Extremal lattices

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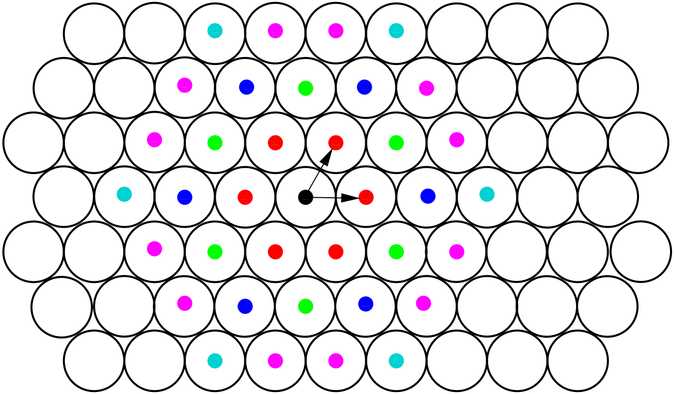
Lattices and sphere packings



Hexagonal Circle Packing

$$\theta = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + \dots.$$

Even unimodular lattices

Definition

- A lattice L in Euclidean n-space $(\mathbb{R}^n, (\,,\,))$ is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis $B = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$

$$L = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}} = \{\sum_{i=1}^{n} a_i b_i \mid a_i \in \mathbb{Z}\}.$$

- The dual lattice is

$$L^{\#} := \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}$$

- L is called unimodular if $L = L^{\#}$.
- $Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, Q(x) := \frac{1}{2}(x, x)$ associated quadratic form
- L is called even if $Q(\ell) \in \mathbb{Z}$ for all $\ell \in L$.
- $\min(L) := \min\{2Q(\ell) \mid 0 \neq \ell \in L\}$ minimum of L.

The sphere packing density of an even unimodular lattice is proportional to its minimum.

Theta-series of lattices

Let (L, Q) be an even unimodular lattice of dimension n so a regular positive definite integral quadratic form $Q : L \to \mathbb{Z}$.

- The theta series of L is

$$\theta_L = \sum_{\ell \in L} q^{Q(\ell)} = 1 + \sum_{k=\min(L)/2}^{\infty} a_k q^k$$

  where $a_k = |\{\ell \in L \mid Q(\ell) = k\}|$.
- θ_L defines a holomorphic function on the upper half plane by substituting $q := \exp(2\pi i z)$.
- Then θ_L is a modular form of weight $\frac{n}{2}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.
- n is a multiple of 8.
- $\theta_L \in \mathcal{M}_{\frac{n}{2}}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, \Delta]_{\frac{n}{2}}$ where $E_4 := \theta_{E_8} = 1 + 240q + \dots$ is the normalized Eisenstein series of weight 4 and

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + \dots \text{ of weight 12}$$

Extremal modular forms

Basis of $\mathcal{M}_{4k}(\mathrm{SL}_2(\mathbb{Z}))$:

$$
\begin{array}{llllll}
E_4^k = & 1+ & 240kq+ & *q^2+ & \ldots & \\
E_4^{k-3}\Delta = & & q+ & *q^2+ & \ldots & \\
E_4^{k-6}\Delta^2 = & & & q^2+ & \ldots & \\
\vdots & & & & & \\
E_4^{k-3m_k}\Delta^{m_k} = & & \ldots & & q^{m_k}+ & \ldots
\end{array}
$$

where $m_k = \lfloor \frac{n}{24} \rfloor = \lfloor \frac{k}{3} \rfloor$.

Definition

This space contains a unique form

$$f^{(k)} := 1 + 0q + 0q^2 + \ldots + 0q^{m_k} + a(f^{(k)})q^{m_k+1} + b(f^{(k)})q^{m_k+2} + \ldots$$

$f^{(k)}$ is called the extremal modular form of weight $4k$.

