# Affine Weyl Groups

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# Crystallographic root systems.

## Definition

A crystallographic root system $\Phi$ is a finite set of non zero vectors in Euclidean space $V$ s.t.

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$

(R2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$, where $s_\alpha : v \mapsto v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$ is the reflection along $\alpha$

(R3) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

$\Phi$ is called irreducible if for all $\alpha, \beta \in \Phi$ there are $m \in \mathbb{N}, \alpha = \alpha_1, \ldots, \alpha_m, \alpha_{m+1} = \beta \in \Phi$ s.t. $\prod_{i=1}^{m}(\alpha_i, \alpha_{i+1}) \neq 0$.

## Remark

$\Phi$ irreducible root system, then there is a basis $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi \subset V$ s.t. for all $\alpha \in \Phi$ there are $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ with

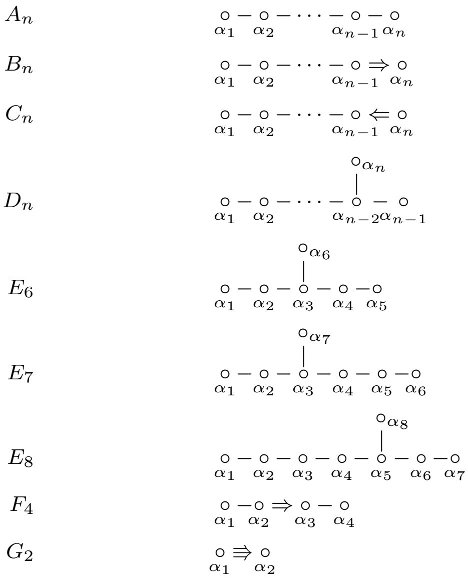
$$\alpha = a_1\alpha_1 + \ldots + a_n\alpha_n \qquad \text{positive root}$$

or

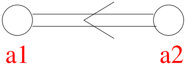
$$\alpha = -a_1\alpha_1 - \ldots - a_n\alpha_n \qquad \text{negative root}$$

$\Phi^+ := \Phi^+(\Delta)$ denotes the set of all positive roots.
$\exists\, !\ \tilde{\alpha} = \sum c_i\alpha_i \in \Phi^+$ (the highest root) with maximal height $\sum_{i=1}^{n} c_i \in \mathbb{N}$.

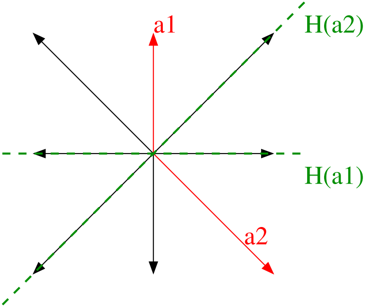
# The irreducible crystallographic root systems



# Example




$\Delta = \{\alpha_1, \alpha_2\}$
$\Phi^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$
$\Phi = \Phi^+ \cup -\Phi^+$
order of $s_{\alpha_1} s_{\alpha_2}$ is $4$
$\langle s_{\alpha_1}, s_{\alpha_2} \rangle \cong D_8$



# Finite Weyl groups

## Definition

Let $\Phi$ be crystallographic root system.
Then $W(\Phi) := \langle s_\alpha : \alpha \in \Phi \rangle$ is called the Weyl group of $\Phi$.

## Remark

Assume that $\Phi$ is irreducible.

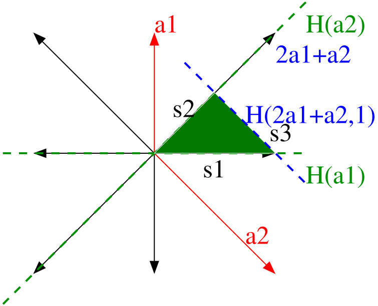
- $W(\Phi) = \langle s_\alpha : \alpha \in \Delta \rangle$
- The Dynkin diagram encodes a presentation of $W(\Phi)$
- $W(\Phi)$ acts irreducibly on $V$.

