Orthogonal Representations of Finite Groups

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joint work with Thomas Breuer, Linda Hoyer, and Richard Parker

Group algebras

- G finite group, K field
- $KG = \bigoplus_{g \in G} Kg$ group algebra
- $\mathbb{Q}G \cong \bigoplus_{i=1}^{h} A_i$ with $A_i \cong D_i^{n_i \times n_i}$ semisimple algebra

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Determine invariants of ι_i

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Invariants of ι_i are the invariants of Q_i resp. H_i that are independent of scaling.

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Determine discriminants for the characters in $\mathrm{Irr}^+(G)$ and $\mathrm{Irr}^o(G)$ for all but the largest few ATLAS groups.

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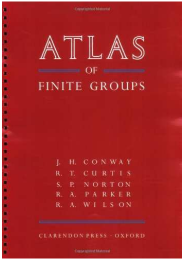

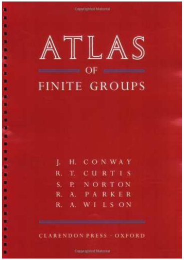

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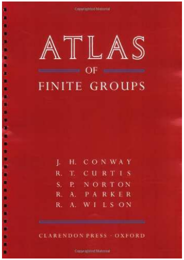
  linear, symplectic, unitary, orthogonal groups over finite fields



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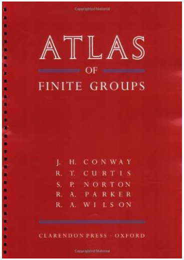

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- ATLAS of finite groups
  ordinary character tables of finite simple groups
  classifying simple $\mathbb{Q}G$-modules

The character table of A_7

		1a	2a	3a	3b	4a	5a	6a	7a	7b
X.1	+	1	1	1	1	1	1	1	1	1
X.2	+	6	2	3	.	.	1	-1	-1	-1
X.3	o	10	-2	1	1	.	.	1	A	B
X.4	o	10	-2	1	1	.	.	1	B	A
X.5	+	14	2	2	-1	.	-1	2	.	.
X.6	+	14	2	-1	2	.	-1	-1	.	.
X.7	+	15	-1	3	.	-1	.	-1	1	1
X.8	+	21	1	-3	.	-1	1	1	.	.
X.9	+	35	-1	-1	-1	1	.	-1	.	.

$$A = (-1 + \sqrt{-7})/2, B = (-1 - \sqrt{-7})/2$$

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An orthogonal character χ is called orthogonally stable if there is a square class $d(\mathbb{Q}(\chi)^\times)^2$ such that for all representations $\rho : G \to \mathrm{GL}_n(L)$ with character χ and all non-degenerate quadratic forms $Q \in \mathcal{F}(\rho)$

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χ orthogonally stable, then

$$\mathrm{disc}(\chi) := d(\mathbb{Q}(\chi)^\times)^2$$

is called the orthogonal discriminant of χ.

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- B non-degenerate symmetric bilinear form on V
- adjoint involution ι_B on $\mathrm{End}(V)$

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- If $\det(X) \neq 0$ then
  (a) χ_ρ is orthogonally stable and
  (b) $\mathrm{disc}(\chi_\rho) = (-1)^{n/2} \det(X)(\mathbb{Q}(\chi)^\times)^2$.