$f^{(1)} = 1 + 240q + \ldots = \theta_{E_8}$, $f^{(2)} = 1 + 480q + \ldots = \theta_{E_8}^2$,
$f^{(3)} = 1 + 196,560q^2 + \ldots = \theta_{\Lambda_{24}}$,
$f^{(6)} = 1 + 52,416,000q^3 + \ldots = \theta_{P_{48p}} = \theta_{P_{48q}} = \theta_{P_{48n}}$,
$f^{(9)} = 1 + 6,218,175,600q^4 + \ldots = \theta_{\Gamma_{72}}$.

Extremal even unimodular lattices

Theorem (Siegel)

$a(f^{(k)}) > 0$ for all k

Corollary

Let L be an n-dimensional even unimodular lattice. Then

$$\min(L) \leq 2 + 2\lfloor \frac{n}{24} \rfloor = 2 + 2m_{n/8}.$$

Lattices achieving this bound are called extremal.

Extremal even unimodular lattices L$\leq \mathbb{R}^n$

n	8	16	24	32	40	48	72	80	$\geq 163,264$
min(L)	2	2	4	4	4	6	8	8	
number of extremal lattices	1	2	1	$\geq 10^7$	$\geq 10^{51}$	≥ 3	≥ 1	≥ 4	0

Extremal even unimodular lattices

Theorem (Siegel)

$a(f^{(k)}) > 0$ for all k and $b(f^{(k)}) < 0$ for large k ($k \geq 20408$).

Corollary

Let L be an n-dimensional even unimodular lattice. Then

$$\min(L) \leq 2 + 2\lfloor \frac{n}{24} \rfloor = 2 + 2m_{n/8}.$$

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Extremal even unimodular lattices in jump dimensions

Let L be an extremal even unimodular lattice of dimension $24m$ so $\min(L) = 2m + 2$

- All non-empty layers $\emptyset \neq \{\ell \in L \mid Q(\ell) = a\}$ form spherical 11-designs.
- The density of the associated sphere packing realises a local maximum of the density function on the space of all $24m$-dimensional lattices.
- If $m = 1$, then $L = \Lambda_{24}$ is unique, Λ_{24} is the Leech lattice.
- The 196560 minimal vectors of the Leech lattice form the unique tight spherical 11-design and realise the maximal kissing number in dimension 24.
- Λ_{24} is the densest 24-dimensional lattice (Cohn, Kumar).
- For $m = 2, 3$ these lattices are the densest known lattices and realise the maximal known kissing number.

Extremal even unimodular lattices in jump dimensions

The extremal theta series

$f^{(3)} = 1 + 196,560q^2 + \ldots = \theta_{\Lambda_{24}}$.
$f^{(6)} = 1 + 52,416,000q^3 + \ldots = \theta_{P_{48p}} = \theta_{P_{48q}} = \theta_{P_{48n}}$.
$f^{(9)} = 1 + 6,218,175,600q^4 + \ldots = \theta_{\Gamma_{72}}$.

The automorphism groups

$\mathrm{Aut}(\Lambda_{24}) \cong 2.Co_1$	order $=$	8315553613086720000 $2^{22}3^9 5^4 7^2 \cdot 11 \cdot 13 \cdot 23$
$\mathrm{Aut}(P_{48p}) \cong (\mathrm{SL}_2(23) \times S_3) : 2$	order	$72864 = 2^5 3^2 11 \cdot 23$
$\mathrm{Aut}(P_{48q}) \cong \mathrm{SL}_2(47)$	order	$103776 = 2^5 3 \cdot 23 \cdot 47$
$\mathrm{Aut}(P_{48n}) \cong (\mathrm{SL}_2(13) \mathsf{Y}\, \mathrm{SL}_2(5)).2^2$	order	$524160 = 2^7 3^2 5 \cdot 7 \cdot 13$
$\mathrm{Aut}(\Gamma_{72}) \cong (\mathrm{SL}_2(25) \times \mathrm{PSL}_2(7)) : 2$	order	$5241600 = 2^8 3^2 5^2 7 \cdot 13$

Construction of extremal lattices

From codes.