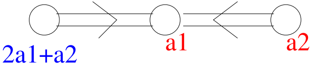
$F_4$ $\qquad \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} \Rightarrow \underset{\alpha_3}{\circ} - \underset{\alpha_4}{\circ}$

$$W(F_4) = \langle s_1, s_2, s_3, s_4 \mid s_i^2, (s_i s_j)^2 (|i-j| > 1), (s_1 s_2)^3, (s_2 s_3)^4, (s_3 s_4)^3 \rangle$$

| $\Phi$ | $A_n$ | $B_n/C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|---|
| $\|\Phi\|$ | $n(n+1)$ | $2n^2$ | $2n(n-1)$ | $72$ | $126$ | $240$ |
| $\|W(\Phi)\|$ | $(n+1)!$ | $2^n n!$ | $2^{n-1} n!$ | $2^7 3^4 5$ | $2^{10} 3^4 5\ 7$ | $2^{14} 3^5 5^2 7$ |
| $W(\Phi)$ | $S_{n+1}$ | $C_2 \wr S_n$ | $(C_2^{n-1}) : S_n$ | $S_4(3) : 2$ | $C_2 \times S_6(2)$ | $2.O_8^+(2) : 2$ |

# Affine Weyl groups




affine Weyl group <s1,s2,s3>

<s1,s2,s3 | s1^2,s2^2,s3^2

(s1s2)^4,(s1s3)^4,(s2s3)^2 >

## Definition

$H_{\alpha,k} := \{v \in V \mid (v,\alpha) = k\}$ (affine hyperplane)

$\alpha^\vee := \frac{2}{(\alpha,\alpha)}\alpha$ the coroot of $\alpha \in \Phi$

$s_{\alpha,k} : V \to V, v \mapsto v - ((v,\alpha) - k)\alpha^\vee$ the reflection in the affine hyperplane $H_{\alpha,k}$

# Affine Weyl groups

## Remark

$\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$, $s_{\alpha,k} : v \mapsto v - ((v,\alpha) - k)\alpha^\vee$, $H_{\alpha,k} := \{v \in V \mid (v,\alpha) = k\}$

- $(\alpha^\vee)^\vee = \alpha$
- $(\alpha^\vee, \alpha) = 2$, $\alpha^\vee = \alpha$ if $(\alpha,\alpha) = 2$
- $H_{\alpha,k} = H_{\alpha,0} + \frac{k}{2}\alpha^\vee$
- $s_{\alpha,k}$ fixes $H_{\alpha,k}$ pointwise and sends $0$ to $k\alpha^\vee$, so it is the reflection in the affine hyperplane $H_{\alpha,k}$.

## Definition

Let $\Phi$ be an irreducible crystallographic root system.

- $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ the dual root system
- $L(\Phi) := \langle \Phi \rangle_{\mathbb{Z}}$ the root lattice, $L(\Phi)^\# = \{v \in V \mid (v,\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi\}$
- $L(\Phi^\vee)^\# =: \widehat{L}(\Phi)$ the weight lattice

## Proposition

The affine Weyl group of $\Phi$

$$W_a(\Phi) := \langle s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z} \rangle \cong L(\Phi^\vee) : W(\Phi)$$

is the semidirect product of $W(\Phi)$ with the translation subgroup $L(\Phi^\vee)$.

# Proof: $W_a(\Phi) = L(\Phi^\vee) : W(\Phi)$

- $s_{\alpha,k} : v \mapsto v - ((v,\alpha) - k)\alpha^\vee$ with
- $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$, $(\alpha,\alpha^\vee) = 2$.
- $W(\Phi) = \langle s_{\alpha,0} \mid \alpha \in \Phi \rangle \leq W_a(\Phi)$.
- $s_{\alpha,0}s_{\alpha,1} = t(\alpha^\vee)$ because both map $v \in V$ to