The character table of J_2

	;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;	@	@	@	@	@	@	@	@	@	@	@	
			1		1																															
		604800	920	240	080	36	96	300	300	50	50	24	12	7	8	20	20	10	10	12	15	15			336	48	12	6	48	16	6	6	7	12	12	
	p power		A	A	A	A	A	A	A	A	A	AA	BB	A	A	BB	AB	DA	CA	AA	BA	AA			A	A	B	BC	A	A	AB	BC	AC	AB	AB	
	p' part		A	A	A	A	A	A	A	A	A	AA	BB	A	A	AB	BB	CA	DA	AA	AA	BA			A	A	A	BC	A	A	AB	BC	AC	AB	AB	
	ind	1A	2A	2B	3A	3B	4A	5A	B*	5C	D*	6A	6B	7A	8A	10A	B*	10C	D*	12A	15A	B*	fus	ind	2C	4B	4C	6C	8B	8C	12B	12C	14A	24A	B*	
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	++	1	1	1	1	1	1	1	1	1	1	1	χ_1
χ_2	+	14	-2	2	5	-1	2	-3b5	*	b5+2	*	1	-1	0	0	b5	*	-b5	*	-1	0	0	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_2
χ_3	+	14	-2	2	5	-1	2	*	-3b5	*	b5+2	1	-1	0	0	*	b5	*	-b5	-1	0	0	|													χ_3
χ_4	+	21	5	-3	3	0	1	b5+4	*	-2b5	*	-1	0	0	-1	b5	*	0	0	1	*	-b5	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_4
χ_5	+	21	5	-3	3	0	1	*	b5+4	*	-2b5	-1	0	0	-1	*	b5	0	0	1	-b5	*	|													χ_5
χ_6	+	36	4	0	9	0	4	-4	-4	1	1	1	0	1	0	0	0	-1	-1	1	-1	-1	:	++	6	-2	0	0	2	2	1	0	-1	-1	-1	χ_6
χ_7	+	63	15	-1	0	3	3	3	3	-2	-2	0	-1	0	1	-1	-1	0	0	0	0	0	:	++	7	3	-1	1	-3	1	0	-1	0	0	0	χ_7
χ_8	+	70	-10	-2	7	1	2	-5b5	*	0	0	-1	1	0	0	-b5	*	0	0	-1	b5	*	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_8
χ_9	+	70	-10	-2	7	1	2	*	-5b5	0	0	-1	1	0	0	*	-b5	0	0	-1	*	b5	|													χ_9
χ_{10}	+	90	10	6	9	0	-2	5	5	0	0	1	0	-1	0	1	1	0	0	1	-1	-1	:	++	6	2	0	0	4	0	-1	0	-1	1	1	χ_{10}
χ_{11}	+	126	14	6	-9	0	2	1	1	1	1	-1	0	0	0	1	1	-1	-1	-1	1	1	:	++	0	4	0	0	2	-2	1	0	0	-1	-1	χ_{11}
χ_{12}	+	160	0	4	16	1	0	-5	-5	0	0	0	1	-1	0	-1	-1	0	0	0	1	1	:	++	8	0	2	-1	0	0	0	-1	1	0	0	χ_{12}
χ_{13}	+	175	15	-5	-5	1	-1	0	0	0	0	3	1	0	-1	0	0	0	0	-1	0	0	:	++	7	-1	1	1	-1	-1	-1	1	0	-1	-1	χ_{13}
χ_{14}	+	189	-3	-3	0	0	-3	-3b5	*	b5+2	*	0	0	0	1	b5	*	b5	*	0	0	0	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{14}
χ_{15}	+	189	-3	-3	0	0	-3	*	-3b5	*	b5+2	0	0	0	1	*	b5	*	b5	0	0	0	|													χ_{15}
χ_{16}	+	224	0	-4	8	-1	0	2r5-1	-2r5-1	2b5	*	0	-1	0	0	1	1	0	0	0	*	-b5	|	+	0	0	0	0	0	0	0	0	0	0	0	χ_{16}
χ_{17}	+	224	0	-4	8	-1	0	-2r5-1	2r5-1	*	2b5	0	-1	0	0	1	1	0	0	0	-b5	*	|													χ_{17}
χ_{18}	+	225	-15	5	0	3	-3	0	0	0	0	0	-1	1	-1	0	0	0	0	0	0	0	:	++	1	-3	-1	1	3	-1	0	-1	1	0	0	χ_{18}
χ_{19}	+	288	0	4	0	-3	0	3	3	-2	-2	0	1	1	0	-1	-1	0	0	0	0	0	:	++	8	0	-2	-1	0	0	0	1	1	0	0	χ_{19}
χ_{20}	+	300	-20	0	-15	0	4	0	0	0	0	1	0	-1	0	0	0	0	0	1	0	0	:	++	6	-2	0	0	-2	-2	1	0	-1	1	1	χ_{20}
χ_{21} $Q_{2,3}$	+	336	16	0	-6	0	0	-4	-4	1	1	-2	0	0	0	0	0	1	1	0	-1	-1	:	++	0	0	0	0	0	0	0	0	0	r6	-r6	χ_{21}
	ind	1	2	4	3	3	4	5	5	5	5	6	12	7	8	20	20	10	10	12	15	15	fus	ind	4	4	8	12	8	8	12	24	28	24	24	
		2	2		6	6	4	10	10	10	10	6		14	8			10	10	12	30	30					8		8		12	24	28	24	24	
		6	2	0	-3	0	2	-2b5	*	*	b5-1	1	0	-1	0	0	0	-b5	*	-1	b5	*	|	-	0	0	0	0	0	0	0	0	0	0	0	χ_{22}
																																				χ_{23}

