Automorphisms of extremal even unimodular lattices

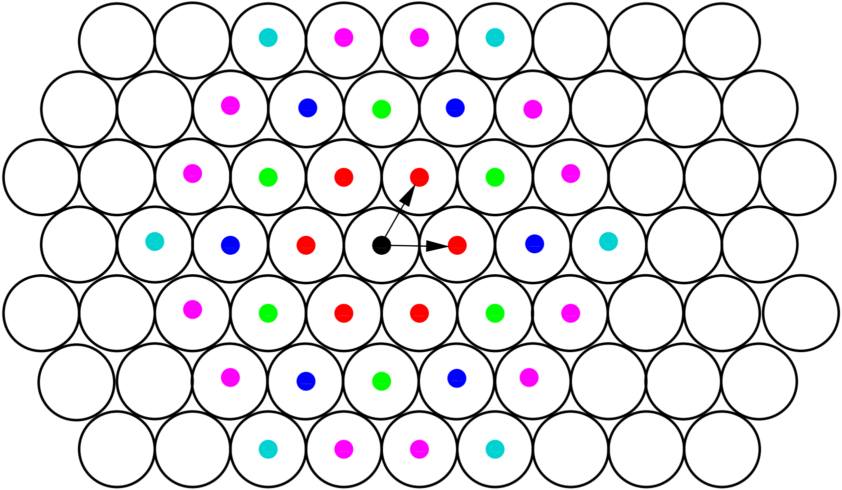
Gabriele Nebe

Lehrstuhl D für Mathematik

CWC 2015 in Barranquilla

RWTH AACHEN UNIVERSITY

The hexagonal lattice



Hexagonal Circle Packing

Even unimodular lattices

Definition

- A lattice L in Euclidean n-space $(\mathbb{R}^n, (,))$ is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis $B = (b_1, \dots, b_n)$ of $\mathbb{R}^n$

$$L = \langle b_1, \dots, b_n \rangle_{\mathbb{Z}} = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in \mathbb{Z}\}.$$

- The dual lattice is

$$L^{\#} := \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}$$

- L is called unimodular if $L = L^{\#}$.
- L is called even if $(\ell, \ell) \in 2\mathbb{Z}$ for all $\ell \in L$.
- Then $Q : L \to \mathbb{Z}, \ell \mapsto \frac{1}{2}(\ell, \ell)$ is an integral quadratic form.
- $\min(L) := \min\{Q(\ell) \mid 0 \neq \ell \in L\}$ the minimum of L.
- $\mathrm{Min}(L) := \{\ell \in L \mid Q(\ell) = \min(L)\}$.
- $\mathrm{Aut}(L) := \{g \in O(\mathbb{R}^n, (,)) \mid g(L) = L\}$ automorphism group of L.

Extremal even unimodular lattices

The sphere packing density of a unimodular lattice is proportional to its minimum.
From the theory of modular forms one gets an upper bound for the minimum:

Extremal lattices

Let L be an n-dimensional even unimodular lattice. Then

$$n \in 8\mathbb{N} \text{ and } \min(L) \leq 1 + \lfloor \frac{n}{24} \rfloor .$$

Lattices achieving equality are called extremal.

Extremal even unimodular lattices.

n	8	24	32	48	72	80	$\geq 163,264$
min(L)	1	2	2	3	4	4	
number of extremal lattices	1	1	$\geq 10^7$	≥ 4	≥ 1	≥ 4	0

Extremal even unimodular lattices in jump dimensions

$$f^{(3)} = 1 + 196,560q^2 + \ldots = \theta_{\Lambda_{24}}.$$
$$f^{(6)} = 1 + 52,416,000q^3 + \ldots = \theta_{P_{48p}} = \theta_{P_{48q}} = \theta_{P_{48n}} = \theta_{P_{48m}}.$$
$$f^{(9)} = 1 + 6,218,175,600q^4 + \ldots = \theta_{\Gamma_{72}}.$$

Let L be an extremal even unimodular lattice of dimension $24m$ so $\min(L) = m + 1$

- All non-empty layers $\emptyset \neq \{\ell \in L \mid Q(\ell) = a\}$ form spherical 11-designs.
- The density of the associated sphere packing realises a local maximum of the density function on the space of all $24m$-dimensional lattices.
- If $m = 1$, then $L = \Lambda_{24}$ is unique, Λ_{24} is the Leech lattice.
- The 196560 minimal vectors of the Leech lattice form the unique tight spherical 11-design and realise the maximal kissing number in dimension 24.
- Λ_{24} is the densest 24-dimensional lattice (Cohn, Kumar).
- For $m = 2, 3$ these lattices are the densest known lattices and realise the maximal known kissing number.