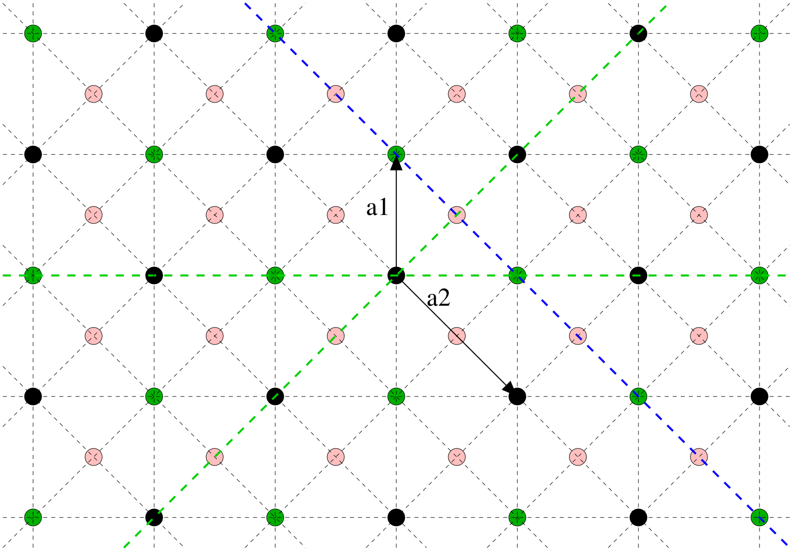
$$(v - (v,\alpha)\alpha^\vee)s_{\alpha,1} = v - (v,\alpha)\alpha^\vee - ((v,\alpha) - (v,\alpha)\underbrace{(\alpha,\alpha^\vee)}_{=2} - 1)\alpha^\vee = v + \alpha^\vee$$

- So $L(\Phi^\vee) = \langle t(\alpha^\vee) \mid \alpha \in \Phi \rangle \leq W_a(\Phi)$.
- $L(\Phi^\vee) \cap W(\Phi) = \{1\}$.
- $L(\Phi^\vee)$ is normalized by $W(\Phi)$ because

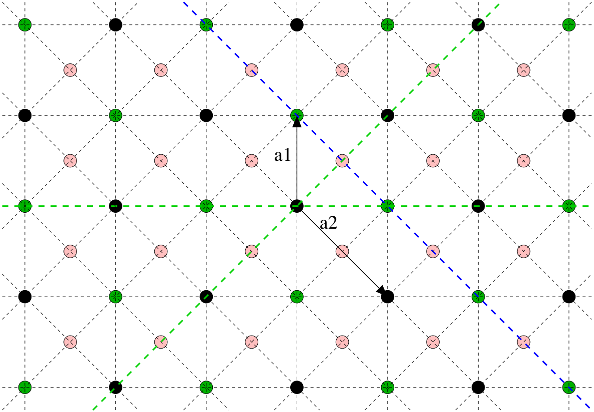
$$s_{\alpha,0}t(v)s_{\alpha,0} = t(s_{\alpha,0}(v))$$

  and $W(\Phi)(\Phi^\vee) = \Phi^\vee$.
- $s_{\alpha,k} = s_{\alpha,0}t(k\alpha^\vee) \in L(\Phi^\vee) : W(\Phi)$.

# Affine Weyl groups



# Root lattice, weight lattice




root lattice

$L(\Phi) = \langle \alpha_1, \alpha_2 \rangle$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

$L(\Phi^\vee) = \langle 2\alpha_1, \alpha_2 \rangle$

$$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}.$$

weight lattice

$\widehat{L}(\Phi) = \langle \alpha_1, \frac{1}{2}\alpha_2 \rangle$

# Alcoves

## Definition

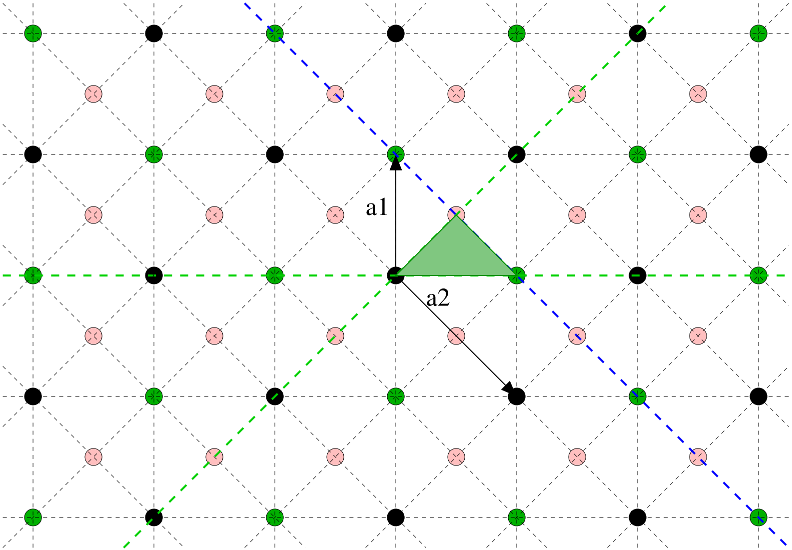
- $\mathcal{H} := \{H_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\}$
- $V^\circ := V - \cup_{H \in \mathcal{H}} H$
- A connected component of $V^\circ$ is called an alcove.
- $A_\circ := \{v \in V \mid 0 < (v, \alpha) < 1 \text{ for all } \alpha \in \Phi^+\}$ the standard alcove.
- $S_\circ := \{s_{\alpha,0} \mid \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}$ the reflections in the faces of the standard alcove.