- Let $(e_1, \ldots, e_n)$ be a p-frame, so $(e_i, e_j) = p\delta_{ij}$.
- $Z := \langle e_1, \ldots, e_n \rangle_{\mathbb{Z}} \cong \sqrt{p}\mathbb{Z}^n$, $Z^{\#} = \frac{1}{p} Z$.
- $Z^{\#}/Z \cong \mathbb{F}_p^n$.
- Given $C \leq \mathbb{F}_p^n$ the codelattice is
- $\Lambda(C) := \{ \frac{1}{p} \sum c_i e_i \mid (\overline{c}_1, \ldots, \overline{c}_n) \in C \}$
- $\Lambda(C)^{\#} = \Lambda(C^{\perp})$.
- $\Lambda(C)$ is even if $p = 2$ and C is doubly even.
- $\min(\Lambda(C)) = \min(p, \frac{d(C)}{p})$.
- $\mathrm{Aut}(C) \leq \mathrm{Aut}(\Lambda(C))$.

Binary extremal codes.

length	8	24	32	40	48	72	80	≥ 3952
$d(C)$	4	8	8	8	12	16	16	
extremal	h_8	$\mathcal{G}_{24}$	5	$16,470$	QR_{48}	$?$	≥ 4	0

Canonical constructions of lattices

- A canonical construction of a lattice is a construction that is respected by (a big subgroup of) its automorphism group.
- The Leech lattice has at least 23 constructions, none of them is really canonical:
- Leech as a neighbor of a code lattice
- Let $\mathcal{G}_{24} \leq \mathbb{F}_2^{24}$ be the binary Golay code (the extended quadratic residue code).
- Then $d(\mathcal{G}_{24}) = 8$.
- $\mathrm{Min}(\Lambda(\mathcal{G}_{24})) = \{\pm e_1, \ldots, \pm e_{24}\}$.
- Neighbor lattice: $v = \frac{1}{2}(3e_1 + \ldots + e_{24})$
- $\Lambda_{24} := \Lambda(\mathcal{G}_{24})^{(v),2} := \langle \{x \in \Lambda(\mathcal{G}_{24}) \mid (x,v) \text{ even } \}, \frac{v}{2} \rangle$
- $2^{12} : M_{24} \leq \mathrm{Aut}(\Lambda_{24}) = 2.Co_1$.

Canonical constructions of the 48-dimensional lattices

Two of the 48-dimensional extremal lattices have a canonical construction with codes:

Theorem (Koch)

Let $C = C^{\perp} \leq \mathbb{F}_3^{48}$ with $d(C) = 15$. Then $\Lambda(C)^{(v),2}$ is an extremal even unimodular lattice, where $v = \frac{1}{3}(e_1 + \ldots + e_{48})$.

Theorem (N)

Let $C = C^{\perp} \leq \mathbb{F}_3^{48}$ with $d(C) = 15$ such that $|\operatorname{Aut}(C)|$ is divisible by some prime $p \geq 5$. Then $C \cong Q_{48}$ or $C \cong P_{48}$. We have $\operatorname{Aut}(Q_{48}) \cong \mathrm{SL}_2(47)$ and $\operatorname{Aut}(P_{48}) \cong (\mathrm{SL}_2(23) \times C_2) : 2$.

Remark

$\Lambda(Q_{48})^{(v),2} \cong P_{48q}$, $\operatorname{Aut}(P_{48q}) \cong \mathrm{SL}_2(47)$
$\Lambda(P_{48})^{(v),2} \cong P_{48p}$, $\operatorname{Aut}(P_{48p}) \cong (\mathrm{SL}_2(23) \times S_3) : 2$

How many 48-dimensional extremal lattices are there?

Let L be an extremal even unimodular lattice of dimension 48 and p be a prime dividing $|\operatorname{Aut}(L)|$.