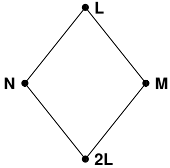
Extremal even unimodular lattices in jump dimensions

The extremal theta series

$f^{(3)} = 1 + 196,560q^2 + \ldots = \theta_{\Lambda_{24}}$.
$f^{(6)} = 1 + 52,416,000q^3 + \ldots = \theta_{P_{48p}} = \ldots = \theta_{P_{48n}} = \theta - P_{48m}$.
$f^{(9)} = 1 + 6,218,175,600q^4 + \ldots = \theta_{\Gamma_{72}}$.

The automorphism groups

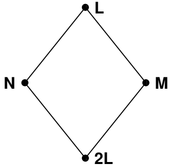
$\mathrm{Aut}(\Lambda_{24}) \cong 2.Co_1$	order	8315553613086720000
	$=$	$2^{22}3^9 5^4 7^2 \cdot 11 \cdot 13 \cdot 23$
$\mathrm{Aut}(P_{48p}) \cong (\mathrm{SL}_2(23) \times S_3) : 2$	order	$72864 = 2^5 3^2 11 \cdot 23$
$\mathrm{Aut}(P_{48q}) \cong \mathrm{SL}_2(47)$	order	$103776 = 2^5 3 \cdot 23 \cdot 47$
$\mathrm{Aut}(P_{48n}) \cong (\mathrm{SL}_2(13) \mathsf{Y}\, \mathrm{SL}_2(5)).2^2$	order	$524160 = 2^7 3^2 5 \cdot 7 \cdot 13$
$\mathrm{Aut}(P_{48m}) \cong (C_5 \times C_5 \times C_3) : (D_8 \mathsf{Y} C_4)$	order	$1200 = 2^4 3\, 5^2$
$\mathrm{Aut}(\Gamma_{72}) \cong (\mathrm{SL}_2(25) \times \mathrm{PSL}_2(7)) : 2$	order	$5241600 = 2^8 3^2 5^2 7 \cdot 13$



Turyn's construction

The extremal lattice in dimension 72

Towards the discovery of the extremal 72-dimensional lattice, whose existence was a longstanding open question.

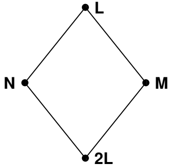


Turyn's construction

The extremal lattice in dimension 72

Towards the discovery of the extremal 72-dimensional lattice, whose existence was a longstanding open question.

- Let (L, Q) be an even unimodular lattice of dimension n.
- Choose sublattices $M, N \leq L$ such that $M + N = L$, $M \cap N = 2L$, and $(M, \frac{1}{2}Q)$, $(N, \frac{1}{2}Q)$ even unimodular.
- Such a pair (M, N) is called a polarisation of L.



Turyn's construction

The extremal lattice in dimension 72

Towards the discovery of the extremal 72-dimensional lattice, whose existence was a longstanding open question.

- Let (L, Q) be an even unimodular lattice of dimension n.
- Choose sublattices $M, N \leq L$ such that $M + N = L$, $M \cap N = 2L$, and $(M, \frac{1}{2}Q)$, $(N, \frac{1}{2}Q)$ even unimodular.
- Such a pair (M, N) is called a polarisation of L.
- $\mathcal{L}(M, N) :=$

$$\{(m+x_1, m+x_2, m+x_3) \in L \perp L \perp L \mid m \in M, x_i \in N, x_1+x_2+x_3 \in 2L\}.$$

- Define $\tilde{Q} : \mathcal{L}(M, N) \to \mathbb{Z}$,

$$\tilde{Q}(y_1, y_2, y_3) := \frac{1}{2}(Q(y_1) + Q(y_2) + Q(y_3)).$$

- $(\mathcal{L}(M, N), \tilde{Q})$ is an even unimodular lattice of dimension $3n$.