## Theorem

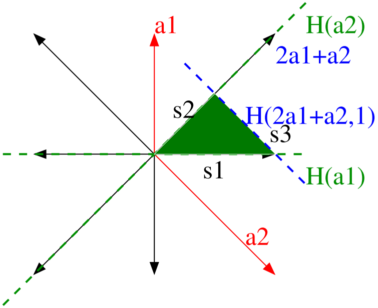
$\Phi$ irreducible crystallographic root system.

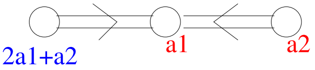
- $A_\circ$ is a simplex.
- $W_a(\Phi) = \langle S_\circ \rangle$.
- $W_a(\Phi)$ acts simply transitively on the set of alcoves.
- $\overline{A_\circ}$ is a fundamental domain for the action of $W_a(\Phi)$ on $V$.

# The standard alcove is a fundamental domain



# Presentation of $W_a(\Phi)$ in standard generators $S_\circ$




affine Weyl group <s1,s2,s3>

<s1,s2,s3 | s1^2,s2^2,s3^2

(s1s2)^4,(s1s3)^4,(s2s3)^2 >

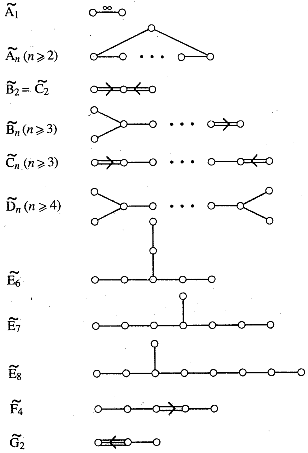



Figure 4.1: Extended Dynkin diagrams

# The length function

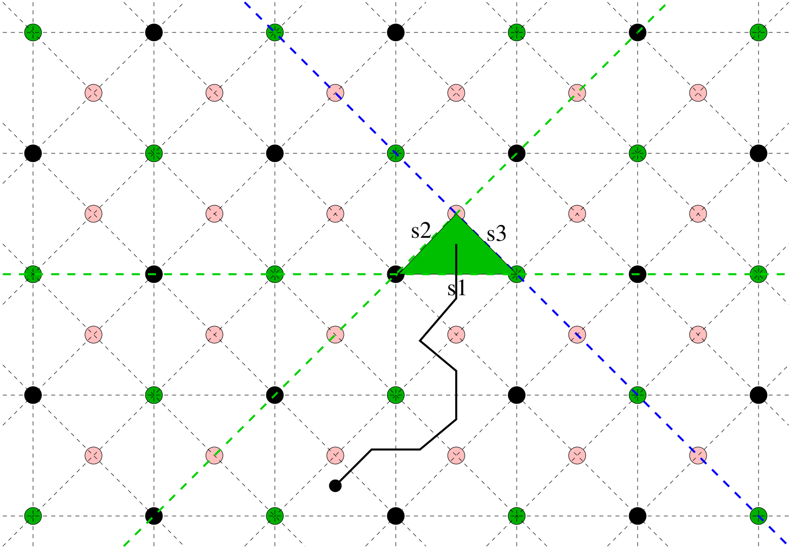
## Definition

Any $w \in W_a(\Phi)$ is a product of elements in $S_\circ$. We put $\ell(w) := \min\{r \mid \exists s_1, \ldots, s_r \in S_\circ; w = s_1 s_2 \cdots s_r\}$ the length of $w$ and call any expression $w = s_1 \cdots s_{\ell(w)}$ a reduced word for $w$.