Theorem

$p = 47, 23$ or $p \leq 13$.
Let $\sigma \in \operatorname{Aut}(L)$ be of order p. Then the fixed lattice $F := Fix(\sigma) := \{v \in L \mid \sigma v = v\}$ is as follows.

p	$\dim \mathbb{F}$	$\det(F)$	F	example
47	2	47	unique	yes
23	4	23^2	unique	yes
13	0		unique	yes
11	8	11^4	unique	yes
7	0		unique	yes
7	6	7^5	$\sqrt{7}A_6$	no
2	24	2^{24}	$\sqrt{2}\Lambda_{24}$	yes
2	24	2^{24}	$\sqrt{2}O_{24}$	yes

Hermitian lattices

Definition

Let K be an imaginary quadratic number field, $\mathbb{Z}_K$ its ring of integers, (V, h) an n-dimensional Hermitian positiv definit K-vectorspace.

- A lattice $P \leq V$ is a finitely generated $\mathbb{Z}_K$-moduls that contains a basis of V.
- The minimum of P is $\min(P) := \min\{h(\ell, \ell) \mid 0 \neq \ell \in P\}$.
- The Hermitian Hermite function $\gamma_h(P) := \frac{\min(P)}{\det(P)^{1/n}}$ measures the density of P. .
- If $P = \langle b_1, \ldots, b_n \rangle_{\mathbb{Z}_K}$ is a free $\mathbb{Z}_K$-module then $\det(P) = \det(h(b_i, b_j))_{i,j}$.
- The Hermitian dual lattice is

$$P^* := \{v \in V \mid h(v, \ell) \in \mathbb{Z}_K \text{ for all } \ell \in P\}$$

We call P Hermitian unimodular, if $P = P^*$ (then $\det(P) = 1$).

Hermitian tensor products (Renaud Coulangeon)

Minimal vectors in tensor products

Let (L, h_L) and (M, h_M) be Hermitian $\mathbb{Z}_K$-lattices, $n = \dim_{\mathbb{Z}_K}(L) \leq m := \dim_{\mathbb{Z}_K}(M)$. Each $v \in L \otimes M$ is the sum of at most n pure tensors

$$v = \sum_{i=1}^{r} \ell_i \otimes m_i, \text{ such that } r =: rk(v) \text{ minimal.}$$

Put $A := (h_L(\ell_i, \ell_j))$ and $B := (h_M(m_i, m_j))$, then

$$h(v, v) = \mathrm{Trace}\, A\overline{B} \geq r \det(A)^{1/r} \det(B)^{1/r}.$$

$$\text{so } \min(L \otimes M) \geq \min\{r d_r(L)^{1/r} d_r(M)^{1/r} \mid r = 1, \ldots, n\}$$

where $d_r(L) = \min\{\det(T) \mid T \leq L, Rg(T) = r\}$.
In particular $d_r(L)^{1/r} \geq \min(L)/\gamma_r(\mathbb{Z}_K)$ where γ_r is the Hermitian Hermite constant.

Trace lattices

Trace lattices

- Any Hermitian $\mathbb{Z}_K$-lattice (P,h) is also a $\mathbb{Z}$-lattice (L,Q) of dimension $2n$,
- where $L = P$ and $Q(x) := h(x,x) \in \mathbb{R} \cap K = \mathbb{Q}$.
- Then the polar form of Q is $(x,y) = \mathrm{Trace}_{K/\mathbb{Q}}(h(x,y))$ and (L,Q) is called the trace lattice of (P,h).
- $\min(L) = 2\min(P)$, $L^{\#} = \mathbb{Z}_K^{\#} P^*$ and $\det(L) = d_K^n \det(P)^2$.
- $\mathbb{Z}_K^{\#} = \{x \in K \mid \mathrm{Trace}_{K/\mathbb{Q}}(x\ell) \in \mathbb{Z}$ for all $\ell \in \mathbb{Z}_K\}$
- $d_K = \det(\mathbb{Z}_K, \mathrm{Trace}(x\overline{y}) = |\mathbb{Z}_K^{\#}/\mathbb{Z}_K|$

$K = \mathbb{Q}[\sqrt{-11}], \mathbb{Z}_K = \mathbb{Z}[\eta], \eta = (1+\sqrt{-11})/2$

Then $\eta^2 - \eta + 3 = 0$, $\mathbb{Z}_[\eta]$ has a Euclidean algorithm, for any $x \in K$ there is some $a \in \mathbb{Z}[\eta]$ such that $N(x-a) \leq \frac{9}{11}$.

