfrak{I}_1, \mathfrak{I}_2$ are maximal in corresponding algebras and parametric parts of p_4, p_5 are indecomposable polynomials in a, b.

3. Now, let us take another ideal $I = \langle x_\beta, x_\gamma, h_\alpha - a, h_\beta - b\rangle$. Then

$$\begin{aligned}\mathfrak{I}_2 = \big\langle\ & p_1 - a,\ \ p_2 - b,\\ & 3p_3 - 4p_4 + (a+2b)(a+2b+6),\\ & 3(a+2b+2)p_4 - p_5 - (a+2b)(a+2b+3)(a+2b+6)\ \big\rangle.\end{aligned}$$

The fourth polynomial of $\mathfrak{I}_2$ generates $\mathfrak{I}_1$. Let $c = a + 2b$. Then the parametric parts of $p = (p_1,p_2,p_3,p_4,p_5)$ form a one–parameter family, depending on t (we *choose $t = p_3$ here):*

$$\left(a,\ b,\ \tfrac{4}{3}t - \tfrac{1}{3}c(c+6),\ t,\ 3(c+2)t + c(c+3)(c+6)\right).$$

3. Kernel of a Homomorphism of Modules

3.1. Syzygies and Homomorphisms of Free Modules

Let $\mathbb{K}$ be a field and A be a G–algebra.

A free A–module A^n could be viewed both as a left and a right A–module. For a vector $v \in A^n$ (respectively a matrix M) we denote by v^t (resp. M^t) a transposed vector (resp. matrix). Consider two free left A–modules A^m, A^n with canonical bases $\{\varepsilon_i\}$ and $\{e_j\}$ respectively. Any left homomorphism ϕ is given by its values on generators:

$$\phi: \quad A^m = \bigoplus_{i=1}^{m} A\varepsilon_i \longrightarrow A^n = \bigoplus_{j=1}^{n} Ae_j, \qquad \varepsilon_i \longmapsto \Phi_i,$$

or, equivalently, by a matrix $\Phi \in A^{n\times m}$ with columns Φ_i. Then, the image of ϕ is a submodule of A^n, generated by the columns of a matrix Φ. In the sequel, a submodule of a free module and a homomorphism will be presented by a matrix, the columns of which constitute the generating set of a module.

Recall, that a syzygy of a k–tuple $(f_1, \ldots, f_k)$, $f_i \in A^n$ is such a k–tuple $(s_1, \ldots, s_k)$, $s_i \in A$, that $\sum_i s_i f_i = 0$.

Consider the kernel of the homomorphism above. Let $I = {}_A\langle \Phi_1, \ldots, \Phi_k\rangle$ be a left submodule of A^n. Then ϕ surjects onto I and $\mathrm{Syz}(I) := \mathrm{Ker}\,\phi$ is called the **(first) module of syzygies** of I with respect to the set of generators $\{\Phi_1, \ldots, \Phi_k\}$. Easy computations ensure that the isomorphism class of $\mathrm{Syz}(I)$ as of A–module does only depend on the isomorphism class of I, in particular, it is independent of the set of generators.

There are several methods for computing syzygy modules ([3,8,10]), which we do not discuss here in details. However, it is worth to note that computation of syzygy module involves Gröbner bases. Implementations of different efficient methods are available in SINGULAR:PLURAL.

3.2. MODULO *Algorithm*

Let A be a G–algebra, T be a proper two–sided ideal $T \subset A$, already given in its two–sided Gröbner basis $\{t_1, \ldots, t_p\} \subset A$ and there is a GR–algebra $\mathcal{A} = A/T$.

For a left ideal $J = {}_A\langle g_1, \ldots, g_p\rangle$ we denote by $\mathcal{M}^s(J) \subset A^s$ a left submodule, generated by the columns of the matrix $J \otimes I_{s\times s}$.

We denote by $\mathbb{I}_{n\times n}$ an $n \times n$ identity matrix.