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- Compute $\operatorname{disc}(\chi_{21})$:
- ρ a 672 dimensional rational representation affording $2\chi_{21}$

- Choose $g, h, k \in \rho(J_2)$, compute $X = X(g,h,k) = g - g^{-1} + h - h^{-1} + k - k^{-1} \in \rho(E_-(\mathbb{Q}J_2))$.
- $\operatorname{disc}(\chi_{21}) = N_{red}(X)(\mathbb{Q}^\times)^2$.

The discriminant field extension

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E.g. $G = J_1$, $\chi(1) = 56$, $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{5})$,

$$\mathrm{disc}(\chi) = (31 + 5\sqrt{5})/2, \ \mathrm{Gal}(\Delta(\chi))/\mathbb{Q} \cong D_8$$

2 ordinaries 20 Dec 2021

	11	19	29	31	41	59		13	17
14	0−	0+	0−	0+	0−	0−	−3	x	x
14	0−	0+	0−	0+	0−	0−	−3	x	x
36	0+	0+	0+	0+	0+	0+	5	0−	0−
70	0−	0+	0−	0+	0−	0−	−3	x	x
70	0−	0+	0−	0+	0−	0−	−3	x	x
90	0+	0−	0+	0−	0−	0−	−7	0−	0−
126	0−	0−	0+	0−	0+	0−	~~−1~~ −5	0−	0−
160	0+	0+	0+	0+	0+	0+	1	0+	0+
224	0+	0+	0+	0+	0+	0+	1	x	x
224	0+	0+	0+	0+	0+	0+	1	x	x
288	0−	0−	0−	0−	0+	0+	~~21~~ 105	0+	0−
300	0−	0−	0−	0−	0+	0+	21	0−	0+
336	0+	0+	0+	0+	0+	0+	1	0+	0+

lab 63 175 189_{ab} 225

1	2	4	8	3	6	13
12	11	9	5	10	7	

<Prime> = 1 + k63 + k36 | e2 k36 = $224_{ab} + 70_{ab} + 21_{ab}$

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A simple algebra with orthogonal involution ι.
$\Sigma^-(A) := \{a \in A \mid a = -\iota(a)\}$.
Subalgebra $B \leq A$ orthogonally stable if and only if
(a) $\iota(B) = B$ and (b) $\Sigma^-(B) \cap A^\times \neq \emptyset$.
Then

$$\mathrm{disc}(\iota) = \mathrm{disc}(\iota_{|B}).$$

Parker's conjecture

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No counterexamples to Parker's conjecture so far.

Hermitian forms

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Discriminant of quaternion algebra

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- $H : V \times V \to K$ Hermitian form
- $H_B := (H(b_i,b_j))_{i,j=1}^n \in K^{n\times n}$, $B = (b_1,\ldots,b_n)$ an K-basis of V
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Discriminant of Hermitian form

- $H : V \times V \to K$ Hermitian form
- $H_B := (H(b_i,b_j))_{i,j=1}^n \in K^{n\times n}$, $B = (b_1,\ldots,b_n)$ an K-basis of V
- $\mathrm{disc}(H) := (-1)^{\binom{n}{2}} \det(H_B) N_{K/F}(K^\times)$ discriminant of H.
- $\Delta(H) := [(K,d)_F] \in \mathrm{Br}_2(K,F)$ discriminant algebra of H.
- $\chi \in \mathrm{Irr}^o(G)$, well defined $\mathrm{disc}(\chi)$ and $\Delta(\chi)$
- Primes that ramify in $\Delta(\chi)$ do divide the group order.

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Fixed algebras of certain group automorphisms are orthogonal subalgebras.

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Conclusion

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Parker's conjecture

$X \in \Sigma^-(G) \Rightarrow \nu(\det(\rho(X)))$ even for all dyadic valuations ν.