The densest 2-dimensional lattice

Let L_K have Gram matrix $\begin{pmatrix} 1 & 3/\sqrt{-11} \\ -3/\sqrt{-11} & 1 \end{pmatrix}$ Then L_K is the densest 2-dimensional $\mathbb{Z}[\eta]$-lattice, the trace lattice is $\mathbb{D}_4$.
Let T be the 2-dimensional unimodular Hermitian $\mathbb{Z}_K$-lattice with Gram matrix $\begin{pmatrix} 2 & \eta \\ \overline{\eta} & 2 \end{pmatrix}$. Then T is Hermitian unimodular, $\min(T) = 2$, $\mathrm{Aut}(T) = \pm S_3$.

The lattice P_{48n} as Hermitian tensor product

Let (P,h) be some 12-dimensional $\mathbb{Z}_K$-lattice such that the trace lattice $\mathrm{Trace}(P)$ is isometric to the Leech lattice. Then the Hermitian tensor product $R := P \otimes_{\mathbb{Z}_K} T$ has Hermitian minimum either 2 or 3. The minimum of R is 3, if and only if (P,h) does not represent one of the lattices L_K or T.

A canonical construction for P_{48n}

Some $\mathbb{Z}[\eta]$-structure of Leech.

Let P be the $\mathbb{Z}[\eta]$-lattice with $\mathrm{Aut}(P) \cong \mathrm{SL}_2(13).2$ such that $\mathrm{Trace}(P) \cong \Lambda_{24}$. Then $\mathrm{Trace}(P \otimes T) \cong P_{48n}$.

Canonical construction for P_{48n}

As a trace lattice of a quaternion tensor product:
$\mathrm{SL}_2(13) \leq \mathrm{GL}_3(\mathcal{Q}_{\sqrt{13},\infty,\infty})$ and $\mathrm{SL}_2(5) \leq \mathrm{GL}_1(\mathcal{Q}_{\sqrt{5},\infty,\infty})$ act on quaternionic lattices L_{13} resp. L_5. Then

$$P_{48n} = \mathrm{Trace}(L_{13} \otimes_{\mathcal{Q}_{\infty,2}} L_5)$$

The discovery of the 72-dimensional extremal even unimodular lattice.

1967 Turyn: Constructed the Golay code $\mathcal{G}_{24}$ from the Hamming code

78,82,84 Tits; Lepowsky, Meurman; Quebbemann: Construction of the Leech lattice Λ_{24} from $\mathbb{E}_8$

1996 Gross, Elkies: Λ_{24} from Hermitian structure of $\mathbb{E}_8$

1996 N.: Tried similar construction of extremal 72-dimensional lattices (Bordeaux).

1998 Bachoc, N.: Two extremal 80-dimensional lattices using Quebbemann's generalization and the Hermitian structure of $\mathbb{E}_8$

2010 Griess: Reminds Lepowsky, Meurman construction of Leech. proposes to construct 72-dimensional lattices from Λ_{24}

2010 N.: Used one of the nine $\mathbb{Z}[\alpha = \frac{1+\sqrt{-7}}{2}]$ structures of Λ_{24} to find extremal 72-dimensional lattice $\Gamma_{72} = \mathcal{L}(\alpha\Lambda_{24}, \overline{\alpha}\Lambda_{24})$

2011 Parker, N.: Check all other polarisations of Λ_{24} to show that Γ_{72} is the unique extremal lattice obtained from Λ_{24} by Turyn's construction.
Chance: $1 : 10^{16}$ to find extremely good polarisation.