Suppose there are left submodules $U \in \mathcal{A}^m = \bigoplus_{i=1}^{m} \mathcal{A}e_i$, $V = {}_A\langle v_1, \ldots, v_k\rangle \subset \mathcal{A}^n$ and left $\mathcal{A}$–modules $M = \mathcal{A}^m/U$ and $N = \mathcal{A}^n/V$.

Consider a homomorphism of left $\mathcal{A}$–modules

$$\phi: \quad \mathcal{A}^m/U \longrightarrow \mathcal{A}^n/V \qquad e_i \longmapsto \Phi_i,$$

given by the matrix $\Phi \in \mathcal{A}^{n\times m}$. We are interested in the computation of the kernel of ϕ.

Then $\mathcal{A}^s = (A/T)^s \cong A^s/\mathcal{M}^s(T)$ as A–modules. Defining $U' := U + \mathcal{M}^m(T)$ and $V' := V + \mathcal{M}^n(T)$, we consider the homomorphism of A–modules

$\psi: \quad A^m \xrightarrow{\Phi} A^n/V'$. Then $\operatorname{Ker}\phi = (\operatorname{Ker}\psi) \mod U'$.

Let $g = \sum_{i=1}^m g_i e_i \in A^m$ and $\mathcal{M}^n(T) = {}_A\langle m_1,\dots,m_{pn}\rangle$. Such g belongs to the $\operatorname{Ker}\psi$ if and only if $\psi(g) \in V'$, that is there exist $\{h_i\},\{r_j\} \subset A$, such that

$$\sum_{i=1}^m g_i\Phi_i + \sum_{l=1}^k h_l v_l + \sum_{j=1}^{pn} r_j m_j = 0.$$

Let $S := \operatorname{Syz}(\{\Phi, V, \mathcal{M}^n(T)\}) \subset A^{m+k+pn}$. Then the previous equality means that $(g_1,\dots,g_m,h_1,\dots,h_k,r_1,\dots,r_{pn}) \in S$. Then $\operatorname{Ker}\psi = S \cap \oplus_{i=1}^m Ae_i$. The latter intersection can be computed with standard "elimination of components" technique ([8,21]).

Computing with S directly as above, we get much overhead (since we do not really need all syzygies of $\{\Phi, V, \mathcal{M}^n(T)\}$ but only those, which are relevant to the Φ part). The next Lemma, inspired by Schönemann ([21]), avoids such extra computations and therefore is used in current implementation.

Lemma 8. *Let $\phi: M \to N$ be a left $\mathcal{A}$–module homomorphism as before. Define the matrix*

$$Y = \left(\begin{array}{c|c|c} \Phi & V & \mathcal{M}^n(T) \\ \hline \mathbb{I}_{m\times m} & 0 & 0 \end{array}\right) \subset A^{(n+m)\times(m+k+pn)}.$$

Let $Z = Y \cap \bigoplus_{i=n+1}^{n+m} Ae_i$ and $U' = U + \mathcal{M}^m(T)$, then

$$\operatorname{Ker}\phi = \operatorname{NF}(Z + U' \mid U') \subseteq M.$$

Further, we refer to this algorithm as to Modulo (in Singular:Plural, the command `modulo` is used for it and we just keep the tradition). For $\mathcal{A}, \psi, \Phi, V$ as above, the kernel of ψ is computed by executing `modulo(`Φ, V`)`.

Example 9 (kernel of a module homomorphism). *Let $A = U(\mathfrak{sl}_2) = \mathbb{K}\langle e,f,h \mid [e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f\rangle$. Let I be the two–sided ideal, given in its two–sided Gröbner basis $\{h^2-1, fh-f, eh+e, f^2, 2ef-h-1, e^2\}$ and $\mathcal{A} = A/I$. Indeed $\mathcal{A}$ is finite–dimensional with the basis $\{1,e,f,h\}$.*

Consider endomorphisms $\tau: \mathcal{A} \to \mathcal{A}$ and let us compute their kernels.

For non–zero $k \in \mathbb{K}$, $\ker(\tau: 1 \mapsto e+k) = \ker(\tau: 1 \mapsto f+k) = 0$.

For $k^2 \neq 1$, $\ker(\tau: 1 \mapsto h+k) = 0$.

