p-adic integral group rings.

Gabriele Nebe

Lehrstuhl D für Mathematik

St. Johns, June 5, 2009

Setup

- G finite group.
- $R \supseteq \mathbb{Z}_p$ (complete) discrete valuation ring.
- $K = \text{frac}(R)$ field of fractions.
- π uniformizer of R, $F = R/\pi R$.

Setup

- G finite group.
- $R \supseteq \mathbb{Z}_p$ (complete) discrete valuation ring.
- $K = \text{frac}(R)$ field of fractions.
- π uniformizer of R, $F = R/\pi R$.

We want to understand the representation theory of RG.

Setup

- G finite group.
- $R \supseteq \mathbb{Z}_p$ (complete) discrete valuation ring.
- $K = \text{frac}(R)$ field of fractions.
- π uniformizer of R, $F = R/\pi R$.

We want to understand the representation theory of RG.
First step: Understand

$$\mathfrak{L}_V(G) := \{L \subset V \mid L \text{ is an } RG\text{-lattice in } V\}$$

for all simple KG-modules V.

Setup

- G finite group.
- $R \supseteq \mathbb{Z}_p$ (complete) discrete valuation ring.
- $K = \text{frac}(R)$ field of fractions.
- π uniformizer of R, $F = R/\pi R$.

We want to understand the representation theory of RG.
First step: Understand

$$\mathfrak{L}_V(G) := \{L \subset V \mid L \text{ is an } RG\text{-lattice in } V\}$$

for all simple KG-modules V.
For the talk assume that R splits RG.

Intersection of maximal orders.

- V a simple KG-module.
- $\epsilon_V \in KG$ the associated central primitive idempotent.

Intersection of maximal orders.

- V a simple KG-module.
- $\epsilon_V \in KG$ the associated central primitive idempotent.
- S a simple FG-module.
- The decomposition number $d_{V,S}$ is the multiplicity of the composition factor S in $L/\pi L$ for any $L \in \mathcal{L}_V(G)$.

Intersection of maximal orders.

- V a simple KG-module.
- $\epsilon_V \in KG$ the associated central primitive idempotent.
- S a simple FG-module.
- The decomposition number $d_{V,S}$ is the multiplicity of the composition factor S in $L/\pi L$ for any $L \in \mathfrak{L}_V(G)$.

Fact.

$$\epsilon_V RG \subseteq \bigcap_{L \in \mathfrak{L}_V(G)} \mathrm{End}_R(L)$$

with equality if $d_{V,S} \in \{0,1\}$ for all S.

Example S_3.

$G = S_3$ the symmetric group of degree 3, $R = \mathbb{Z}_3$, $\dim(V) = 2$.

$$\mathcal{L}_V(G) = \{3^n L_1, 3^n L_2 \mid n \in \mathbb{Z}\}$$

where

$$L_1 := \langle b_1, b_2 \rangle \supset L_2 = \langle 3b_1, b_2 \rangle \supset 3L_1$$

$L_1/L_2 \cong \mathbb{F}_3$ trivial module, $L_2/3L_1 \cong \mathbb{F}_3$ sign module.

Example S_3.

$G = S_3$ the symmetric group of degree 3, $R = \mathbb{Z}_3$, $\dim(V) = 2$.

$$\mathfrak{L}_V(G) = \{3^n L_1, 3^n L_2 \mid n \in \mathbb{Z}\}$$

where

$$L_1 := \langle b_1, b_2 \rangle \supset L_2 = \langle 3b_1, b_2 \rangle \supset 3L_1$$

$L_1/L_2 \cong \mathbb{F}_3$ trivial module, $L_2/3L_1 \cong \mathbb{F}_3$ sign module.

$$\mathrm{End}_R(L_1) = \begin{pmatrix} R & R \\ R & R \end{pmatrix}, \ \mathrm{End}_R(L_2) = \begin{pmatrix} R & 3R \\ 3^{-1}R & R \end{pmatrix}$$

and

$$\epsilon_V RG = \mathrm{End}_R(L_1) \cap \mathrm{End}_R(L_2) = \begin{pmatrix} R & 3R \\ R & R \end{pmatrix}.$$

Example D_8.

$G = D_8$ the dihedral group of order 8, $R = \mathbb{Z}_2$, $\dim(V) = 2$.

$$\mathcal{L}_V(G) = \{2^n L_1, 2^n L_2 \mid n \in \mathbb{Z}\}$$

where

$$L_1 := \langle b_1, b_2 \rangle \supset L_2 = \langle 2b_1, b_2 \rangle \supset 2L_1$$

and $L_1/L_2 \cong L_2/2L_1 \cong \mathbb{F}_2$.