Leech from E_8

$$(m+a, m+b, m+c) \text{ in } \begin{cases} L \perp L \perp L \\ L(M,N) \\ 2L \perp 2L \perp 2L \end{cases}$$

m in M

a,b,c in N

a+b+c in 2L

Theorem (Lepowsky, Meurman; Tits)

Let $(L, Q) \cong E_8$ be the unique even unimodular lattice of dimension 8. Then for any polarisation (M, N) of E_8 the lattice $\mathcal{L}(M, N)$ has minimum ≥ 2.

Leech from E_8

$$(m+a,m+b,m+c) \text{ in } \begin{array}{l} L \perp L \perp L \\ L(M,N) \\ 2L \perp 2L \perp 2L \end{array} \qquad \begin{array}{l} m \text{ in } M \\ a,b,c \text{ in } N \\ a+b+c \text{ in } 2L \end{array}$$

Theorem (Lepowsky, Meurman; Tits)

Let $(L, Q) \cong E_8$ be the unique even unimodular lattice of dimension 8. Then for any polarisation (M, N) of E_8 the lattice $\mathcal{L}(M, N)$ has minimum ≥ 2.

Note that $\mathrm{Aut}(E_8)$ acts transitively on the polarisations of E_8.

Leech from E_8

(m+a,m+b,m+c) in

- **L ⊥ L ⊥ L**
- **L(M,N)**
- **2L⊥2L⊥2L**

m in M

a,b,c in N

a+b+c in 2L

Theorem (Lepowsky, Meurman; Tits)

Let $(L, Q) \cong E_8$ be the unique even unimodular lattice of dimension 8. Then for any polarisation (M, N) of E_8 the lattice $\mathcal{L}(M, N)$ has minimum ≥ 2.

Note that $\mathrm{Aut}(E_8)$ acts transitively on the polarisations of E_8.
Proof: Let $y := (y_1, y_2, y_3) \in \mathcal{L}(M, N)$.
All $y_i \neq 0$:

$$\tilde{Q}(y_1, y_2, y_3) = \frac{1}{2}\sum_{i=1}^{3} Q(y_i) \geq \lceil \frac{3}{2} \rceil = 2.$$

(m+a,m+b,m+c) in

- **L ⊥ L ⊥ L**
- **L(M,N)**
- **2L⊥2L⊥2L**

m in M

a,b,c in N

a+b+c in 2L

Leech from E_8

Theorem (Lepowsky, Meurman; Tits)

Let $(L, Q) \cong E_8$ be the unique even unimodular lattice of dimension 8. Then for any polarisation (M, N) of E_8 the lattice $\mathcal{L}(M, N)$ has minimum ≥ 2.

Note that $\mathrm{Aut}(E_8)$ acts transitively on the polarisations of E_8.
Proof: Let $y := (y_1, y_2, y_3) \in \mathcal{L}(M, N)$.
All $y_i \neq 0$:

$$\tilde{Q}(y_1, y_2, y_3) = \frac{1}{2}\sum_{i=1}^{3} Q(y_i) \geq \lceil \frac{3}{2} \rceil = 2.$$

$y_1 \neq 0 \neq y_2$: Then $y_i \in N$ and

$$\tilde{Q}(y) \geq 1 + 1 + 0 = 2.$$

Leech from E_8

(m+a,m+b,m+c) in

- **L ⊥ L ⊥ L**
- **L(M,N)**
- **2L⊥2L⊥2L**

m in M

a,b,c in N

a+b+c in 2L

Theorem (Lepowsky, Meurman; Tits)

Let $(L, Q) \cong E_8$ be the unique even unimodular lattice of dimension 8. Then for any polarisation (M, N) of E_8 the lattice $\mathcal{L}(M, N)$ has minimum ≥ 2.

Note that $\mathrm{Aut}(E_8)$ acts transitively on the polarisations of E_8.
Proof: Let $y := (y_1, y_2, y_3) \in \mathcal{L}(M, N)$.
All $y_i \neq 0$:

$$\tilde{Q}(y_1, y_2, y_3) = \frac{1}{2}\sum_{i=1}^{3} Q(y_i) \geq \lceil \frac{3}{2} \rceil = 2.$$

$y_1 \neq 0 \neq y_2$: Then $y_i \in N$ and

$$\tilde{Q}(y) \geq 1 + 1 + 0 = 2.$$

Only one $y_i \neq 0$ then $y_i \in 2L$ and $\tilde{Q}(y) \geq 2$.