$K = \mathbb{Q}[\sqrt{-7}]$, $\mathbb{Z}_K = \mathbb{Z}[\alpha]$, $\alpha = (1+\sqrt{-7})/2$

Then $\alpha^2 - \alpha + 2 = 0$, $\beta = \overline{\alpha} = 1 - \alpha$, $\alpha\beta = 2$ and $\mathbb{Z}[\alpha]$ has a Euclidean algorithm, for any $x \in K$ there is some $a \in \mathbb{Z}[\alpha]$ such that $N(x-a) \leq \frac{4}{7}$.

The densest 2-dimensional lattice

Denote by P_a the $\mathbb{Z}[\alpha]$-lattice with Gram matrix $\begin{pmatrix} 1 & 2/\sqrt{-7} \\ -2/\sqrt{-7} & 1 \end{pmatrix}$. Then $\min(P_a) = 1$ and $\det(P_a) = 3/7$.

The Barnes-lattice

$P_b = \langle(\beta,\beta,0),(0,\beta,\beta),(\alpha,\alpha,\alpha)\rangle \leq \mathbb{Z}[\alpha]^3$ with Hermitian form $h : P_b \times P_b \to \mathbb{Z}[\alpha], h((a_1,a_2,a_3),(b_1,b_2,b_3)) = \frac{1}{2}\sum_{i=1}^{3} a_i\overline{b_i}$ is Hermitian unimodular, $\mathrm{Aut}_{\mathbb{Z}[\alpha]}(P_b) \cong \pm\mathrm{PSL}_2(7)$,

$\gamma_h(P_b) = \min(P_b) = 2$. Gram matrix $\begin{pmatrix} 2 & 1 & \beta \\ 1 & 2 & \beta \\ \alpha & \alpha & 3 \end{pmatrix}$

Densest $\mathbb{Z}[\alpha]$-lattices

$\mathbb{E}_8$ as trace lattice

$$P_c := \mathbb{Z}[\alpha]^4 + \langle \frac{1}{\sqrt{-7}}(1,1,1,3), \frac{1}{\sqrt{-7}}(0,1,3,-2)\rangle \leq K^4$$

Then $\min(P_c) = 1$, $\det(P_c) = (1/7)^2$, $P_c^* = \sqrt{-7}P_c$, $\mathrm{Trace}(P_c) = \mathbb{E}_8$

Theorem

P_a, P_b and P_c are the densest $\mathbb{Z}[\alpha]$-lattices in dimension 2,3,4.
$\gamma_2(\mathbb{Z}[\alpha]) = \sqrt{7/3}$, $\gamma_3(\mathbb{Z}[\alpha]) = 2$, $\gamma_4(\mathbb{Z}[\alpha]) = \sqrt{7}$.

Proof: $\mathrm{Trace}(P_c)$ is densest $\mathbb{Z}$-lattice.
For P_a enough to apply reduction theory.
For P_b explicit application of Voronoi's algorithm.

Extremal lattices as Hermitian tensor products

The Leech lattice

Let $P := P_b \otimes_{\mathbb{Z}[\alpha]} P_c$. Then $\min(P) = 2$ and $\mathrm{Trace}(P)$ is an extremal even unimodular lattice of dimension 24, so $\mathrm{Trace}(P) \cong \Lambda_{24}$.

Proof: $\mathrm{Trace}(P)$ is even unimodular, since P_b Hermitian unimodular and $\mathbb{E}_8 = \mathrm{Trace}(P_c)$ even unimodular. Show that $\min(P) \geq 2$:

r	1	2	3
$d_r(P_b)$	2	2	1
$d_r(P_c)$	1	$3/7$	$\geq 1/8$
$rd_r(P_b)^{1/r}d_r(P_c)^{1/r}$	2	$1,85$	$1,5$

$$\min(L \otimes M) \geq \min\{rd_r(L)^{1/r}d_r(M)^{1/r} \mid r = 1, \ldots, n\}$$

Dimension 72.