Example D_8.

$G = D_8$ the dihedral group of order 8, $R = \mathbb{Z}_2$, $\dim(V) = 2$.

$$\mathcal{L}_V(G) = \{2^n L_1, 2^n L_2 \mid n \in \mathbb{Z}\}$$

where

$$L_1 := \langle b_1, b_2 \rangle \supset L_2 = \langle 2b_1, b_2 \rangle \supset 2L_1$$

and $L_1/L_2 \cong L_2/2L_1 \cong \mathbb{F}_2$.

$$\mathrm{End}_R(L_1) \cap \mathrm{End}_R(L_2) = \begin{pmatrix} R & 2R \\ R & R \end{pmatrix}$$

and

$$\epsilon_V RG = \{\begin{pmatrix} a & 2b \\ c & d \end{pmatrix} \mid a, b, c, d \in R, a \equiv d \pmod 2\}$$

Exponent matrices.

Definition.

For $M \in \mathbb{Z}^{k \times k}$, $(n_1, \ldots, n_k) \in \mathbb{N}^k$, $n := \sum_{i=1}^{k} n_i$ let

$$\Lambda(n_1, \ldots, n_k; M) := \{(X_{ij}) \in R^{n \times n} \mid X_{ij} \in \pi^{m_{ij}} R^{n_i \times n_j}\}$$

be the graduated order with exponent matrix M.

Exponent matrices.

Definition.

For $M \in \mathbb{Z}^{k\times k}$, $(n_1, \dots, n_k) \in \mathbb{N}^k$, $n := \sum_{i=1}^{k} n_i$ let

$$\Lambda(n_1, \dots, n_k; M) := \{(X_{ij}) \in R^{n\times n} \mid X_{ij} \in \pi^{m_{ij}} R^{n_i \times n_j}\}$$

be the graduated order with exponent matrix M.

$G = S_3, \dim(V) = 2$ then $\epsilon_V \mathbb{Z}_3 G = \Lambda(1, 1; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

Exponent matrices.

Definition.

For $M \in \mathbb{Z}^{k\times k}$, $(n_1,\dots,n_k) \in \mathbb{N}^k$, $n := \sum_{i=1}^k n_i$ let

$$\Lambda(n_1,\dots,n_k;M) := \{(X_{ij}) \in R^{n\times n} \mid X_{ij} \in \pi^{m_{ij}} R^{n_i\times n_j}\}$$

be the graduated order with exponent matrix M.

$G = S_3$, $\dim(V) = 2$ then $\epsilon_V \mathbb{Z}_3 G = \Lambda(1,1;\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$.

Theorem.

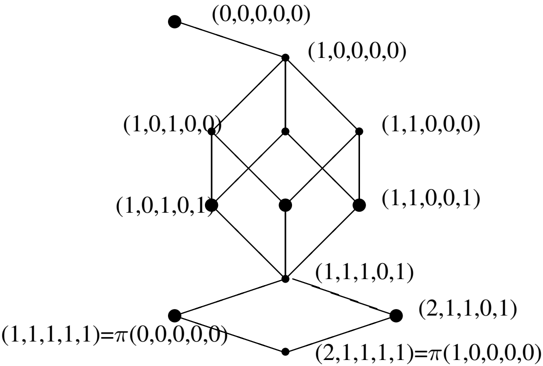
$\epsilon_V RG$ is graduated $\Leftrightarrow$ $d_{V,S} \in \{0,1\}$ for all S.
Then $(\mathcal{L}_V(G), \subset)$ is a distributive lattice.

From exponent matrices to lattices.

$$\Lambda = \Lambda(M) \text{ with } M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

From exponent matrices to lattices.

$$\Lambda = \Lambda(M) \text{ with } M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$



Obvious properties of exponent matrices.

- Since $\Lambda := \Lambda(n_1, \ldots, n_k; M)$ is an order we have

$$m_{ii} = 0 \text{ and } m_{ij} + m_{j\ell} \geq m_{i\ell} \text{ for all } i, j, \ell$$

Obvious properties of exponent matrices.