Turyn's construction

(m+a,m+b,m+c) in

- **L ⊥ L ⊥ L**
- **L(M,N)**
- **2L⊥2L⊥2L**

m in M

a,b,c in N

a+b+c in 2L

$d := \min(L, Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M, N)) \leq 2d$.

Turyn's construction

$L \perp L \perp L$

(m+a,m+b,m+c) in $L(M,N)$

$2L \perp 2L \perp 2L$

m in M

a,b,c in N

a+b+c in 2L

$d := \min(L, Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M, N)) \leq 2d$.

Proof:

$(a, 0, 0)$ $a = 2\ell \in 2L$ with $\frac{1}{2}Q(2\ell) = 2Q(\ell) \geq 2d$.

Turyn's construction

$$
(m+a,m+b,m+c) \text{ in } \begin{cases} L \perp L \perp L \\ L(M,N) \\ 2L \perp 2L \perp 2L \end{cases}
$$

m in M

a,b,c in N

a+b+c in 2L

$d := \min(L,Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M,N)) \leq 2d.$

Proof:

$(a,0,0)$ $a = 2\ell \in 2L$ with $\frac{1}{2}Q(2\ell) = 2Q(\ell) \geq 2d.$

$(a,b,0)$ $a,b \in N$ with $\frac{1}{2}Q(a) + \frac{1}{2}Q(b) \geq 2d.$

Turyn's construction

$$
\textbf{(m+a,m+b,m+c) in}\quad
\begin{array}{l}
\bullet\ L \perp L \perp L \\
\vert \\
\bullet\ L(M,N) \\
\vert \\
\bullet\ 2L \perp 2L \perp 2L
\end{array}
$$

m in M

a,b,c in N

a+b+c in 2L

$d := \min(L, Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M, N)) \leq 2d$.

Proof:

$(a, 0, 0)$ $a = 2\ell \in 2L$ with $\frac{1}{2}Q(2\ell) = 2Q(\ell) \geq 2d$.

$(a, b, 0)$ $a, b \in N$ with $\frac{1}{2}Q(a) + \frac{1}{2}Q(b) \geq 2d$.

(a, b, c) then $\frac{1}{2}(Q(a) + Q(b) + Q(c)) \geq \frac{3}{2}d$.

Turyn's construction

$$(m+a, m+b, m+c) \text{ in } \begin{cases} L \perp L \perp L \\ L(M,N) \\ 2L \perp 2L \perp 2L \end{cases}$$

m in M

a,b,c in N

a+b+c in 2L

$d := \min(L,Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M,N)) \leq 2d$.

Proof:

$(a,0,0)$ $a = 2\ell \in 2L$ with $\frac{1}{2}Q(2\ell) = 2Q(\ell) \geq 2d$.

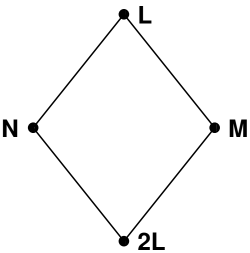
$(a,b,0)$ $a,b \in N$ with $\frac{1}{2}Q(a) + \frac{1}{2}Q(b) \geq 2d$.

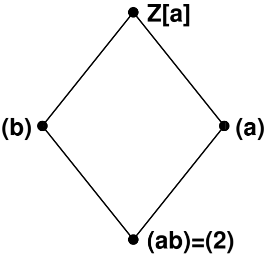
(a,b,c) then $\frac{1}{2}(Q(a) + Q(b) + Q(c)) \geq \frac{3}{2}d$.

72-dimensional lattices from Leech (Griess)

If $(L,Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$ then $3 \leq \min(\mathcal{L}(M,N)) \leq 4$.

Hermitian polarisations



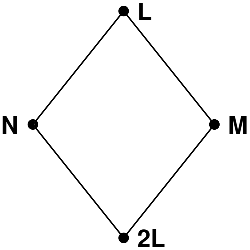


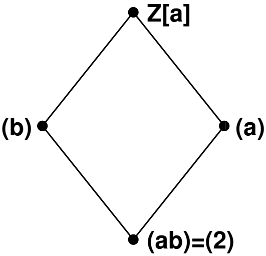
Let $\alpha \in \mathrm{End}(L)$ such that

- $\alpha^2 - \alpha + 2 = 0$ ($\mathbb{Z}[\alpha]$ = integers in $\mathbb{Q}[\sqrt{-7}]$).
- $(\alpha x, y) = (x, \beta y)$ where $\beta = 1 - \alpha = \overline{\alpha}$.