## Definition

For $w \in W_a(\Phi)$ let $\mathcal{L}(w) := \{H \in \mathcal{H} \mid H$ separates $A_\circ$ and $A_\circ w\}$ and $n(w) := |\mathcal{L}(w)|$.

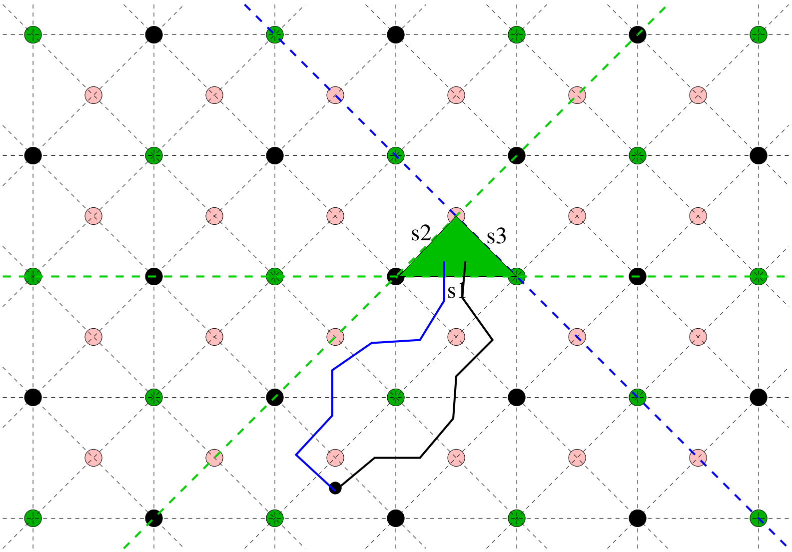
# Words of minimal length




s1(s1s2s1)(s1s2s3s2s1)(s1s2s3s1s3s2s1)(s1s2s3s1s3s1s3s2s1) ...

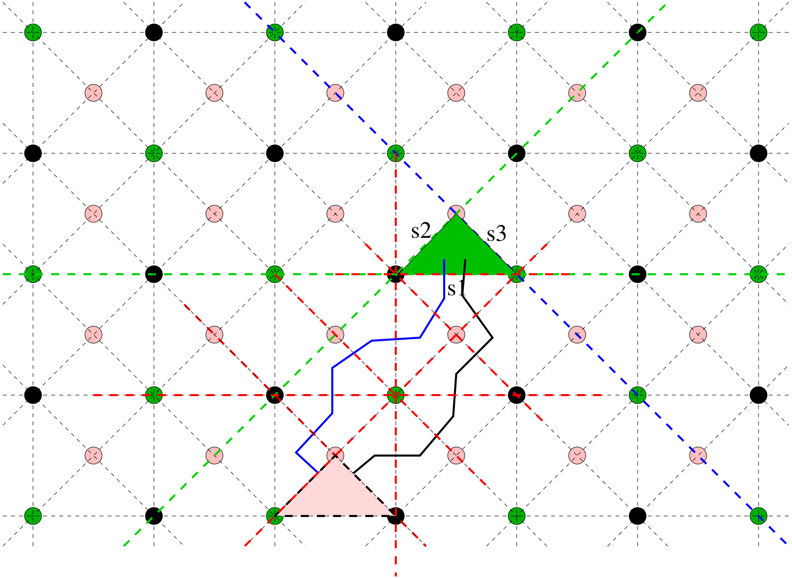
=s2s1s3s1s3s2s1

# Words of minimal length




w = s3s2s1s3s1s2s1 = s2s1s3s1s2s3s1

# Separating hyperplanes




w = s3s2s1s3s1s2s1 = s2s1s3s1s2s3s1

L(w) contains 7 hyperplanes length of reduced word of w = 7

# The length function

## Definition

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## Definition

For $w \in W_a(\Phi)$ let $\mathcal{L}(w) := \{H \in \mathcal{H} \mid H$ separates $A_\circ$ and $A_\circ w\}$ and $n(w) := |\mathcal{L}(w)|$.