- Since $\Lambda := \Lambda(n_1, \ldots, n_k; M)$ is an order we have

$$m_{ii} = 0 \text{ and } m_{ij} + m_{j\ell} \geq m_{i\ell} \text{ for all } i, j, \ell$$

- If
$$e_1 = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n_1}, 0 \ldots, 0), \ldots, e_k = \mathrm{diag}(0, \ldots, 0, \underbrace{1, \ldots, 1}_{n_k})$$
are lifts of the central primitive idempotents of $\Lambda / J(\Lambda)$, then

$$J(\Lambda(n_1, \ldots, n_k; M)) = \Lambda(n_1, \ldots, n_k; M + I_k)$$

and hence $m_{ij} + m_{ji} > 0$ for $i \neq j$.

Obvious properties of exponent matrices.

- Since $\Lambda := \Lambda(n_1, \dots, n_k; M)$ is an order we have

$$m_{ii} = 0 \text{ and } m_{ij} + m_{j\ell} \geq m_{i\ell} \text{ for all } i, j, \ell$$

- If
$$e_1 = \mathrm{diag}(\underbrace{1, \dots, 1}_{n_1}, 0 \dots, 0), \dots, e_k = \mathrm{diag}(0, \dots, 0, \underbrace{1, \dots, 1}_{n_k})$$
are lifts of the central primitive idempotents of $\Lambda/J(\Lambda)$, then

$$J(\Lambda(n_1, \dots, n_k; M)) = \Lambda(n_1, \dots, n_k; M + I_k)$$

and hence $m_{ij} + m_{ji} > 0$ for $i \neq j$.

- W.l.o.g. write matrices with respect to a suitable basis of $L := \Lambda\, \mathrm{diag}(1, 0 \dots, 0)$. Then

$$m_{i1} = 0 \text{ for all } i \text{ and } m_{ij} \geq 0 \text{ for all } i, j.$$

Duality.

$< x, y >:= \frac{1}{|G|} \mathrm{trace}_{reg}(xy)$ is an associative non degenerate symmetric bilinear form on KG so that RG is self-dual

$$RG = RG^{\#} = \{x \in KG \mid < x, RG > \subset R\}.$$

$$\Gamma^{\#} \subset RG \subset \Gamma := \bigoplus_{V} \epsilon_V RG.$$

Duality.

$< x, y >:= \frac{1}{|G|} \text{trace}_{reg}(xy)$ is an associative non degenerate symmetric bilinear form on KG so that RG is self-dual

$$RG = RG^{\#} = \{x \in KG \mid < x, RG > \subset R\}.$$

$$\Gamma^{\#} \subset RG \subset \Gamma := \bigoplus_V \epsilon_V RG.$$

Remark.

If $\epsilon_V RG = \Lambda(n_1, \dots, n_k; M) =: \Lambda$, then

$$\Lambda^{\#} = \Lambda(n_1, \dots, n_k; aJ - M^{\text{tr}}) \subset \Lambda$$

with $a := v_\pi(|G|) - v_\pi(\dim(V))$ and $J = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$.

Duality.

$< x, y >:= \frac{1}{|G|} \mathrm{trace}_{reg}(xy)$ is an associative non degenerate symmetric bilinear form on KG so that RG is self-dual

$$RG = RG^{\#} = \{x \in KG \mid < x, RG > \subset R\}.$$

$$\Gamma^{\#} \subset RG \subset \Gamma := \bigoplus_V \epsilon_V RG.$$

Remark.

If $\epsilon_V RG = \Lambda(n_1, \ldots, n_k; M) =: \Lambda$, then

$$\Lambda^{\#} = \Lambda(n_1, \ldots, n_k; aJ - M^{\mathrm{tr}}) \subset \Lambda$$

with $a := v_\pi(|G|) - v_\pi(\dim(V))$ and $J = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \ldots & 1 \end{pmatrix}$.

So $a - m_{ij} \geq m_{ji}$ or equivalently

$$m_{ij} + m_{ji} \leq a.$$

Involution.

RG is an R-order with a canonical R-linear involution

$$^\circ : RG \to RG, g \mapsto g^{-1}.$$

Involution.

RG is an R-order with a canonical R-linear involution

$$^\circ : RG \to RG, g \mapsto g^{-1}.$$

If $\epsilon_V = \epsilon_V^\circ$ then choose $0 \neq \phi = \phi^{tr} \in K^{n \times n}$ such that

$$g \phi g^{\mathrm{tr}} = \phi, \text{ so } g^{-1} = \phi g^{\mathrm{tr}} \phi^{-1} \text{ for all } g \in G.$$

Then

$$\epsilon_V RG = (\epsilon_V RG)^\circ = \{\phi X^{\mathrm{tr}} \phi^{-1} \mid X \in \epsilon_V RG\} \subset R^{n \times n}.$$

Involution.