Then $M := \alpha L$, $N := \beta L$ defines a polarisation of L such that $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q)$.

Hermitian polarisations





Let $\alpha \in \mathrm{End}(L)$ such that

- $\alpha^2 - \alpha + 2 = 0$ ($\mathbb{Z}[\alpha]$ = integers in $\mathbb{Q}[\sqrt{-7}]$).
- $(\alpha x, y) = (x, \beta y)$ where $\beta = 1 - \alpha = \overline{\alpha}$.

Then $M := \alpha L$, $N := \beta L$ defines a polarisation of L such that $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q)$.

Remark

$\mathcal{L}(\alpha L, \beta L) = L \otimes_{\mathbb{Z}[\alpha]} P_b$ where

$$P_b = \langle (\beta, \beta, 0), (0, \beta, \beta), (\alpha, \alpha, \alpha) \rangle \leq \mathbb{Z}[\alpha]^3$$

P_b is Hermitian unimodular and $\mathrm{Aut}_{\mathbb{Z}[\alpha]}(P_b) \cong \pm \mathrm{PSL}_2(7)$. So $\mathrm{Aut}(\mathcal{L}(\alpha L, \beta L)) \geq \mathrm{Aut}_{\mathbb{Z}[\alpha]}(L) \times \mathrm{PSL}_2(7)$.

Hermitian structures of the Leech lattice

Theorem (M. Hentschel, 2009)

There are exactly nine $\mathbb{Z}[\alpha]$-structures of the Leech lattice.

	group	order	
1	$\mathrm{SL}_2(25)$	$2^4 3 \cdot 5^2 13$	
2	$2.A_6 \times D_8$	$2^7 3^2 5$	
3	$\mathrm{SL}_2(13).2$	$2^4 3 \cdot 7 \cdot 13$	
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	
6	soluble	$2^9 3^3$	
7	$\pm \mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2^4 3^2 7^2$	
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2^7 3^3 5 \cdot 7^2$	
9	$2.J_2.2$	$2^9 3^3 5^2 7$	

Hermitian structures of the Leech lattice

Theorem (M. Hentschel, 2009)

There are exactly nine $\mathbb{Z}[\alpha]$-structures of the Leech lattice.

	group	order	# Q(v) = 3
1	$\mathrm{SL}_2(25)$	$2^4 3 \cdot 5^2 13$	0
2	$2.A_6 \times D_8$	$2^7 3^2 5$	$2 \cdot 20,160$
3	$\mathrm{SL}_2(13).2$	$2^4 3 \cdot 7 \cdot 13$	$2 \cdot 52,416$
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	$2 \cdot 100,800$
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	$2 \cdot 100,800$
6	soluble	$2^9 3^3$	$2 \cdot 177,408$
7	$\pm\mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2^4 3^2 7^2$	$2 \cdot 306,432$
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2^7 3^3 5 \cdot 7^2$	$2 \cdot 504,000$
9	$2.J_2.2$	$2^9 3^3 5^2 7$	$2 \cdot 1,209,600$

The extremal 72-dimensional lattice Γ_{72}

Main result

- $\Gamma_{72} = \Lambda_{24} \otimes_{\mathbb{Z}[\alpha]]} P_b$ is an extremal even unimodular lattice of dimension 72.
- $\mathrm{Aut}(\Gamma_{72}) \cong (\mathrm{PSL}_2(7) \times \mathrm{SL}_2(25)) : 2$ (uses the classification of finite simple groups).
- Γ_{72} realises the densest known sphere packing
- and maximal known kissing number in dimension 72.
- Γ_{72} is the unique extremal even unimodular lattice that admits an automorphism g for which μ_g has an irreducible factor of degree > 36 (see below).

Theorem (R. Parker, N)

If (M, N) is a polarisation of the Leech lattice such that $\mathcal{L}(M, N)$ is extremal, then $\mathcal{L}(M, N) \cong \Gamma_{72}$.

The Type of an automorphism.

Lattices with large automorphisms

We now use automorphisms to classify extremal even unimodular lattices of dimension 48 and 72. The motivation comes from coding theory, where one tries to construct an extremal code of length 72 using automorphisms. In the meantime we know that if an extremal $[72, 36, 16]$ code exists, then its automorphism group has order ≤ 5.