$\ker(\tau: 1 \mapsto e) = \ker(\tau: 1 \mapsto h+1) = {}_{\mathcal{A}}\langle e, h-1\rangle$.

$\ker(\tau: 1 \mapsto f) = \ker(\tau: 1 \mapsto h-1) = {}_{\mathcal{A}}\langle f, h+1\rangle$.

3.3. Applications

With the help of an algorithm MODULO we can solve some useful problems.

2nd Isomorphism Theorem. Let $M_1, M_2 \in \mathcal{A}^\ell$ be two left submodules. By the classical theorem, we have $M_1/(M_1 \cap M_2) \cong (M_1 + M_2)/M_2$.

Illustrating the situation with the diagram $\mathcal{A}^k \xrightarrow{M_1} \mathcal{A}^\ell \xleftarrow{M_2} \mathcal{A}^m$, we see that indeed, $M_1/(M_1 \cap M_2) \cong \mathcal{A}^\ell / \operatorname{Ker}\phi$, where $\phi : \mathcal{A}^k \xrightarrow{M_1} \mathcal{A}^\ell/M_2$.

The presentation matrix for $M_1/(M_1 \cap M_2)$ equals $\operatorname{Ker}\phi$ and hence can be computed by MODULO(M_1, M_2).

Intersect Many Submodules via MODULO. We can compute the intersection of a finite set of submodules with the MODULO algorithm in an efficient manner, generalizing [21].

Proposition 10. *Let $\mathcal{A}$ be a GR–algebra and $\{M_i = {}_{\mathcal{A}}\langle f_1^i, \ldots, f_{N_i}^i\rangle \subset \mathcal{A}^r,\ i \le m\}$ be the finite set of submodules. Assume, that each M_i is actually a submodule of $\mathcal{A}^{n_i}$, where $n_i \le n_{i+1} \le r$. Consider the left homomorphism of $\mathcal{A}$–modules*

$$\phi : \mathcal{A}^m \longrightarrow \mathcal{A}^{n_1}/M_1 \oplus \cdots \oplus \mathcal{A}^{n_m}/M_m, \quad e_i \mapsto I_{n_i \times n_i}.$$

Then $\bigcap_{i=1}^{m} M_i$ can be computed by

$$\text{MODULO}(\begin{pmatrix} I_{n_1\times n_1} \\ \vdots \\ I_{n_m \times n_m} \end{pmatrix}, \begin{pmatrix} M_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & M_m \end{pmatrix}).$$

4. Central Character Decomposition of the Module

Decompositions of modules are of big interest for many branches of algebra. In the non–commutative case, especially in the representation theory, a particularly important role is played by the decomposition by central characters. The algorithmic treatment of this problem goes back to [11], which we follow.

For the whole section we assume $\mathbb{K}$ to be algebraically closed.

Let A be a G–algebra and C be a finitely generated commutative subalgebra of A. Denote by $C^* = \operatorname{Hom}(C, \mathbb{K})$ the set of maximal ideals of C.

Definition 11. *Let M be a finite generated A-module and $\chi \in C^*$.*

- *The χ–weight subspace of M with respect to C is defined to be*

$$M_\chi = \{v \in M \mid \ \forall c \in C,\ (c - \chi(c))v = 0\}.$$

- *The generalized χ–weight subspace of M with respect to C is defined to be*

$$M^\chi = \left\{v \in M \mid \ \exists n(v) \in \mathbb{N}, \forall c \in C,\ (c - \chi(c))^{n(v)} v = 0\right\}.$$

- *We will say that M possesses a weight decomposition (resp. generalized weight decomposition) if*

$$M = \bigoplus_{\chi \in C^*} M_\chi \quad (\textit{resp. } M = \bigoplus_{\chi \in C^*} M^\chi).$$

- $\operatorname{Supp}_C M = \{\chi \in C^* | M^\chi \neq 0\}$ *is called a* **support of M with respect to C**.
- *We will say that M possesses a finite (generalized) weight decomposition with respect to C if M possesses a (generalized) weight decomposition, and its support is finite.*

One can determine, whether a given element $m \in M$ belongs to M_χ (resp. M^χ) for some $\chi \in C$ by analyzing the ideal $\operatorname{Ann}_A^M m \cap C \subset C$. The last can be computed by the Theorem 4.