RG is an R-order with a canonical R-linear involution

$$\circ : RG \to RG, g \mapsto g^{-1}.$$

If $\epsilon_V = \epsilon_V^\circ$ then choose $0 \neq \phi = \phi^{tr} \in K^{n\times n}$ such that

$$g\phi g^{\mathrm{tr}} = \phi, \text{ so } g^{-1} = \phi g^{\mathrm{tr}} \phi^{-1} \text{ for all } g \in G.$$

Then

$$\epsilon_V RG = (\epsilon_V RG)^\circ = \{\phi X^{\mathrm{tr}} \phi^{-1} \mid X \in \epsilon_V RG\} \subset R^{n\times n}.$$

Remark.

If $\epsilon_V RG = \Lambda(n_1, \ldots, n_k; M)$ and all simple modules S are selfdual, then ϕ can be chosen as

$$\phi = \mathrm{diag}(f_1, \pi^{a_2} f_2, \ldots, \pi^{a_k} f_k) \text{ with } a_i \in \mathbb{N}, f_i \in \mathrm{GL}_{n_i}(R)$$

and $m_{ij} - m_{ji} = a_j - a_i$.

Summary: Properties of exponent matrices.

Theorem.

Let V be a simple KG-module such that $\epsilon_V RG = \Lambda(n_1, \ldots, n_k; M)$ is a graduated order.
Let $a := v_\pi(|G|) - v_\pi(\dim(V))$. Then w.r.t. a suitable basis of L as above for all $i, j, \ell \in \{1, \ldots, k\}$

- **wlog** $m_{i1} = 0$, $m_{ij} \geq 0$.
- **order** $m_{ii} = 0$, $m_{ij} + m_{j\ell} \geq m_{i\ell}$
- **rad** $m_{ij} + m_{ji} > 0$ if $i \neq j$.
- **dual** $m_{ij} + m_{ji} \leq a$
- **invo** $m_{ij} - m_{ji} = m_{1j} - m_{1i} = a_j - a_i$ if $\epsilon_V^\circ = \epsilon_V$ and all simple FG-modules S with $d_{V,S} > 0$ are selfdual.

Symmetric groups.

- Irreducible representations in characteristic 0 are parametrized by the partitions λ of n.
- $S^\lambda \leq 1_{S_{\lambda_1}\times\ldots\times S_{\lambda_s}}^{S_n}$ Specht lattice.

Symmetric groups.

- Irreducible representations in characteristic 0 are parametrized by the partitions λ of n.
- $S^\lambda \leq 1_{S_{\lambda_1}\times \ldots \times S_{\lambda_s}}^{S_n}$ Specht lattice.
- Irreducible representations D^λ in characteristic p are parametrized by the p-regular partitions λ of n.
- $D^\lambda = FS^\lambda/(FS^\lambda)^\perp$

Symmetric groups.

- Irreducible representations in characteristic 0 are parametrized by the partitions λ of n.
- $S^\lambda \leq 1_{S_{\lambda_1} \times \ldots \times S_{\lambda_s}}^{S_n}$ Specht lattice.
- Irreducible representations D^λ in characteristic p are parametrized by the p-regular partitions λ of n.
- $D^\lambda = FS^\lambda/(FS^\lambda)^\perp$
- Jantzen-Schaper-formula: multiplicity of D^μ as composition factor in $(S^\lambda)^\#/S^\lambda$ if decomposition matrix is known.
- This formula yields the exponents $a_2, \ldots, a_k$ of the invariant form ϕ and hence the first row of the exponent matrices.

The decomposition matrix of the principal block of $\mathbb{Z}_3 S_6$.

	(6)	$(5,1)$	$(4,1^2)$	(3^2)	$(3,2,1)$
(6)	1	.	.	.	.
$(5,1)$	1	1	.	.	.
$(4,1^2)$	.	1	1	.	.
(3^2)	.	1	.	1	.
$(3,2,1)$	1	1	1	1	1
$(3,1^3)$	.	.	1	.	1
(2^3)	1	.	.	.	1
$(2,1^4)$	.	.	.	1	1
(1^6)	.	.	.	1	.

Example $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6$.