## Theorem

Let $w = s_1 \cdots s_{\ell(w)}$ and $H_i := H_{s_i} = \{v \in V \mid v s_i = v\}$. Then

$$\mathcal{L}(w) = \{H_1, H_2 s_1, H_3 s_2 s_1, \ldots, H_{\ell(w)} s_{\ell(w)-1} \cdots s_2 s_1\}.$$

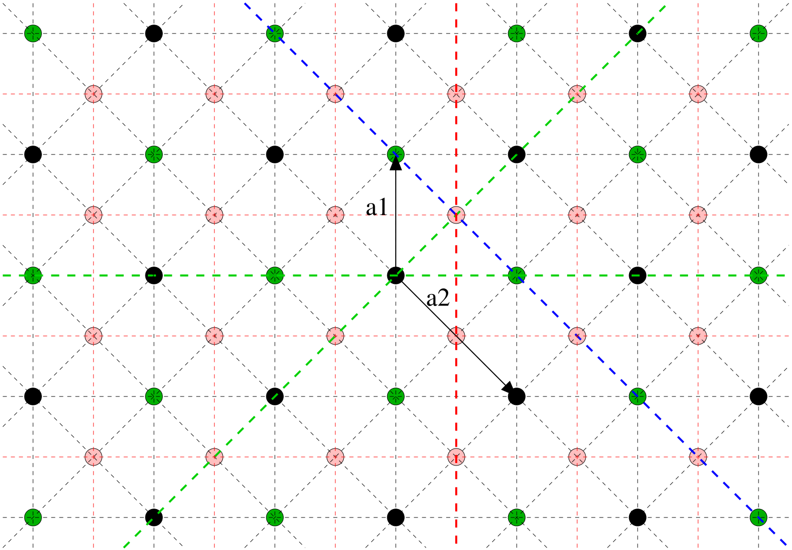
In particular $n(w) = \ell(w)$.

# Coweights modulo coroots and diagram automorphisms

- $W_a(\Phi) = L(\Phi^\vee) : W(\Phi)$
- $L(\Phi^\vee)$ coroot lattice
- $L(\Phi)^\# = \{v \in V \mid (v, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi\}$ coweight lattice
- $L(\Phi^\vee) \subseteq L(\Phi)^\#$
- $\hat{W}_a(\Phi) := L(\Phi)^\# : W(\Phi)$ acts on the set of alcoves.
- $W_a(\Phi)$ acts simply transitively on the set of alcoves.
- So $\hat{W}_a(\Phi) = W_a(\Phi) : \Omega$ where $\Omega = \mathrm{Stab}_{\hat{W}_a(\Phi)}(A_\circ)$.
- $\Omega \cong \hat{W}_a(\Phi)/W_a(\Phi) \cong L(\Phi)^\#/L(\Phi^\vee)$ acts faithfully on the simplex $A_\circ$.
- $\Omega$ acts as diagram automorphisms on the extended Dynkin diagram.

| $\Phi$ | $A_n$ | $B_n$ | $C_n$ | $D_{2n}$ | $D_{2n+1}$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\Omega$ | $C_{n+1}$ | $C_2$ | $C_2$ | $C_2 \times C_2$ | $C_4$ | $C_3$ | $C_2$ | $1$ | $1$ | $1$ |