Let us now concentrate our attention on computing generalized weight decomposition and Zariski closure of the support with respect to the center $Z = Z(A)$ of A. In this case the subspaces M_χ and M^χ are submodules for any $\chi \in Z^*$, what should not be true, for example, for Gel'fand–Zetlin subalgebras. The generalized weight decomposition with respect to the center will be called **the central character decomposition**.

Lemma 12. *Let $\mathcal{A}$ be a GR–algebra and $M \cong \mathcal{A}^N / I_M$ for a left submodule $I_M \subset \mathcal{A}^N$. We define a module*

$$J_M := \bigcap_{j=1}^{N} \operatorname{Ann}_{\mathcal{A}}^M e_j.$$

Then $Z(\mathcal{A}) \cap J_M = Z(\mathcal{A}) \cap \operatorname{Ann}_{\mathcal{A}} M$ holds.

Proof. Note, that if I_M is an ideal, $J_M = I_M$ and $\operatorname{Ann}_{\mathcal{A}} M \subset I_M$. In general, we have $J_M \supset \operatorname{Ann}_{\mathcal{A}} M$ too, hence $Z(\mathcal{A}) \cap J_M \supset Z(\mathcal{A}) \cap \operatorname{Ann}_{\mathcal{A}} M$. Now, suppose $z \in Z(\mathcal{A}) \cap J_M$.

$$\forall v \in M,\ \exists \{a_j\} \subset \mathcal{A} \text{ such that } v = \sum_{j=1}^{N} a_j e_j. \text{ Then } zv = \sum_{j=1}^{N} a_j z e_j = 0,$$

and hence, $z \in \operatorname{Ann}_{\mathcal{A}} M$. □

Using Nullstellensatz we obtain a corollary, describing the set $\operatorname{Supp}_Z M$ in terms of ideal $J_M \cap Z(A)$, which can be computed with the Algorithm 1.

Corollary 13. *Let A be a G–algebra and M be an A–module. Let, moreover, $\mathfrak{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal and $V(\mathfrak{I}) \subset \mathbb{A}^n_{\mathbb{K}}$ denotes the set of zeros of $\mathfrak{I}$. Then the Zariski closure of $\operatorname{Supp}_Z M$ equals $V(J_M \cap Z(A))$.*

To proceed with the discussion of algorithm for computation of M^χ, notions of central quotient ideal and central quotient module are needed. These notions are quite different from the usual ([3,8]) quotient ideals. We denote the central quotient by $(I : J)$ instead of $(I :_Z J)$, since classical quotients will not appear in the sequel.

Definition 14. *Let $I \subset \mathcal{A}^N$ be a left submodule and $Z = Z(\mathcal{A})$ be a center of $\mathcal{A}$.*

- *For $z \in Z$ the left submodule $(I : z) := \{v \in \mathcal{A}^N \mid zv \in I\}$.*
- *For an ideal $\mathfrak{I} \subset Z$ the submodule $I : \mathfrak{I}$ is defined to be*

$$(I : \mathfrak{I}) := \{v \in \mathcal{A}^N \mid zv \in I \text{ for all } z \in \mathfrak{I}\}.$$

- *The submodule $I : z^\infty$ is defined to be $\varinjlim_{n\in\mathbb{N}} I : z^n$.*
- *The submodule $I : \mathfrak{I}^\infty$ is called a* **central saturation** *of I by $\mathfrak{I}$ and is defined to be $\varinjlim_{n\in\mathbb{N}} I : \mathfrak{I}^n$.*

The usefulness of central quotient modules in our context is indicated by the following proposition.