The Type of an automorphism.

Lattices with large automorphisms

We now use automorphisms to classify extremal even unimodular lattices of dimension 48 and 72. The motivation comes from coding theory, where one tries to construct an extremal code of length 72 using automorphisms. In the meantime we know that if an extremal $[72, 36, 16]$ code exists, then its automorphism group has order ≤ 5.

Let $L \leq \mathbb{R}^n$ be some even unimodular lattice and $\sigma \in \mathrm{Aut}(L)$ of prime order p. The fixed lattice

$$F := \mathrm{Fix}_L(\sigma) := \{v \in L \mid \sigma v = v\} \leq L$$

has dimension d, and σ acts on $M := \mathrm{Cyc}_L(\sigma) := F^{\perp}$ as a pth root of unity, so $n = d + z(p-1)$.

$$F^{\#} \perp M^{\#} \geq L = L^{\#} \geq F \perp M \geq pL$$

with $\det(F) = |F^{\#}/F| = |M^{\#}/M| = \det(M) = p^s$

Definition: p-(z,d)-s is called the Type of σ.

Proposition: $s \leq \min(d, z)$ and $z - s$ is even.

48-dimensional extremal lattices

Theorem

Let L be an extremal even unimodular lattice of dimension 48 and p be a prime dividing $|\mathrm{Aut}(L)|$. Then $p = 47, 23$ or $p \leq 13$.

The possible types of automorphisms of prime order $p > 3$				
Type	$\mathrm{Fix}(\sigma)$	$\mathrm{Cyc}(\sigma)$	example	class.
47-(1,2)-1	unique	unique	P_{48q}	yes
23-(2,4)-2	unique	at least 2	P_{48q}, P_{48p}	
13-(4,0)-0	$\{0\}$	at least 1	P_{48n}	
11-(4,8)-4	unique	at least 1	P_{48p}	
7-(8,0)-0	$\{0\}$	at least 1	P_{48n}	
7-(7,6)-5	$\sqrt{7}A_6^{\#}$	not known	not known	
5-(12,0)-0	$\{0\}$	at least 2	P_{48n}, P_{48m}	
5-(10,8)-8	$\sqrt{5}E_8$	at least 1	P_{48m}	
5-(8,16)-8	$[2.\mathrm{Alt}_{10}]_{16}$	Λ_{32}	P_{48m}	yes

Prime order automorphisms

Possible types of prime order automorphisms of extremal lattices

Dimension 24	Dimension 48	Dimension 72	Dimension 96
	47-(1,2)-1	37-(2,0)-0	
23-(1,2)-1	23-(2,4)-2	19-(4,0)-0	17-(6,0)-0
13-(2,0)-0	13-(4,0)-0	13-(6,0)-0	13-(8,0)-0
11-(2,4)-2	11-(4,8)-4	7-(12,0)-0	13-(7,12)-7
7,5,3,2	7,5,3,2	5,3,2	7,5,3,2

Prime divisors

Let L be an extremal even unimodular lattice of dimension 24m and p be a prime dividing $|\operatorname{Aut}(L)|$. Then

m=1: $p = 23$ or $p \leq 13$.

m=2: $p = 47, 23$ or $p \leq 13$.

m=3: $p = 37, 19, 13$ or 7, and μ_σ is irreducible,
or $p \leq 5$

Large automorphisms of extremal lattices

Definition

$\sigma \in \mathrm{Aut}(L)$ is called large, if μ_σ has an irreducible factor Φ_a of degree $d = \varphi(a) > \frac{1}{2}\dim(L)$.

Remark

Let $\sigma \in \mathrm{Aut}(\Lambda_{24})$ be large. Then

a	23	33	35	39	40	52	56	60	84
d	22	20	24	24	16	24	24	16	24

Theorem

Let L be an extremal unimodular lattice of dimension 48, $\sigma \in \mathrm{Aut}(L)$ large. Then

a	120	132	69	47	65	104
d	32	40	44	46	48	48
L	P_{48n}	P_{48p}	P_{48p}	P_{48q}	P_{48n}	P_{48n}

Theorem

Let Γ be an extremal unimodular lattice of dimension 72, $\sigma \in \mathrm{Aut}(\Gamma)$ large. Then $\Gamma = \Gamma_{72}$ and either $a = 91$ $(d = 72)$ or $a = 168$ $(d = 48)$.