- Jantzen-Schaper yields:
  $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6 = \Lambda((3,2,1),(3^2),(4,1^2),(5,1),(6);M)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & a & b & c \\ 0 & a' & 0 & d & e \\ 0 & b' & d' & 0 & f \\ 0 & c' & e' & f' & 0 \end{pmatrix}$$

Example $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6$.

- Jantzen-Schaper yields:
  $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6 = \Lambda((3,2,1),(3^2),(4,1^2),(5,1),(6);M)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & a & b & c \\ 0 & a' & 0 & d & e \\ 0 & b' & d' & 0 & f \\ 0 & c' & e' & f' & 0 \end{pmatrix}$$

- For all x we have

$$0 < x + x' \leq 2, \text{ so } x + x' \in \{1, 2\}$$

Example $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6$.

- Jantzen-Schaper yields:
  $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6 = \Lambda((3,2,1),(3^2),(4,1^2),(5,1),(6);M)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & a & b & c \\ 0 & a' & 0 & d & e \\ 0 & b' & d' & 0 & f \\ 0 & c' & e' & f' & 0 \end{pmatrix}$$

- For all x we have

$$0 < x + x' \leq 2, \text{ so } x + x' \in \{1, 2\}$$

- Invariance under the involution $\circ$ yields

  $a - a' = c - c' = e - e' = 0$ and $b - b' = d - d' = f' - f = 1$

  hence $a = a' = c = c' = e = e' = 1$, $b = d = f' = 1$, and $b' = d' = f = 0$.

Example $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6$.

- Jantzen-Schaper yields:
  $\epsilon_{(3,2,1)}\mathbb{Z}_3 S_6 = \Lambda((3,2,1),(3^2),(4,1^2),(5,1),(6);M)$ where

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

- For all x we have

$$0 < x + x' \leq 2, \text{ so } x + x' \in \{1, 2\}$$

- Invariance under the involution $\circ$ yields

$$a - a' = c - c' = e - e' = 0 \text{ and } b - b' = d - d' = f' - f = 1$$

hence $a = a' = c = c' = e = e' = 1$, $b = d = f' = 1$, and $b' = d' = f = 0$.

Symmetric groups of degree 2p, exponent matrices.

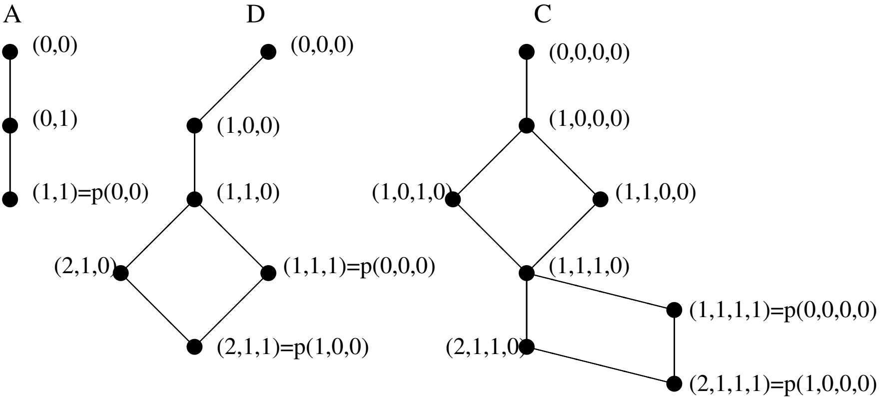
Theorem.

Let $G = S_{2p}$, $R = \mathbb{Z}_p$, V a simple KG-module. Then $\epsilon_V RG$ is graduated with exponent matrix X, A, B, C, or D:

$$X := \begin{pmatrix} 0 \end{pmatrix}, \qquad A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad D := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B := \begin{pmatrix} 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \qquad C := \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Symmetric groups of degree 2p, lattices.



Second step: Description of RG.

$$RG/J(RG) = \bigoplus_{i=1}^{s} \underbrace{\overline{e}'_i RG/J(RG)}_{M_{n_i}(F)}$$