Proposition 15. *Let A be a G–algebra and M be an A–module. Suppose M possesses a finite central character decomposition and $\mid \operatorname{Supp}_Z M \mid = s$. If $s = 1$, we have $M \cong M^\chi$. Otherwise,*

$$M^\chi \cong A^N/(I_M : \mathfrak{I}_\chi^\infty), \text{ where } \mathfrak{I}_\chi = \bigcap_{\substack{\psi\in\operatorname{Supp}_Z M\\ \psi\neq\chi}} \ker\psi.$$

Proof. By assumption, $M = \bigoplus_{\psi\in Z^*} M^\psi$. Define a left submodule

$$I_\chi = \sum_{\psi\in Z^*\setminus\{\chi\}} M^\psi + I_M \subset A^N.$$

Obviously $M^\chi \cong A^N/I_\chi$. One has to show that $I_M : \mathfrak{I}_\chi^\infty = I_\chi$.

Since $\operatorname{Supp}_Z M$ is finite, there exists such $n \in \mathbb{N}$, that for all $\psi \in \operatorname{Supp}_Z M$ holds $(\ker\psi)^n M^\psi = 0$. For all $x \in I_\chi$ one has $\mathfrak{I}_\chi^n x \in I$. Thus $I_\chi \subset I_M : \mathfrak{I}_\chi^\infty$.

Taking $x \in A^N \setminus I_\chi$, we see that the image v of x in $M^\chi \cong A^N/I_\chi$ is non–zero. Suppose $x \in I_M : \mathfrak{I}_\chi^\infty$, then there exists such $m \in \mathbb{N}$, that $\mathfrak{I}_\chi^m x \in I_M$. Hence we have also $\mathfrak{I}_\chi^m v = 0$, what contradicts the definition of $\mathfrak{I}_\chi$. □

The computation of a central quotient is much easier than the computation of a classical quotient module (see, for example, [3]).

Lemma 16. *Let $\mathcal{A}$ be a GR–algebra, $z \in Z(\mathcal{A})$ be a central element in $\mathcal{A}$ and let $F \subset \mathcal{A}^N$ be a left submodule, generated by $\{f_1, \dots, f_m\}$. Then the central quotient $(F : z) \subseteq \mathcal{A}^N$ is generated by the first N components of generators of the syzygy module $\operatorname{Syz}(ze_1, \dots, ze_N, f_1, \dots, f_m)$ and hence, can be computed by* Modulo$(z \cdot \mathbb{I}_{N\times N}, F)$.

Proof. Let $\bar{a} = (a_1, \dots, a_{N+m}) \in \mathrm{Syz}(ze_1, \dots, ze_N, f_1, \dots, f_m) \subset \mathcal{A}^{N+m}$. Then

$$\sum_{i=1}^{N} z a_i e_i = -\sum_{i=1}^{m} a_{i+N} f_i.$$

Hence, the tuple $(a_1, \dots, a_N)$ is an element of $(F : z)$ if and only if

$$\bar{a} \in \mathrm{Syz}(ze_1, \dots, ze_N, f_1, \dots, f_m).$$

□

We can also compute the annihilator of an element of a module:

Lemma 17. *Let $m \in M = \mathcal{A}^N / I_M$, and I_M be a left submodule of $\mathcal{A}^N$, generated by $\{m_1, \dots, m_k\}$, then $\mathrm{Ann}^M_{\mathcal{A}}(m)$ is the left ideal generated by the first components of generators of the syzygy module $\mathrm{Syz}(m, m_1, \dots, m_k) \subseteq \mathcal{A}^{k+1}$ and hence, could be computed by* Modulo(m, I_M).

Proof.

$$\forall \bar{a} = (a_0, a_1, \dots, a_k) \in \mathrm{Syz}(m, m_1, \dots, m_k),\ a_0 m + \sum_{i=1}^{k} a_i m_i = 0,$$

hence $a_0 m = 0 \mod I_M$. □

The advantage of the situation we are in is indicated by the lemma, which follows from the fact that $\mathfrak{I}$ is an ideal in the center of A.

Lemma 18. *Let $\{c_1, \dots, c_n\}$ be the Gröbner basis of $\mathfrak{I} \subset Z$, then*

$$(I : \mathfrak{I}) = \bigcap_{i=1}^{n} (I : c_i).$$

In the following algorithms we formalize the described approach.