## $\hat{W}_a(\Phi)$



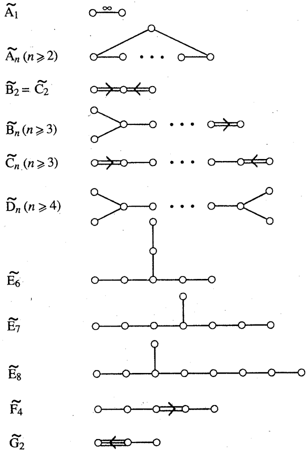
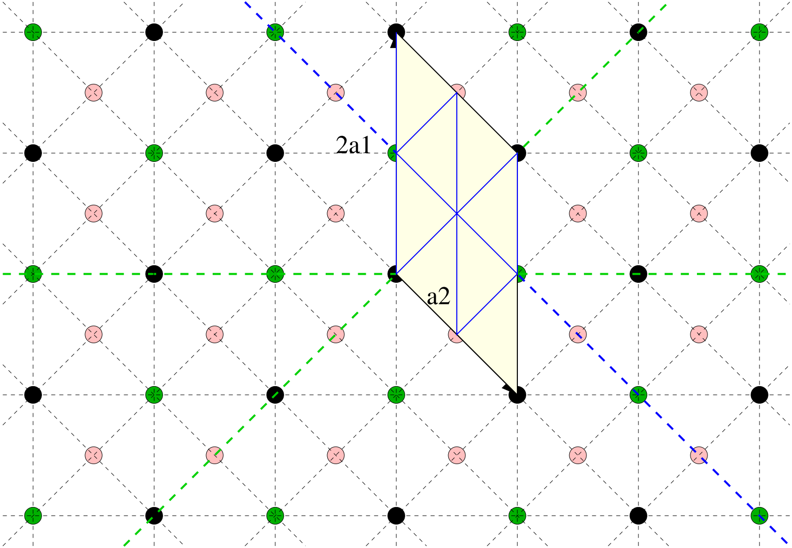



Figure 4.1: Extended Dynkin diagrams

# The order of the finite Weyl group




Wa=L(Phi^):W(Phi) => |W(Phi)| = 8

# The order of the finite Weyl group

- $\Phi$ irreducible crystallographic root system.
- $W_a(\Phi) = L(\Phi^\vee) : W(\Phi)$
- $A_\circ$ is a fundamental domain for the action of $W_a(\Phi)$.
- $P(\Delta^\vee) := \{\sum_{i=1}^n \lambda_i \alpha_i^\vee \mid 0 < \lambda_i < 1\}$ is a fundamental domain for $L(\Phi^\vee)$.
- $L(\Phi^\vee)$ is a normal subgroup of $W_a(\Phi)$ with
- $W_a(\Phi)/L(\Phi^\vee) \cong W(\Phi)$.

## Theorem

$P(\Delta^\vee)$ is the union of $|W(\Phi)|$ alcoves.
(up to a set of measure 0).
Write the highest root $\tilde{\alpha} = \sum_{i=1}^n c_i \alpha_i$. Then

$$\mathrm{vol}(P(\Delta^\vee))/\,\mathrm{vol}(A_\circ) = n!|\Omega|c_1 \cdots c_n.$$

So $|W(\Phi)| = n!|\Omega|c_1 \cdots c_n$.

# The order of the finite Weyl group

$$|W(\Phi)| = n!|\Omega|c_1 \cdots c_n.$$

| $\Phi$ | $c_1, \ldots, c_n$ | $\|\Omega\|$ | $\|W(\Phi)\|$ | $W(\Phi)$ |
|---|---|---|---|---|
| $A_n$ | $1, 1, \ldots, 1$ | $n+1$ | $(n+1)!$ | $S_{n+1}$ |
| $B_n$ | $1, 2, \ldots, 2$ | $2$ | $2^n n!$ | $C_2 \wr S_n$ |
| $C_n$ | $2, \ldots, 2, 1$ | $2$ | $2^n n!$ | $C_2 \wr S_n$ |
| $D_n$ | $1, 2, \ldots, 2, 1, 1$ | $4$ | $2^{n-1} n!$ | $C_2^{n-1} : S_n$ |
| $E_6$ | $1, 2, 2, 3, 2, 1$ | $3$ | $2^7 3^4 5$ | $S_4(3) : 2$ |
| $E_7$ | $2, 2, 3, 4, 3, 2, 1$ | $2$ | $2^{10} 3^4 5\ 7$ | $C_2 \times S_6(2)$ |
| $E_8$ | $2, 3, 4, 6, 5, 4, 3, 2$ | $1$ | $2^{14} 3^5 5^2 7$ | $2.O_8^+(2).2$ |
| $F_4$ | $2, 3, 4, 2$ | $1$ | $2^7 3^2$ | $1152$ |
| $G_2$ | $3, 2$ | $1$ | $2^2 3$ | $D_{12}$ |