$P_1, \dots, P_s$ the projective indecomposable RG-lattices.
Morita equivalence:

$$RG \sim \mathrm{End}_{RG}(\bigoplus_{i=1}^{s} P_i) =: \Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) == \bigoplus_{i,j=1}^{s} \underbrace{e_j \Delta e_i}_{\Delta_{ji}}$$

where $\Delta/J(\Delta) = \bigoplus_{i=1}^{s} \underbrace{\overline{e}_i \Delta/J(\Delta)}_{F}$ and the e_i are orthognal primitive idempotents in Δ that lift the $\overline{e}_i$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.
- $\Delta_{ij}\Delta_{ji} \subseteq J(\Delta_{ii})$ if $i \neq j$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.
- $\Delta_{ij}\Delta_{ji} \subseteq J(\Delta_{ii})$ if $i \neq j$.
- $\Delta_{ij}^{\#} = \Delta_{ji}$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.
- $\Delta_{ij}\Delta_{ji} \subseteq J(\Delta_{ii})$ if $i \neq j$.
- $\Delta_{ij}^{\#} = \Delta_{ji}$.
- $(e_i \Delta e_j)^{\circ} = e_j^{\circ} \Delta e_i^{\circ}$

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.
- $\Delta_{ij}\Delta_{ji} \subseteq J(\Delta_{ii})$ if $i \neq j$.
- $\Delta_{ij}^{\#} = \Delta_{ji}$.
- $(e_i \Delta e_j)^{\circ} = e_j^{\circ} \Delta e_i^{\circ}$
- $\Delta_{ij} \subset \bigoplus_{V} \epsilon_V \Delta_{ij}$.

The basic order Δ.

Remark.

Let $\Delta = \bigoplus_{i,j=1}^{s} \mathrm{Hom}_{RG}(P_i, P_j) = \bigoplus_{i,j=1}^{s} \Delta_{ji}$. Then

- Δ_{ii} are self-dual local R-orders.
- Δ_{ij} is a $\Delta_{ii} - \Delta_{jj}$-bimodule.
- $\Delta_{ij}\Delta_{jk} \subseteq \Delta_{ik}$.
- $\Delta_{ij}\Delta_{ji} \subseteq J(\Delta_{ii})$ if $i \neq j$.
- $\Delta_{ij}^{\#} = \Delta_{ji}$.
- $(e_i \Delta e_j)^{\circ} = e_j^{\circ} \Delta e_i^{\circ}$
- $\Delta_{ij} \subset \bigoplus_V \epsilon_V \Delta_{ij}$.
- If $d_{V,S_i} \in \{0,1\}$ for all V, then Δ_{ii} is commutative and

$$\bigoplus_{V, d_{V,S_i}=1} \epsilon_V \Delta_{ii} \cong \bigoplus_{V, d_{V,S_i}=1} R$$

is the unique maximal order in $K\Delta_{ii}$.

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)
- $G = S_{2p}$, $R = \mathbb{Z}_p$.

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)
- $G = S_{2p}$, $R = \mathbb{Z}_p$.
- $G = \mathrm{SL}_2(p^2)$.

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)
- $G = S_{2p}$, $R = \mathbb{Z}_p$.
- $G = \mathrm{SL}_2(p^2)$.
- $G = \mathrm{SL}_2(2^f)$, $f \leq 6$.

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)
- $G = S_{2p}$, $R = \mathbb{Z}_p$.
- $G = \mathrm{SL}_2(p^2)$.
- $G = \mathrm{SL}_2(2^f)$, $f \leq 6$.
- $G = S_n$, $n \leq 9$.
- some other examples.

Results.

Theorem.

We know RG for

- blocks with cyclic defect groups (Plesken, 1983)
- $G = S_{2p}$, $R = \mathbb{Z}_p$.
- $G = \mathrm{SL}_2(p^2)$.
- $G = \mathrm{SL}_2(2^f)$, $f \leq 6$.
- $G = S_n$, $n \leq 9$.
- some other examples.

Theorem.

We know $\Gamma = \bigoplus_V \epsilon_V RG$ for $\mathrm{SL}_2(p^f)$.

The decomposition matrix of the principal block of $\mathbb{Z}_3 S_6$.

$\chi(1)$		(6)	$(5,1)$	$(4,1^2)$	(3^2)	$(3,2,1)$
1	(6)	1	.	.	.	.
5	$(5,1)$	1	1	.	.	.
10	$(4,1^2)$	.	1	1	.	.
5	(3^2)	.	1	.	1	.
16	$(3,2,1)$	1	1	1	1	1
10	$(3,1^3)$	.	.	1	.	1
5	(2^3)	1	.	.	.	1
5	$(2,1^4)$	.	.	.	1	1
1	(1^6)	.	.	.	1	.

Example $\mathrm{End}(P_{(6)})$.

$\chi(1)$	1	5	16	5
$\chi(1) \pmod 3$	1	-1	1	-1
	1	1	1	1
	0	3	0	-3
	0	0	3	3
	0	0	0	9