Algorithm 2 CentralSaturation(M, T);

```
Input :  M, a left A^N–submodule, T, an ideal in Z(A);
Output:  S, a left A^N–submodule;                       ▷ S = M : T^∞
function CentralQuotient(M, T)
    Int s := size(T);
    Matrix E := IdentityMatrix(s);
    for i=1  to  s do
        N[i] := Modulo(T[i] · E, M);
    end for
    S := IntersectManyModules(N[1], ..., N[s]);
    return S;
end function

Module Q := 0;
Ideal T := GröbnerBasis(T);
S := M;
repeat
    Q := CentralQuotient(S, T);
    S := CentralQuotient(Q, T);
until (S == Q)
return S;
```

Proof. (of Algorithm 2).
Termination: The algorithm CentralQuotient clearly terminates. As for CentralSaturation, we see that due to the obvious property $(I : \mathfrak{I}) : \mathfrak{I} = I : \mathfrak{I}^2$, one has an increasing sequence $I : \mathfrak{I} \subset I : \mathfrak{I}^2 \subset \dots$ of submodules in $\mathcal{A}^N$. It stabilizes by the Noetherian property of $\mathcal{A}$, so the computation of the $I : \mathfrak{I}^\infty$ will be finished after a finite number of steps.
Correctness: Lemmata 16, 18 imply the correctness of CentralQuotient. □

In algorithms we have used the following auxiliary procedures:

- SetRing(ring A): sets the ring A active;
- Ann(module M, vector v): the annihilator of v in M (Lemma 17);
- IntersectManyModules(module $P_1, \dots, P_m$) (Proposition 10);
- MinAssPrimes(ideal I): minimal associated prime ideals for the zero–dimensional ideal $I \subset \mathbb{K}[z]$; (see [20]).

All of them are implemented in Singular:Plural.

Algorithm 3 CentralCharDecomposition(A, Z, M);

Input 1: A, a G–algebra;
Input 2: $Z = \{Z_1, \dots, Z_m\} \subset A$, generators of $Z(A)$;
Input 3: I_M, a left A^N–submodule; ▷ $M \cong A^N/I_M$
Output: R, a list of pairs $\{(\chi, I_\chi)\}$.

```
InitRing 𝕂[z] := 𝕂[z_1, ..., z_m];
InitMap φ : 𝕂[z] → A;  φ(z_i) = Z_i;
SetRing A;
for i=1 to N do
    P[i] := Ann(M, e_i);                ▷ e_i is the i–th basis vector of A^N
end for
J_M := IntersectManyModules(P[1], ..., P[N]);
SetRing 𝕂[z];
J_z := PreimageInCommutativeAlgebra(𝕂[z], A, J_M, φ);
if (Dim (J_z) > 0) then
    ErrorMessage = "There is no finite decomposition";
    return ERROR;
else
    List L_0 := MinAssPrimes(J_z);
end if
SetRing A;
List L := φ(L_0);   Int s = Size(L);   List S;
for i=1 to s do
    P := IntersectManyModules(L[1], ..., L[î], ..., L[s]);
    S[i] := TwoSidedGröbnerBasis(P);
end for
List R;
for i=1 to s do
    R[i][1] := S[i];
    R[i][2] := CentralSaturation(I_M, S[i]);
end for
return R;
```

Algorithms 2 and 3 have been recently implemented by the author in the Singular:Plural library `ncdecomp.lib` ([19]); all the examples from the article have been computed with this implementation.

Example 19. *Let us continue with the example 7.*

The central support of the parametric module $M = A/I$, $I = \langle x_\alpha, x_\beta, h_\alpha - a, h_\beta - b\rangle$ *equals* $\chi_1 = \langle p_4 - a^2 - ab - b^2 - 3a - 3b, p_5 - 2a^3 - 3a^2b + 3ab^2 + 2b^3 - 6a^2 + 3ab + 12b^2 + 18b\rangle$, *a maximal ideal in* $\mathbb{K}[p_4, p_5]$ *for any value of parameters* a, b. *Hence,* $M \cong M^{\chi_1}$.