Theorem (R. Coulangeon, N)

Let P be an Hermitian $\mathbb{Z}[\alpha]$-lattice with $\min(P) = 2$. Then $\min(P \otimes P_b) \geq 3$ and $\min(P \otimes P_b) > 3$ if and only if P has no sublattice isometric to P_b.

Proof.

r	1	2	3
$d_r(P_b)^{1/r}$	2	$\sqrt{2}$	1
$d_r(P)^{1/r}$	2	$\geq 2\sqrt{3/7}$	≥ 1
$rd_r(P_b)^{1/r}d_r(P)^{1/r}$	4	$\geq 3,7$	≥ 3

And $d_3(P) > 1$ if P_b is not a sublattice of P.

Corollary

Let P be some 12-dimensional $\mathbb{Z}[\alpha]$-lattice such that $\mathrm{Trace}(P) \cong \Lambda_{24}$. Then $\min(P \otimes P_b) \geq 3$ and $\min(P \otimes P_b) = 4$ if P does not contain P_b.

Hermitian structures of the Leech lattice

Theorem (M. Hentschel, 2009)

There are exactly nine $\mathbb{Z}[\alpha]$-structures of the Leech lattice.

	group	$\#P_b \leq P_i\}$
1	$\mathrm{SL}_2(25)$	0
2	$2.A_6 \times D_8$	$2 \cdot 20,160$
3	$\mathrm{SL}_2(13).2$	$2 \cdot 52,416$
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
6	$2^9 3^3$	$2 \cdot 177,408$
7	$\pm\mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2 \cdot 306,432$
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2 \cdot 504,000$
9	$2.J_2.2$	$2 \cdot 1,209,600$

Theorem (R. Coulangeon, N)

$d_3(P_i) = 1$ for $i = 2, \ldots, 9$ and $d_3(P_1) > 1$, so $\min(P_1 \otimes P_b) = 4$.

Stehlé, Watkins proof of extremality

Theorem (Stehlé, Watkins (2010))

Let L be an even unimodular lattice of dimension 72 with $\min(L) \geq 6$. Then L is extremal, if and only if it contains at least $6,218,175,600$ vectors v with $Q(v) = 4$.

Proof: L is an even unimodular lattice of minimum ≥ 6, so its theta series is

$$\theta_L = 1 + a_3 q^3 + a_4 q^4 + \ldots = f^{(9)} + a_3 \Delta^3.$$

$$\begin{array}{ccccccc} f^{(9)} & = & 1 & + & 6,218,175,600 q^4 & + \ldots \\ \Delta^3 & = & & q^3 & -72 q^4 & + \ldots \end{array}$$

So $a_4 = 6,218,175,600 - 72 a_3 \geq 6,218,175,600$ if and only if $a_3 = 0$.

Remark

A similar proof works in all jump dimensions $24k$ (extremal minimum = $2k+2$) for lattices of minimum $\geq 2k$.
For dimensions $24k+8$ and lattices of minimum $\geq 2k$ one needs to count vectors v with $Q(v) = k+2$.

The extremal 72-dimensional lattice Γ_{72}

Main result

- Γ_{72} is an extremal even unimodular lattice of dimension 72.
- Γ_{72} has a canonical construction as trace lattice of Hermitian tensor product.
- $\mathrm{Aut}(\Gamma_{72}) = \mathcal{U} := (\mathrm{PSL}_2(7) \times \mathrm{SL}_2(25)) : 2$.
- $\mathcal{U}$ is an absolutely irreducible subgroup of $\mathrm{GL}_{72}(\mathbb{Q})$.
- All $\mathcal{U}$-invariant lattices are similar to Γ_{72}.
- Γ_{72} is an ideal lattice in the 91st cyclotomic number field.
- Γ_{72} realises the densest known sphere packing
- and maximal known kissing number in dimension 72.
- Γ_{72} is a $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$-lattice. This gives (n^2+5n+5)-modular lattices of minimum $8+4n$ ($n \in \mathbb{N}_0$).