On automorphism groups of Type II codes.

Gabriele Nebe

Lehrstuhl D für Mathematik

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RWTH AACHEN UNIVERSITY

Binary Codes

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Clear: $C \subset C^{\perp} \Rightarrow \mathrm{wt}(c) \in 2\mathbb{Z}$ for all $c \in C$.

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Facts:

- $C = C^\perp \leq \mathbb{F}_2^n \Rightarrow n = 2\dim(C)$ is even.
- $C = C^\perp \leq \mathbb{F}_2^n \Rightarrow \mathbf{1} = (1, \ldots, 1) \in C$.
- $C \leq \mathbb{F}_2^n$ Type II $\Rightarrow n \in 8\mathbb{Z}$.

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Question:

- Is there $C = C^\perp \in \mathcal{C}(G)$?
- Is there a Type II code $C \in \mathcal{C}(G)$?

Group ring codes.

Thompson, Sloane, Willems and others treat group ring codes, so $G \leq S_G$ via its regular representation.

Then $\mathcal{C}(G) =: \mathcal{C}_{reg}(G)$ are the left ideals of $\mathbb{F}_2 G$.

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Theorem 2 (Sloane, Thompson)
$\exists C = C^\perp \in \mathcal{C}_{reg}(G)$ of Type II
$\Leftrightarrow$
$|G| \in 8\mathbb{Z}$ and the Sylow 2-subgroups of G are not cyclic.

Proof of Theorem 1.

$\Rightarrow$: $C = C^{\perp} \leq \mathbb{F}_2 G \cong \mathbb{F}_2^{|G|}$, then $\dim(C) = \frac{|G|}{2}$, so $|G|$ is even.

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More precisely, write $G = \dot{\bigcup} \{h_i, h_i g\}$, then with respect to $h_1, h_1 g, h_2, h_2 g, \ldots$ C is the rowspace of

$$\begin{array}{cccc} 1\,1 & 0\,0 & \ldots & 0\,0 \\ 0\,0 & 1\,1 & \ddots & 0\,0 \\ \ddots & \ddots & \ddots & \vdots \\ 0\,0 & 0\,0 & \ldots & 1\,1 \end{array}$$

General permutation representations

In joint work with Annika Günther we treat arbitrary permutation groups $G \leq S_n$.

Theorem A $\exists C = C^{\perp} \in \mathcal{C}(G) \Leftrightarrow$ condition (E) is satisfied.

(E) every simple $\mathbb{F}_2G$-module S with $S \cong S^* = \mathrm{Hom}(S, \mathbb{F}_2)$ occurs in $\mathbb{F}_2^n$ with even multiplicity.

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Remark. Condition (E) is fulfilled, if $|N_G(H_i)/H_i|$ is even where $H_i := \mathrm{Stab}_G(i)$ for $i \in \{1, \ldots, n\}$.

Clear. Theorem A implies Theorem 1.

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Theorem B If $C = C^\perp$ is of Type II, then $P(C) \leq \mathrm{Alt}_n$.

Theorem C $\exists C = C^\perp \in \mathcal{C}(G)$ of Type II $\Leftrightarrow$
(a) $n \in 8\mathbb{Z}$,
(b) condition (E) is satisfied, and
(c) $G \leq \mathrm{Alt}_n$.

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Proof:

- Condition (E) for group ring codes is equivalent to even group order.
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Proof of Theorem A

$\Rightarrow$: $\mathbb{F}_2^n/C^\perp \cong \mathrm{Hom}(C, \mathbb{F}_2)$, so if S is a composition factor of C, then S^* is a composition factor of $\mathbb{F}_2^n/C^\perp$.

$$\underbrace{\mathbb{F}_2^n \supseteq C^\perp}_{S^*} = \underbrace{C \supseteq \{0\}}_{S}$$

$\Leftarrow$: $C \subset C^\perp$ maximal self-orthogonal, then $C^\perp/C$ anisotropic and hence semi-simple,

$$C^\perp/C \cong \perp S_j \text{ with } S_j \cong S_j^* \forall j.$$

$S \perp S$ is hyperbolic since $\mathbb{F}_2 = \{x^2 \mid x \in \mathbb{F}_2\}$.

Theorem A $\exists C = C^\perp \in \mathcal{C}(G) \Leftrightarrow$ condition (E) is satisfied.

Orthogonal groups

Let K be any field, $V = K^{2m}$, $q : V \to K$ a non-degenerate quadratic form of Witt defect 0. This means that there is

$$U \leq V,\ \dim(U) = m,\ q(U) = \{0\}.$$

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Theorem. $\mathrm{Stab}_{O(V,q)}(U) \leq \ker(D)$

Proof of Theorem B

Let $n \in 8\mathbb{Z}$.

$$V := \mathbf{1}^\perp / \langle \mathbf{1} \rangle = \{x + \langle \mathbf{1} \rangle \mid x \in \mathbb{F}_2^n, \mathrm{wt}(x) \in 2\mathbb{Z}\}$$

$$q : V \to \mathbb{F}_2, q(x + \langle \mathbf{1} \rangle) := \frac{\mathrm{wt}(x)}{2} + 2\mathbb{Z}.$$

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- The maximal isotropic subspaces of (V, q) are precisely the images of the Type II codes in $\mathbb{F}_2^n$.
- S_n fixes $\mathbf{1}$ and preserves the weight hence embeds into $O(V, q)$.
- The restriction of the Dickson homomorphism $D : S_n \to \{1, -1\}$ is the sign.

Generalization of Theorem B

This shows more general:
Theorem B'. Let $C \leq \mathbb{F}_{2^d}^n$ be a self-dual generalized doubly-even code. Then $P(C) \leq \mathrm{Alt}_n$.

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Theorem B'. Let $C \leq \mathbb{F}_{2^d}^n$ be a self-dual generalized doubly-even code. Then $P(C) \leq \mathrm{Alt}_n$.

For odd characteristic, the weight preserving mappings that preserve orthogonality are all permutations and sign changes

$$\{\pm 1\}^n : S_n$$

and one obtains

Theorem B". Let $p > 2$ and $C = C^{\perp} \leq \mathbb{F}_{p^d}^n$. Then $\det(\mathrm{Aut}(C)) = \{1\}$.

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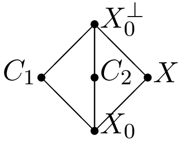

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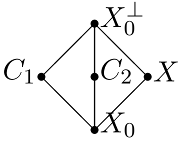

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- C_1 and C_2 are doubly-even.
- $\dim(C_1) - \dim(C_1 \cap C_2) = 1$ is odd.
- $G \leq P(X) \leq P(X_0)$ acts on $\{C_1, C_2\}$.
- $D(G) = \{1\}$ so $C_i g = C_i$ for all $g \in G$, $i = 1, 2$.
- $C_i \in \mathcal{C}(G)$ are Type II.