As for the parametric module $M' = A/I$, $I = \langle x_\beta, x_\gamma, h_\alpha - a, h_\beta - b\rangle$, *we have* $\operatorname{Supp}_Z M' = \langle 3(a+2b+2)p_4 - p_5 - (a+2b)(a+2b+3)(a+2b+6)\rangle \subset \mathbb{K}[p_4, p_5]$, *an ideal of dimension 1 for any value of parameters* a, b. *Hence, there exists no finite central decomposition.*

Example 20. *Let* $A = U(\mathfrak{sl}_2)$ *(cf. Example 9). Consider a set of generators* $S = \{e^3, f^3, h^3 - 4h\} \subset A$ *and two ideals therein:* I_L*, a left ideal and* I_T*, a two–sided ideal, both generated by* S*. Gröbner basis computations show* $I_L \supset I_T$*.*

We draw our attention at two finite–dimensional modules:
$M_L = U(\mathfrak{sl}_2)/I_L$ *(of dimension 15) and*
$M_T = U(\mathfrak{sl}_2)/I_T$ *(of dimension 10).*

Intersection with the center of A*, generated by the polynomial* $4ef + h^2 - 2h$*, gives us the following supports:*
$\mathrm{Supp}_Z\, M_L = \{z, z-8, z-24\}$ *and* $\mathrm{Supp}_Z\, M_T = \{z, z-8\}$.
Then, $M_T = M_T^{(z)} \oplus M_T^{(z-8)} = U(\mathfrak{sl}_2)/\mathfrak{m} \,\oplus\, U(\mathfrak{sl}_2)/I_9$ *and*
$M_L = M_L^{(z)} \oplus M_L^{(z-8)} \oplus M_L^{(z-24)} = U(\mathfrak{sl}_2)/\mathfrak{m} \,\oplus\, U(\mathfrak{sl}_2)/I_9 \,\oplus\, U(\mathfrak{sl}_2)/I_5$.
Here, we used the ideals $\mathfrak{m} = \langle e, f, h\rangle$*,* $I_5 = \langle e^3, f^3, ef - 6, h\rangle$ *and*
$I_9 = \langle 4ef+h^2-2h-8, h^3-4h, e^3, f^3, fh^2-2fh, eh^2+2eh, f^2h-2f^2, e^2h+2e^2\rangle$.
The $\mathbb{K}$*–dimensions of corresponding modules are* $1, 5, 9$ *respectively.*

Note, that modules $U(\mathfrak{sl}_2)/\mathfrak{m}$ *and* $U(\mathfrak{sl}_2)/I_5$ *are simple modules, whereas* $U(\mathfrak{sl}_2)/I_9$ *is a sum of three following* 3*–dimensional simple modules*
$U(\mathfrak{sl}_2)/\langle e^2, f^2, ef-2, h\rangle \;\oplus\; U(\mathfrak{sl}_2)/\langle e, f^3, h-2\rangle \;\oplus\; U(\mathfrak{sl}_2)/\langle e^3, f, h+2\rangle$.

Conclusion and Future Work

An algorithm, computing a preimage of an ideal under the map between a commutative and a non–commutative GR–algebra (Algorithm 1) is a building block for the whole family of algorithms, like algebraic dependency of pairwise commuting polynomials (2.3), membership of a polynomial in a commutative subalgebra (2.4) and central character decomposition (Algorithm 3). The latter uses an algorithm for computation of the kernel of a homomorphism of modules (Lemma 8), which has its own applications.

We hope that nontrivial examples, computed and described in details, help to understand both attractivity and computational complexity of treated problems. More applications like the investigation of singularities of polynomials, describing algebraic dependency of generators of the center (in particular, this is quite interesting in universal enveloping algebras of Lie algebras over fields of positive characteristic) can be effectively supported by proposed methods.

One of advantages of our implementation in Singular:Plural is that this computer algebra system is freely distributed. One can download it together with its libraries and detailed documentation from `http://www.singular.uni-kl.de`.

Concerning the preimage of modules under a general morphism between two GR–algebras, the situation is more complicated; we are investigating it further and hope to report on progress in future publications. It requires the development of tools for handling opposite algebras together with the effective treatment of bimodules.

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